# Coq - introduction

Daria Walukiewicz-Chrząszcz

19 March 2024

# Coq — interactive proof assistant

http://coq.inria.fr/

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- rich (pure) functional programming language
- rich logical language
- user writes proofs
- Coq makes sure every step is correct
- and solves subgoals for which automated proving algorithms have been implemented
- (proved to be correct) program can be extracted to Ocaml, Haskell, Scheme...

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# Proving in Coq

Intuitionistic logic

Curry-Howard isomorphism

| proof | of a given | formula |
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| $\updownarrow$ | | $\updownarrow$ |
| term | of the corresponding | type |

$$\lambda x^{A\to B\to C}\lambda y^{A\to B}\lambda z^{A}\, xz(yz) \;:\; (A\to B\to C)\to(A\to B)\to(A\to C)$$

Correctness of Coq relies on correctness of type-checking

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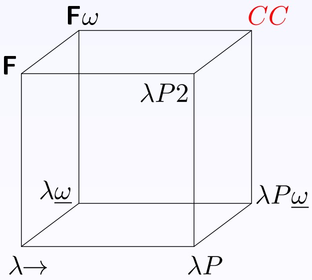
# Coq — formalism

Coq — calculus of constructions (CC) + inductive definitions




- $\uparrow$ polimorphism
- $\nearrow$ type constructors
- $\rightarrow$ dependent types

# Big picture of Coq architecture

The De Bruijn principle (“small” core, externally checkable terms)

- core / kernel, responsible for:
  - CIC typing
  - reduction
  - environment (definitions, axioms etc).
  - modules
- the rest, responsible for:
  - user interface
  - file management
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# Coq — a bit of history

1984 CoC - calculus of constructions - G. Huet, T. Coquand

1989 first public release (version 4.10)

1991 Coq - calculus of inductive constructions - C. Paulin (version 5.6)

⋮

2000 version 7.0 with new (safer) architecture

2003 version 7.4 with modules

2004 version 8.0 with new syntax

2009 version 8.2 with "type classes"

2012 version 8.4 with eta-reduction, structural proof syntax...

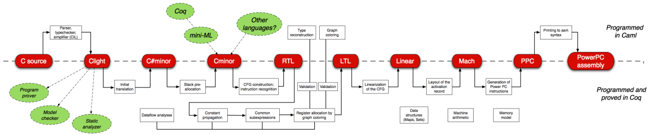
2019 version 8.11 with new tactic language Ltac2

2020 version 8.13 with definitional proof irrelevance for the equality type in SProp (introduced in 8.10)

# Coq — famous formalizations

- Fundamental theorem of algebra, Nijmegen 2000
- JavaCard Platform formalization, Trusted Logic 2003
- Four color theorem, Cambridge 2004
- CompCert certified Clight compiler, 2008-now

  *For the development of CompCert, the 2021 ACM Software System Award is awarded to Xavier Leroy, Sandrine Blazy, Zaynah Dargaye, Jacques-Henri Jourdan, Michael Schmidt, Bernhard Schommer, and Jean-Baptiste Tristan. This award recognizes "a software system that has had a lasting influence, reflected in contributions to concepts, in commercial acceptance, or both" and is considered the highest award for software originating from computer science research.*



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- predicative sorts `Set` and `Type`
- abstraction and application
- inductive types,
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- dependant types and dependent pattern-matching
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- interactive proof mode (goal management)
- built-in tactics (constructing a bit of proof-term): intro, apply, etc.
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- decision procedures: lia, ring, field, tauto, etc.
- tactic language (`Ltac mytactic:=...`)

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- extracted program satisfies its specification *by definition*
- extraction — "elimination" of logical parts from the proof-term
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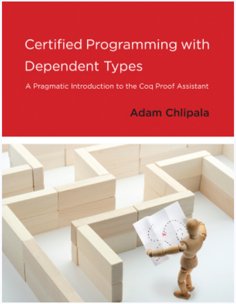


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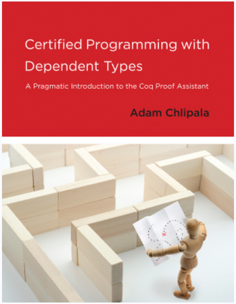


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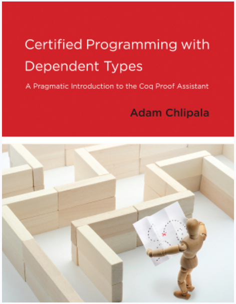


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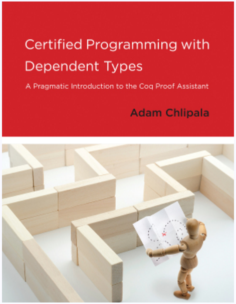


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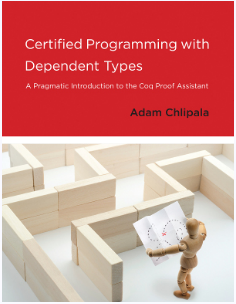


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# Typing

$$\texttt{environment} \vdash \texttt{term} : \texttt{type}$$

environment: global and local declarations and definitions

types are terms and have types, ex. `nat:Set`, `Set:Type`$_1$

but there are terms that are not types, ex. `fun n:nat => n`

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# Coq — formalism: fun for all

simple types abstraction rule: $$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M \; : \; A \to B}$$

dependent types abstraction rule: $$\frac{\Gamma, x{:}A \vdash M : B(x)}{\Gamma \vdash \lambda x{:}A.M \; : \; \forall x{:}A.B(x)}$$

Shorthand: $A \to B$ is $\forall x{:}A.B$, where $x \notin FV(B)$

concrete Coq syntax:

`fun n:nat =>` $M$ `: forall n:nat, vector n`

application rule:

$$\frac{\Gamma \vdash F : A \to B \quad \Gamma \vdash G : A}{\Gamma \vdash F\,G : B} \qquad \frac{\Gamma \vdash F : \forall x{:}A.B(x) \quad \Gamma \vdash G : A}{\Gamma \vdash F\,G : B[G/x]}$$

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# Coq — typing rules: sorts

- Sorts in Coq:

$$\begin{array}{l} Prop \\ Set \end{array} : Type_1 : Type_2 : \ldots$$

- Cummulativity (or sub-sorting):

$$Prop \leq Set \leq Type_1 \leq Type_2 \leq \ldots$$

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# Coq — products

product rule

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \forall x{:}A.B : s_2}$$ if $s_1$ and $s_2$ satisfy ...

- $s_1 \leq s_2$, or
- $s_2 = \mathit{Prop}$

cummulativity rule

$$\frac{\Gamma \vdash M : s_1}{\Gamma \vdash M : s_2}$$ jeśli $s_1 \leq s_2$

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# Coq — reductions

- beta

  $(\lambda x{:}A.M)N \longrightarrow_\beta M[N/x]$

- eta expansion (if $M$ is of a functional type)

  $M \longrightarrow_\eta \lambda x{:}A.Mx$

- delta

  (definition unfolding)

- zeta

  `(let x:=N in M)` $\longrightarrow_\zeta$ `M[N/x]`

- iota

  (inductive types reductions — soon :)

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# Coq — conversion

conversion rule

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash A =_{\beta\eta\delta\zeta\iota} A' \qquad \Gamma \vdash A' : s}{\Gamma \vdash M : A'}$$

`vector nat 4` $=_{iota}$ `vector nat (2+2)`

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# Coq implements intuitionistic logic

- forall and implication are built-in
- in intuitionistic logic $False$, $\wedge$, $\vee$ cannot be defined from $\rightarrow$
- they are defined as inductive types
- negation is defined $\neg\phi \equiv \phi \rightarrow False$
- in intuitionistic logic $\exists$ cannot be defined from $\forall$
- existential quantifier is defined as an inductive type
- one can use classical logic - axioms needed (ex: excluded middle)
- proof-checking is decidable (not provability)

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# True and False

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Inductive False : Prop :=.
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```
False_ind
   :forall P : Prop, False -> P
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Inductive True : Prop :=
   I : True.
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# Conjunction and disjunction

```
Inductive and (A B : Prop) : Prop :=
   conj : A -> B -> and A B
```

```
and_ind
   :forall A B P : Prop, (A -> B -> P) -> and A B -> P
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```
Inductive or (A B : Prop) : Prop :=
   or_introl : A -> or A B
 | or_intror : B -> or A B.
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or_ind
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$\wedge$ is an infix notation for and, $\vee$ is an infix notation for or

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# Existential quantifier

```
Inductive ex (A : Type) (P : A -> Prop) : Prop :=
   ex_intro : forall x : A, P x -> ex A P.
```

`exists y, P y` is a notation for `ex A P`

```
ex_ind
   : forall (A : Type) (P : A -> Prop) (P0 : Prop),
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