Dependent pattern-matching in Coq

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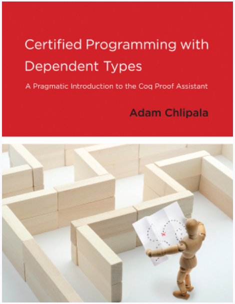
Dependent types in programming

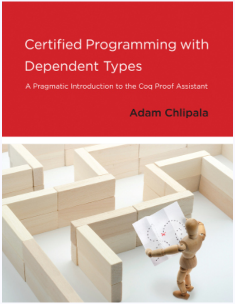
Compilation correctness — example from "Certified Programming with Dependent Types" by Adam Chlipala (MIT)



- available: http://adam.chlipala.net/cpdt/

Dependent types in programming

Compilation correctness — example from "Certified Programming with Dependent Types" by Adam Chlipala (MIT)



- available: `http://adam.chlipala.net/cpdt/`

Single sort: Source language

```
Inductive binop : Set := Plus | Times.

Inductive exp : Set :=
| Const : Z → exp
| Binop : binop → exp → exp → exp.

Check Const 42.

Check Binop Plus (Const 2) (Const 2).

Check Binop Times (Binop Plus (Const 2) (Const 2)) (Const 7).
```

Single sort: Source language

$\texttt{Inductive}\ \mathit{binop} : \texttt{Set} := \mathit{Plus} \mid \mathit{Times}.$

$\texttt{Inductive}\ \mathit{exp} : \texttt{Set} :=$
$\mid \mathit{Const} : Z \to \mathit{exp}$
$\mid \mathit{Binop} : \mathit{binop} \to \mathit{exp} \to \mathit{exp} \to \mathit{exp}.$

$\texttt{Check}\ \mathit{Const}\ 42.$

$\texttt{Check}\ \mathit{Binop}\ \mathit{Plus}\ (\mathit{Const}\ 2)\ (\mathit{Const}\ 2).$

$\texttt{Check}\ \mathit{Binop}\ \mathit{Times}\ (\mathit{Binop}\ \mathit{Plus}\ (\mathit{Const}\ 2)\ (\mathit{Const}\ 2))\ (\mathit{Const}\ 7).$

Single sort: Source language

```
Inductive binop : Set := Plus | Times.

Inductive exp : Set :=
| Const : Z → exp
| Binop : binop → exp → exp → exp.

Check Const 42.

Check Binop Plus (Const 2) (Const 2).

Check Binop Times (Binop Plus (Const 2) (Const 2)) (Const 7).
```

Single sort: Source language

$\texttt{Inductive}$ $\mathit{binop} : \texttt{Set} := \mathit{Plus} \mid \mathit{Times}.$

$\texttt{Inductive}$ $\mathit{exp} : \texttt{Set} :=$
$\mid \mathit{Const} : Z \to \mathit{exp}$
$\mid \mathit{Binop} : \mathit{binop} \to \mathit{exp} \to \mathit{exp} \to \mathit{exp}.$

$\texttt{Check}$ $\mathit{Const}\ 42.$

$\texttt{Check}$ $\mathit{Binop}\ \mathit{Plus}\ (\mathit{Const}\ 2)\ (\mathit{Const}\ 2).$

$\texttt{Check}$ $\mathit{Binop}\ \mathit{Times}\ (\mathit{Binop}\ \mathit{Plus}\ (\mathit{Const}\ 2)\ (\mathit{Const}\ 2))\ (\mathit{Const}\ 7).$

Single sort: Source language

Inductive $binop$: Set := $Plus$ | $Times$.

Inductive exp : Set :=
| $Const$: $Z \to exp$
| $Binop$: $binop \to exp \to exp \to exp$.

Check $Const$ 42.

Check $Binop$ $Plus$ ($Const$ 2) ($Const$ 2).

Check $Binop$ $Times$ ($Binop$ $Plus$ ($Const$ 2) ($Const$ 2)) ($Const$ 7).

Single sort: Source language

$\texttt{Inductive}\ \mathit{binop} : \texttt{Set} := \mathit{Plus} \mid \mathit{Times}.$

$\texttt{Inductive}\ \mathit{exp} : \texttt{Set} :=$
$\mid \mathit{Const} : Z \rightarrow \mathit{exp}$
$\mid \mathit{Binop} : \mathit{binop} \rightarrow \mathit{exp} \rightarrow \mathit{exp} \rightarrow \mathit{exp}.$

$\texttt{Check}\ \mathit{Const}\ 42.$

$\texttt{Check}\ \mathit{Binop}\ \mathit{Plus}\ (\mathit{Const}\ 2)\ (\mathit{Const}\ 2).$

$\texttt{Check}\ \mathit{Binop}\ \mathit{Times}\ (\mathit{Binop}\ \mathit{Plus}\ (\mathit{Const}\ 2)\ (\mathit{Const}\ 2))\ (\mathit{Const}\ 7).$

Single sort: Source language denotation

$\texttt{Inductive}\ \mathit{binop} : \texttt{Set} := \mathit{Plus} \mid \mathit{Times}.$

$\texttt{Inductive}\ \mathit{exp} : \texttt{Set} :=$
$\mid \mathit{Const} : Z \rightarrow \mathit{exp}$
$\mid \mathit{Binop} : \mathit{binop} \rightarrow \mathit{exp} \rightarrow \mathit{exp} \rightarrow \mathit{exp}.$

$\texttt{Definition}\ \mathit{binopDenote}\ (b : \mathit{binop}) : Z \rightarrow Z \rightarrow Z :=$
$\quad \texttt{match}\ b\ \texttt{with}$
$\quad \mid \mathit{Plus} \Rightarrow \texttt{fun}\ x\ y \Rightarrow x + y$
$\quad \mid \mathit{Times} \Rightarrow \texttt{fun}\ x\ y \Rightarrow x \times y$
$\quad \texttt{end}.$

$\texttt{Fixpoint}\ \mathit{expDenote}\ (e : \mathit{exp}) : Z :=$
$\quad \texttt{match}\ e\ \texttt{with}$
$\quad \mid \mathit{Const}\ n \Rightarrow n$
$\quad \mid \mathit{Binop}\ b\ e1\ e2 \Rightarrow (\mathit{binopDenote}\ b)\ (\mathit{expDenote}\ e1)\ (\mathit{expDenote}\ e2)$
$\quad \texttt{end}.$

Single sort: Source language denotation

```
Inductive binop : Set := Plus | Times.

Inductive exp : Set :=
| Const : Z → exp
| Binop : binop → exp → exp → exp.

Definition binopDenote (b : binop) : Z → Z → Z :=
  match b with
  | Plus ⇒ fun x y ⇒ x + y
  | Times ⇒ fun x y ⇒ x × y
  end.

Fixpoint expDenote (e : exp) : Z :=
  match e with
  | Const n ⇒ n
  | Binop b e1 e2 ⇒ (binopDenote b) (expDenote e1) (expDenote e2)
  end.
```

Single sort: Source language denotation

Inductive $binop$: Set := $Plus \mid Times$.

Inductive exp : Set :=
| $Const$: $Z \rightarrow exp$
| $Binop$: $binop \rightarrow exp \rightarrow exp \rightarrow exp$.

Definition $binopDenote$ $(b : binop) : Z \rightarrow Z \rightarrow Z$:=
  match b with
  | $Plus \Rightarrow$ fun x y $\Rightarrow x + y$
  | $Times \Rightarrow$ fun x y $\Rightarrow x \times y$
  end.

Fixpoint $expDenote$ $(e : exp) : Z$:=
  match e with
  | $Const\ n \Rightarrow n$
  | $Binop\ b\ e1\ e2 \Rightarrow (binopDenote\ b)\ (expDenote\ e1)\ (expDenote\ e2)$
  end.

Single sort: Target language

```
Inductive instr : Set :=
| iConst : Z → instr
| iBinop : binop → instr.
```

```
Definition prog := list instr.
```

```
Check iConst 42 :: nil.
```

```
Check iConst 2 :: iConst 2 :: iBinop Plus :: nil.
```

```
Check iConst 7 :: iConst 2 :: iConst 2 :: iBinop Plus :: iBinop Times :: nil.
```

Single sort: Target language

```
Inductive instr : Set :=
| iConst : Z → instr
| iBinop : binop → instr.

Definition prog := list instr.

Check iConst 42 :: nil.

Check iConst 2 :: iConst 2 :: iBinop Plus :: nil.

Check iConst 7 :: iConst 2 :: iConst 2 :: iBinop Plus :: iBinop Times :: nil.
```

Single sort: Target language

Inductive $instr$: Set :=
| $iConst : Z \to instr$
| $iBinop : binop \to instr.$

Definition $prog := list\ instr.$

Check $iConst\ 42 :: nil.$

Check $iConst\ 2 :: iConst\ 2 :: iBinop\ Plus :: nil.$

Check $iConst\ 7 :: iConst\ 2 :: iConst\ 2 :: iBinop\ Plus :: iBinop\ Times :: nil.$

Single sort: Target language

Inductive *instr* : Set :=
| *iConst* : $Z \to$ *instr*
| *iBinop* : *binop* $\to$ *instr*.

Definition *prog* := *list instr*.

Check *iConst* 42 :: *nil*.

Check *iConst* 2 :: *iConst* 2 :: *iBinop Plus* :: *nil*.

Check *iConst* 7 :: *iConst* 2 :: *iConst* 2 :: *iBinop Plus* :: *iBinop Times* :: *nil*.

Single sort: Target language

```
Inductive instr : Set :=
| iConst : Z → instr
| iBinop : binop → instr.
```

```
Definition prog := list instr.
```

```
Check iConst 42 :: nil.
```

```
Check iConst 2 :: iConst 2 :: iBinop Plus :: nil.
```

```
Check iConst 7 :: iConst 2 :: iConst 2 :: iBinop Plus :: iBinop Times :: nil.
```

Single sort: Target language denotation

```
Inductive instr : Set :=
| iConst : Z → instr
| iBinop : binop → instr.

Definition prog := list instr.

Definition stack := list Z.

Definition instrDenote (i : instr) (s : stack) : option stack :=
  match i with
  | iConst n ⇒ Some (n :: s)
  | iBinop b ⇒
    match s with
    | arg1 :: arg2 :: s' ⇒ Some ((binopDenote b) arg1 arg2 :: s')
    | _ ⇒ None
    end
  end.
```

Single sort: Target language denotation

Inductive $instr$: Set :=
| $iConst$: $Z \rightarrow instr$
| $iBinop$: $binop \rightarrow instr$.

Definition $prog$:= $list\ instr$.

Definition $stack$:= $list\ Z$.

```
Definition instrDenote (i : instr) (s : stack) : option stack :=
  match i with
  | iConst n ⇒ Some (n :: s)
  | iBinop b ⇒
    match s with
    | arg1 :: arg2 :: s' ⇒ Some ((binopDenote b) arg1 arg2 :: s')
    | _ ⇒ None
    end
  end.
```

Single sort: Target language denotation

```
Inductive instr : Set :=
| iConst : Z → instr
| iBinop : binop → instr.

Definition prog := list instr.

Definition stack := list Z.

Definition instrDenote (i : instr) (s : stack) : option stack :=
  match i with
  | iConst n ⇒ Some (n :: s)
  | iBinop b ⇒
    match s with
    | arg1 :: arg2 :: s' ⇒ Some ((binopDenote b) arg1 arg2 :: s')
    | _ ⇒ None
    end
  end.
```

Single sort: Target language denotation

Inductive $instr$: Set :=
| $iConst : Z \to instr$
| $iBinop : binop \to instr$.

Definition $prog$:= $list\ instr$.

Definition $stack$:= $list\ Z$.

Definition $instrDenote\ (i : instr)\ (s : stack) : option\ stack$:= ...

Single sort: Target language denotation

```
Inductive instr : Set :=
| iConst : Z → instr
| iBinop : binop → instr.

Definition prog := list instr.

Definition stack := list Z.

Definition instrDenote (i : instr) (s : stack) : option stack := ...

Fixpoint progDenote (p : prog) (s : stack) : option stack :=
  match p with
    | nil ⇒ Some s
    | i :: p' ⇒ match instrDenote i s with
                  | None ⇒ None
                  | Some s' ⇒ progDenote p' s'
                end
  end.
```

Single sort: Compilation

```
Fixpoint compile (e : exp) : prog :=
  match e with
  | Const n ⇒ iConst n :: nil
  | Binop b e1 e2 ⇒ compile e2 ++ compile e1 ++ iBinop b :: nil
  end.
```

Single sort: Compilation correctness

```
Fixpoint compile (e : exp) : prog :=
  match e with
  | Const n ⇒ iConst n :: nil
  | Binop b e1 e2 ⇒ compile e2 ++ compile e1 ++ iBinop b :: nil
  end.
```

```
Theorem compile_correct :
  ∀ e, progDenote (compile e) nil = Some (expDenote e :: nil).
```

```
Lemma compile_correct' : ∀ e p s,
  progDenote (compile e ++ p) s = progDenote p (expDenote e :: s).
```

Single sort: Compilation correctness

Fixpoint $compile\ (e : exp) : prog :=$
  match e with
  $|\ Const\ n \Rightarrow iConst\ n :: nil$
  $|\ Binop\ b\ e1\ e2 \Rightarrow compile\ e2 ++ compile\ e1 ++ iBinop\ b :: nil$
  end.

Theorem $compile\_correct$:
  $\forall\ e,\ progDenote\ (compile\ e)\ nil = Some\ (expDenote\ e :: nil)$.

Lemma $compile\_correct' : \forall\ e\ p\ s,$
  $progDenote\ (compile\ e ++ p)\ s = progDenote\ p\ (expDenote\ e :: s)$.

Single sort: Compilation correctness

```
Fixpoint compile (e : exp) : prog :=
  match e with
  | Const n ⇒ iConst n :: nil
  | Binop b e1 e2 ⇒ compile e2 ++ compile e1 ++ iBinop b :: nil
  end.
```

```
Theorem compile_correct :
  ∀ e, progDenote (compile e) nil = Some (expDenote e :: nil).
```

```
Lemma compile_correct' : ∀ e p s,
  progDenote (compile e ++ p) s = progDenote p (expDenote e :: s).
```

Many sorts: Source language

```
Inductive sort : Set := Mint | Mbool.

Inductive mbinop : sort → sort → sort → Set :=
| MPlus : mbinop Mint Mint Mint
| MTimes : mbinop Mint Mint Mint
| MEq : ∀ s, mbinop s s Mbool
| MLt : mbinop Mint Mint Mbool.

Inductive mexp : sort → Set :=
| MZConst : Z → mexp Mint
| MBConst : bool → mexp Mbool
| MBinop : ∀ s1 s2 s, mbinop s1 s2 s → mexp s1 → mexp s2 → mexp s.

Check MBinop MPlus (MZConst 2) (MZConst 2) : mexp Mint.

Check MBinop (MEq Mint) (MBinop MPlus (MZConst 2) (MZConst 2))
                        (MZConst 7) : mexp Mbool.
```

Many sorts: Source language

Inductive *sort* : Set := *Mint* | *Mbool*.

Inductive *mbinop* : *sort* → *sort* → *sort* → Set :=
| *MPlus* : *mbinop Mint Mint Mint*
| *MTimes* : *mbinop Mint Mint Mint*
| *MEq* : ∀ *s*, *mbinop s s Mbool*
| *MLt* : *mbinop Mint Mint Mbool*.

Inductive *mexp* : *sort* → Set :=
| *MZConst* : *Z* → *mexp Mint*
| *MBConst* : *bool* → *mexp Mbool*
| *MBinop* : ∀ *s1 s2 s*, *mbinop s1 s2 s* → *mexp s1* → *mexp s2* → *mexp s*.

Check *MBinop MPlus* (*MZConst* 2) (*MZConst* 2) : *mexp Mint*.

Check *MBinop* (*MEq Mint*) (*MBinop MPlus* (*MZConst* 2) (*MZConst* 2))
(*MZConst* 7) : *mexp Mbool*.

Many sorts: Source language

Inductive sort : Set := Mint | Mbool.

Inductive mbinop : $\mathit{sort} \to \mathit{sort} \to \mathit{sort} \to$ Set :=
| MPlus : $\mathit{mbinop}\ \mathit{Mint}\ \mathit{Mint}\ \mathit{Mint}$
| MTimes : $\mathit{mbinop}\ \mathit{Mint}\ \mathit{Mint}\ \mathit{Mint}$
| MEq : $\forall\ s,\ \mathit{mbinop}\ s\ s\ \mathit{Mbool}$
| MLt : $\mathit{mbinop}\ \mathit{Mint}\ \mathit{Mint}\ \mathit{Mbool}$.

Inductive mexp : $\mathit{sort} \to$ Set :=
| $\mathit{MZConst}$: $Z \to \mathit{mexp}\ \mathit{Mint}$
| $\mathit{MBConst}$: $\mathit{bool} \to \mathit{mexp}\ \mathit{Mbool}$
| MBinop : $\forall\ s1\ s2\ s,\ \mathit{mbinop}\ s1\ s2\ s \to \mathit{mexp}\ s1 \to \mathit{mexp}\ s2 \to \mathit{mexp}\ s$.

Check $\mathit{MBinop}\ \mathit{MPlus}\ (\mathit{MZConst}\ 2)\ (\mathit{MZConst}\ 2)$: $\mathit{mexp}\ \mathit{Mint}$.

Check $\mathit{MBinop}\ (\mathit{MEq}\ \mathit{Mint})\ (\mathit{MBinop}\ \mathit{MPlus}\ (\mathit{MZConst}\ 2)\ (\mathit{MZConst}\ 2))$
$(\mathit{MZConst}\ 7)$: $\mathit{mexp}\ \mathit{Mbool}$.

Many sorts: Source language

Inductive *sort* : Set := *Mint* | *Mbool*.

Inductive *mbinop* : *sort* → *sort* → *sort* → Set :=
| *MPlus* : *mbinop Mint Mint Mint*
| *MTimes* : *mbinop Mint Mint Mint*
| *MEq* : ∀ *s*, *mbinop s s Mbool*
| *MLt* : *mbinop Mint Mint Mbool*.

Inductive *mexp* : *sort* → Set :=
| *MZConst* : *Z* → *mexp Mint*
| *MBConst* : *bool* → *mexp Mbool*
| *MBinop* : ∀ *s1 s2 s*, *mbinop s1 s2 s* → *mexp s1* → *mexp s2* → *mexp s*.

Check *MBinop MPlus* (*MZConst* 2) (*MZConst* 2) : *mexp Mint*.

Check *MBinop* (*MEq Mint*) (*MBinop MPlus* (*MZConst* 2) (*MZConst* 2))
(*MZConst* 7) : *mexp Mbool*.

Many sorts: Source language

Inductive $sort$: Set := $Mint$ | $Mbool$.

Inductive $mbinop$: $sort \rightarrow sort \rightarrow sort \rightarrow$ Set :=
| $MPlus$: $mbinop\ Mint\ Mint\ Mint$
| $MTimes$: $mbinop\ Mint\ Mint\ Mint$
| MEq : $\forall\ s,\ mbinop\ s\ s\ Mbool$
| MLt : $mbinop\ Mint\ Mint\ Mbool$.

Inductive $mexp$: $sort \rightarrow$ Set :=
| $MZConst$: $Z \rightarrow mexp\ Mint$
| $MBConst$: $bool \rightarrow mexp\ Mbool$
| $MBinop$: $\forall\ s1\ s2\ s,\ mbinop\ s1\ s2\ s \rightarrow mexp\ s1 \rightarrow mexp\ s2 \rightarrow mexp\ s$.

Check $MBinop\ MPlus\ (MZConst\ 2)\ (MZConst\ 2) : mexp\ Mint$.

Check $MBinop\ (MEq\ Mint)\ (MBinop\ MPlus\ (MZConst\ 2)\ (MZConst\ 2))$
$(MZConst\ 7) : mexp\ Mbool$.

Many sorts: Source language denotation

```
Definition sortDenote (s : sort) : Set :=
  match s with
    | Mint ⇒ Z
    | Mbool ⇒ bool
  end.

Definition mbinopDenote arg1 arg2 res (b : mbinop arg1 arg2 res)
  : sortDenote arg1 → sortDenote arg2 → sortDenote res :=
  match b with
    | MPlus ⇒ Z.add
    | MTimes ⇒ Z.mul
    | MEq Mint ⇒ Z.eqb
    | MEq Mbool ⇒ Bool.eqb
    | MLt ⇒ Z.ltb
  end.

Fixpoint mexpDenote s (e : mexp s) : sortDenote s :=
  match e with
    | MZConst n ⇒ n
    | MBConst b ⇒ b
    | MBinop b e1 e2 ⇒
```

Many sorts: Source language denotation

```
Definition sortDenote (s : sort) : Set :=
  match s with Mint ⇒ Z | Mbool ⇒ bool end.

Definition mbinopDenote arg1 arg2 res (b : mbinop arg1 arg2 res)
  : sortDenote arg1 → sortDenote arg2 → sortDenote res :=
  match b with
    | MPlus ⇒ Z.add
    | MTimes ⇒ Z.mul
    | MEq Mint ⇒ Z.eqb
    | MEq Mbool ⇒ Bool.eqb
    | MLt ⇒ Z.ltb
  end.

Fixpoint mexpDenote s (e : mexp s) : sortDenote s :=
  match e with
    | MZConst n ⇒ n
    | MBConst b ⇒ b
    | MBinop b e1 e2 ⇒
        (mbinopDenote b) (mexpDenote e1) (mexpDenote e2)
  end.
```

Many sorts: Source language denotation

```
Definition sortDenote (s : sort) : Set :=
  match s with Mint ⇒ Z | Mbool ⇒ bool end.

Definition mbinopDenote arg1 arg2 res (b : mbinop arg1 arg2 res)
  : sortDenote arg1 → sortDenote arg2 → sortDenote res :=
  match b with
    | MPlus ⇒ Z.add
    | MTimes ⇒ Z.mul
    | MEq Mint ⇒ Z.eqb
    | MEq Mbool ⇒ Bool.eqb
    | MLt ⇒ Z.ltb
  end.

Fixpoint mexpDenote s (e : mexp s) : sortDenote s :=
  match e with
    | MZConst n ⇒ n
    | MBConst b ⇒ b
    | MBinop b e1 e2 ⇒
        (mbinopDenote b) (mexpDenote e1) (mexpDenote e2)
  end.
```

Many sorts: Source language denotation

```
Definition sortDenote (s : sort) : Set :=
  match s with Mint ⇒ Z | Mbool ⇒ bool end.

Definition mbinopDenote arg1 arg2 res (b : mbinop arg1 arg2 res)
  : sortDenote arg1 → sortDenote arg2 → sortDenote res :=
  match b with
    | MPlus ⇒ Z.add
    | MTimes ⇒ Z.mul
    | MEq Mint ⇒ Z.eqb
    | MEq Mbool ⇒ Bool.eqb
    | MLt ⇒ Z.ltb
  end.

Fixpoint mexpDenote s (e : mexp s) : sortDenote s :=
  match e with
    | MZConst n ⇒ n
    | MBConst b ⇒ b
    | MBinop b e1 e2 ⇒
        (mbinopDenote b) (mexpDenote e1) (mexpDenote e2)
  end.
```

Many sorts: Target language

```
Definition sstack := list sort.

Inductive minstr : sstack → sstack → Set :=
| MiZConst : ∀ ss, Z → minstr ss (Mint :: ss)
| MiBConst : ∀ ss, bool → minstr ss (Mbool :: ss)
| MiBinop : ∀ arg1 arg2 res ss,
    mbinop arg1 arg2 res → minstr (arg1 :: arg2 :: ss) (res :: ss).

Inductive mprog : sstack → sstack → Set :=
| MNil : ∀ ss, mprog ss ss
| MCons : ∀ ss1 ss2 ss3, minstr ss1 ss2 → mprog ss2 ss3 → mprog ss1 ss3.
```

Many sorts: Target language

Definition *sstack* := *list sort*.

Inductive *minstr* : *sstack* → *sstack* → Set :=
| *MiZConst* : ∀ *ss*, *Z* → *minstr ss* (*Mint* :: *ss*)
| *MiBConst* : ∀ *ss*, *bool* → *minstr ss* (*Mbool* :: *ss*)
| *MiBinop* : ∀ *arg1 arg2 res ss*,
  mbinop arg1 arg2 res → *minstr* (*arg1* :: *arg2* :: *ss*) (*res* :: *ss*).

Inductive *mprog* : *sstack* → *sstack* → Set :=
| *MNil* : ∀ *ss*, *mprog ss ss*
| *MCons* : ∀ *ss1 ss2 ss3*, *minstr ss1 ss2* → *mprog ss2 ss3* → *mprog ss1 ss3*.

Many sorts: Target language

```
Definition sstack := list sort.

Inductive minstr : sstack → sstack → Set :=
| MiZConst : ∀ ss, Z → minstr ss (Mint :: ss)
| MiBConst : ∀ ss, bool → minstr ss (Mbool :: ss)
| MiBinop : ∀ arg1 arg2 res ss,
   mbinop arg1 arg2 res → minstr (arg1 :: arg2 :: ss) (res :: ss).

Inductive mprog : sstack → sstack → Set :=
| MNil : ∀ ss, mprog ss ss
| MCons : ∀ ss1 ss2 ss3, minstr ss1 ss2 → mprog ss2 ss3 → mprog ss1 ss3.
```

Many sorts: Target language denotation

```
Fixpoint vstack (ss : sstack) : Set :=
  match ss with
    | nil ⇒ unit
    | s :: ss' ⇒ sortDenote s × vstack ss'
  end.

Check (5, (true, (false, tt))) : vstack (Mint::Mbool::Mbool::nil).

Definition minstrDenote ss ss' (i : minstr ss ss') : vstack ss → vstack ss' :=
  match i with
    | MiZConst _ n ⇒ fun vs ⇒ (n, vs)
    | MiBConst _ b ⇒ fun vs ⇒ (b, vs)
    | MiBinop _ b ⇒ fun vs ⇒
        let '(v1, (v2, vs')) := vs in ((mbinopDenote b) v1 v2, vs')
  end.

Fixpoint mprogDenote ss ss' (p : mprog ss ss') : vstack ss → vstack ss' :=
  match p with
    | MNil _ ⇒ fun vs ⇒ vs
    | MCons i p' ⇒ fun vs ⇒ mprogDenote p' (minstrDenote i vs)
  end.
```

Many sorts: Target language denotation

```
Fixpoint vstack (ss : sstack) : Set :=
  match ss with
    | nil ⇒ unit
    | s :: ss' ⇒ sortDenote s × vstack ss'
  end.

Check (5, (true, (false, tt))) : vstack (Mint::Mbool::Mbool::nil).

Definition minstrDenote ss ss' (i : minstr ss ss') : vstack ss → vstack ss' :=
  match i with
    | MiZConst _ n ⇒ fun vs ⇒ (n, vs)
    | MiBConst _ b ⇒ fun vs ⇒ (b, vs)
    | MiBinop _ b ⇒ fun vs ⇒
      let '(v1, (v2, vs')) := vs in ((mbinopDenote b) v1 v2, vs')
  end.

Fixpoint mprogDenote ss ss' (p : mprog ss ss') : vstack ss → vstack ss' :=
  match p with
    | MNil _ ⇒ fun vs ⇒ vs
    | MCons i p' ⇒ fun vs ⇒ mprogDenote p' (minstrDenote i vs)
  end.
```

Many sorts: Target language denotation

Fixpoint $vstack$ $(ss : sstack)$: Set :=
  match ss with $nil \Rightarrow unit$ | $s :: ss' \Rightarrow sortDenote\ s \times vstack\ ss'$ end.

Check $(5, (true, (false, tt)))$: $vstack\ (Mint::Mbool::Mbool::nil)$.

Definition $minstrDenote\ ss\ ss'\ (i : minstr\ ss\ ss') : vstack\ ss \rightarrow vstack\ ss'$:=
  match i with
    | $MiZConst\ \_\ n \Rightarrow$ fun $vs \Rightarrow (n, vs)$
    | $MiBConst\ \_\ b \Rightarrow$ fun $vs \Rightarrow (b, vs)$
    | $MiBinop\ \_\ b \Rightarrow$ fun $vs \Rightarrow$
      let $'(v1, (v2, vs'))$:= vs in $((mbinopDenote\ b)\ v1\ v2, vs')$
  end.

Fixpoint $mprogDenote\ ss\ ss'\ (p : mprog\ ss\ ss') : vstack\ ss \rightarrow vstack\ ss'$:=
  match p with
    | $MNil\ \_ \Rightarrow$ fun $vs \Rightarrow vs$
    | $MCons\ i\ p' \Rightarrow$ fun $vs \Rightarrow mprogDenote\ p'\ (minstrDenote\ i\ vs)$
  end.

Many sorts: Target language denotation

```
Fixpoint vstack (ss : sstack) : Set :=
  match ss with nil ⇒ unit | s :: ss' ⇒ sortDenote s × vstack ss' end.

Check (5, (true, (false, tt))) : vstack (Mint::Mbool::Mbool::nil).

Definition minstrDenote ss ss' (i : minstr ss ss') : vstack ss → vstack ss' :=
  match i with
    | MiZConst _ n ⇒ fun vs ⇒ (n, vs)
    | MiBConst _ b ⇒ fun vs ⇒ (b, vs)
    | MiBinop _ b ⇒ fun vs ⇒
      let '(v1, (v2, vs')) := vs in ((mbinopDenote b) v1 v2, vs')
  end.

Fixpoint mprogDenote ss ss' (p : mprog ss ss') : vstack ss → vstack ss' :=
  match p with
    | MNil _ ⇒ fun vs ⇒ vs
    | MCons i p' ⇒ fun vs ⇒ mprogDenote p' (minstrDenote i vs)
  end.
```

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  end.
```

Many sorts: Compilation

$\texttt{Fixpoint}\ \mathit{mcompile}\ s\ (e : \mathit{mexp}\ s)\ (\mathit{ss} : \mathit{sstack}) : \mathit{mprog}\ \mathit{ss}\ (s :: \mathit{ss}) :=$
$\quad\texttt{match}\ e\ \texttt{with}$
$\qquad |\ \mathit{MZConst}\ n \Rightarrow \mathit{MCons}\ (\mathit{MiZConst}\ \_\ n)\ (\mathit{MNil}\ \_)$
$\qquad |\ \mathit{MBConst}\ b \Rightarrow \mathit{MCons}\ (\mathit{MiBConst}\ \_\ b)\ (\mathit{MNil}\ \_)$
$\qquad |\ \mathit{MBinop}\ b\ e1\ e2 \Rightarrow \mathit{mconcat}\ (\mathit{mcompile}\ e2\ \_)$
$\qquad\quad (\mathit{mconcat}\ (\mathit{mcompile}\ e1\ \_)\ (\mathit{MCons}\ (\mathit{MiBinop}\ \_\ b)\ (\mathit{MNil}\ \_)))$
$\quad\texttt{end}.$

Many sorts: Compilation correctness

```
Fixpoint mcompile s (e : mexp s) (ss : sstack) : mprog ss (s :: ss) :=
  match e with
    | MZConst n ⇒ MCons (MiZConst _ n) (MNil _)
    | MBConst b ⇒ MCons (MiBConst _ b) (MNil _)
    | MBinop b e1 e2 ⇒ mconcat (mcompile e2 _)
        (mconcat (mcompile e1 _) (MCons (MiBinop _ b) (MNil _)))
  end.
```

```
Theorem mcompile_correct : ∀ s (e : mexp s),
  mprogDenote (mcompile e nil) tt = (mexpDenote e, tt).
```

```
Lemma mcompile_correct' : ∀ s (e : mexp s) ss (s : vstack ss),
  mprogDenote (mcompile e ss) s = (mexpDenote e, s).
```

Many sorts: Compilation correctness

Fixpoint $mcompile\ s\ (e : mexp\ s)\ (ss : sstack) : mprog\ ss\ (s :: ss) :=$
  match e with
    $|\ MZConst\ n \Rightarrow MCons\ (MiZConst\ \_\ n)\ (MNil\ \_)$
    $|\ MBConst\ b \Rightarrow MCons\ (MiBConst\ \_\ b)\ (MNil\ \_)$
    $|\ MBinop\ b\ e1\ e2 \Rightarrow mconcat\ (mcompile\ e2\ \_)$
      $(mconcat\ (mcompile\ e1\ \_)\ (MCons\ (MiBinop\ \_\ b)\ (MNil\ \_)))$
  end.

Theorem $mcompile\_correct : \forall\ s\ (e : mexp\ s),$
  $mprogDenote\ (mcompile\ e\ nil)\ tt = (mexpDenote\ e, tt).$

Lemma $mcompile\_correct' : \forall\ s\ (e : mexp\ s)\ ss\ (s : vstack\ ss),$
  $mprogDenote\ (mcompile\ e\ ss)\ s = (mexpDenote\ e, s).$

Many sorts: Compilation correctness

$\texttt{Fixpoint}\ \mathit{mcompile}\ s\ (e : \mathit{mexp}\ s)\ (\mathit{ss} : \mathit{sstack}) : \mathit{mprog}\ \mathit{ss}\ (s :: \mathit{ss}) :=$
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$\qquad|\ \mathit{MZConst}\ n \Rightarrow \mathit{MCons}\ (\mathit{MiZConst}\ \_\ n)\ (\mathit{MNil}\ \_)$
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$\qquad|\ \mathit{MBinop}\ b\ e1\ e2 \Rightarrow \mathit{mconcat}\ (\mathit{mcompile}\ e2\ \_)$
$\qquad\quad(\mathit{mconcat}\ (\mathit{mcompile}\ e1\ \_)\ (\mathit{MCons}\ (\mathit{MiBinop}\ \_\ b)\ (\mathit{MNil}\ \_)))$
$\quad\texttt{end}.$

$\texttt{Theorem}\ \mathit{mcompile\_correct} : \forall\ s\ (e : \mathit{mexp}\ s),$
$\quad\mathit{mprogDenote}\ (\mathit{mcompile}\ e\ \mathit{nil})\ \mathit{tt} = (\mathit{mexpDenote}\ e, \mathit{tt}).$

$\texttt{Lemma}\ \mathit{mcompile\_correct}' : \forall\ s\ (e : \mathit{mexp}\ s)\ \mathit{ss}\ (s : \mathit{vstack}\ \mathit{ss}),$
$\quad\mathit{mprogDenote}\ (\mathit{mcompile}\ e\ \mathit{ss})\ s = (\mathit{mexpDenote}\ e, s).$

Function pred

Print *pred*.

$$\begin{array}{l} \mathrm{pred} = \mathtt{fun}\ n : \mathbf{nat} \Rightarrow \mathtt{match}\ n\ \mathtt{with} \\ \qquad\qquad\qquad\qquad\quad \mid 0 \Rightarrow 0 \\ \qquad\qquad\qquad\qquad\quad \mid \mathsf{S}u \Rightarrow u \\ \qquad\qquad\qquad\qquad\quad \mathrm{end} \\ \quad : \mathbf{nat} \rightarrow \mathbf{nat} \end{array}$$

Function pred_strong1

```
Lemma zgtz : 0 > 0 → False.
  intro H.
  inversion H.
Qed.

Definition pred_strong1 (n : nat) : n > 0 → nat :=
  match n with
    | O ⇒ fun pf : 0 > 0 ⇒ match zgtz pf with end
    | S n' ⇒ fun _ ⇒ n'
  end.

Theorem two_gt0 : 2 > 0.
  auto.
Qed.

Eval compute in pred_strong1 two_gt0.

    = 1
    : nat
```

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Theorem two_gt0 : 2 > 0.
  auto.
Qed.

Eval compute in pred_strong1 two_gt0.

     = 1
     : nat
```

Function pred_strong1'

```
Definition pred_strong1' (n : nat) (pf : n > 0) : nat :=
  match n with
    | O => match zgtz pf with end
    | S n' => n'
  end.
```

```
Error: In environment
n : nat
pf : n > 0
The term "pf" has type " n > 0" while it is expected to have
type "0 > 0"
```

Function pred_strong1'

```
Definition pred_strong1' (n : nat) (pf : n > 0) : nat :=
  match n with
    | O => match zgtz pf with end
    | S n' => n'
  end.
```

```
Error: In environment
n : nat
pf : n > 0
The term "pf" has type " n > 0" while it is expected to have
type "0 > 0"
```

Function pred_strong1 with explicit return

```
Definition pred_strong1 (n : nat) : n > 0 → nat :=
  match n return n > 0 → nat with
    | O ⇒ fun pf : 0 > 0 ⇒ match zgtz pf with end
    | S n' ⇒ fun _ ⇒ n'
  end.
```

Lists with length

```
Section ilist.
  Variable A : Set.

  Inductive ilist : nat → Set :=
  | INil : ilist O
  | ICons : ∀ n, A → ilist n → ilist (S n).
```

Dependent pattern-matching for `ilist` (1)

```
match l:ilist m

with
  | INil ⇒ ...
  | ICons n x l' ⇒ ...
end.   : P m
```

Dependent pattern-matching for `ilist` (2)

```
match l:ilist m
  in (ilist k)
  return (P k)
with
  | lNil ⇒ ... : P 0
  | lCons n x l' ⇒ ... : P (S n)
end.   : P m
```

Dependent pattern-matching for `ilist` (3)

```
match l:ilist m
  as v:ilist k
  in (ilist k)
  return (P k v)
with
  | INil ⇒ ... : P 0 INil
  | ICons n x l' ⇒ ... : P (S n) (ICons n x l')
end.   : P m l
```

Destruction - match

Simple form:

$$\text{match } m \text{ with } (c_1\ x_{11}\ \ldots\ x_{1p_1}) \Rightarrow f_1 \mid \ldots \mid (c_n\ x_{n1} \ldots x_{np_n}) \Rightarrow f_n \text{ end}$$

Full form:

$$\text{match } m \text{ as } x \text{ in } I\ \_\ a \text{ return } (P\ a\ x) \text{ with}$$
$$(c_1\ x_{11}\ \ldots\ x_{1p_1}) \Rightarrow f_1 \mid \ldots \mid (c_n\ x_{n1} \ldots x_{np_n}) \Rightarrow f_n \text{ end}$$

For the purpose of presenting the inference rules, we use a more compact notation :

$$\text{case}(m, P, \lambda x_{11}\ \ldots\ x_{1p_1}.f_1 \mid \ldots \mid \lambda x_{n1} \ldots x_{np_n}.f_n)$$

Destruction - match (2)

Type of branches. Let c be a term of type C, we assume C is a type of constructor for an inductive definition I. Let P be a term that represents the property to be proved. We assume r is the number of parameters. We define a new type $\{c : C\}^P$ which represents the type of the branch corresponding to the $c : C$ constructor.

$$\begin{aligned} \{c : (I_i\ p_1 \ldots p_r\ t_1 \ldots t_p)\}^P & \equiv (P\ t_1 \ldots\ t_p\ c) \\ \{c : \forall\ x : T, C\}^P & \equiv \forall\ x : T, \{(c\ x) : C\}^P \end{aligned}$$

We write $\{c\}^P$ for $\{c : C\}^P$ with C the type of c.

Example for `ilist` and $s \in \mathcal{S}$:
$P :\ \forall n : nat, \texttt{ilist}\ n \to s$
$\{(\texttt{ICons})\}^P \equiv \forall n : nat, a : A, l : \texttt{ilist}\ n, P\ (\texttt{S}\ n)\ (\texttt{ICons}\ n\ a\ l)$.

Allowed elimination sorts

`[I:A |B]` is the smallest relation satisfying the following rules:

- if `[(I x):A'|B']` then `[I:forall x:A, A'|forall x:A, B']`
- `[I:s1|I -> s2]` for any $s1 \in \mathsf{Set}, \mathsf{Type}(j)$, $s2 \in S$
- `[I:Prop|I -> Prop]`
- `[I:Prop|I -> s]` for `I` empty or singleton definition, $s \in S$

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- `[I:Prop|I -> s]` for `I` empty or singleton definition, $s \in S$

Destruction - match (3)

Typing rule.
Our very general destructor for inductive definition enjoys the following typing rule

$$\frac{\begin{array}{c} E[\Gamma] \vdash m : (I\ q_1 \ldots q_r\ t_1 \ldots t_s) \\ E[\Gamma] \vdash P : B \\ [(I\ q_1 \ldots q_r)|B] \\ (E[\Gamma] \vdash f_i : \{(c_{p_i}\ q_1 \ldots q_r)\}^P)_{i=1\ldots l} \end{array}}{E[\Gamma] \vdash \mathsf{case}(m, P, f_1|\ldots|f_l) : (P\ t_1 \ldots t_s\ m)}$$

provided I is an inductive type in a declaration $\mathsf{Ind}(\Delta)[r](\Gamma_I := \Gamma_C)$ with $\Gamma_C = [c_1 : C_1; \ldots ; c_n : C_n]$ and $c_{p_1} \ldots c_{p_l}$ are the only constructors of I.

Iota reduction

A ι-reduction has the following form

$$\text{case}((c_{p_i}\ q_1 \ldots q_r\ a_1 \ldots a_m), P, f_1 | \ldots | f_n) \triangleright_\iota (f_i\ a_1 \ldots a_m)$$

with c_{p_i} the i-th constructor of the inductive type I with r parameters.

Typing: pred_strong1

```
Definition pred_strong1 (n : nat) : n > 0 → nat :=
  match n with
    | O ⇒ fun pf : 0 > 0 ⇒ match zgtz pf with end
    | S n' ⇒ fun _ ⇒ n'
  end.
```

```
Definition pred_strong1 (n : nat) : n > 0 → nat :=
  match n return n > 0 → nat with
    | O ⇒ fun pf : 0 > 0 ⇒ match zgtz pf with end     : 0>0 → nat
    | S n' ⇒ fun _ ⇒ n'   : S n'> 0 → nat
  end.
```

Typing: pred_strong1

$\texttt{Definition pred\_strong1}\ (n : \textbf{nat}) : n > 0 \to \textbf{nat} :=$
$\quad \texttt{match}\ n\ \texttt{with}$
$\qquad \mid \mathsf{O} \Rightarrow \texttt{fun}\ pf : 0 > 0 \Rightarrow \texttt{match zgtz}\ pf\ \texttt{with end}$
$\qquad \mid \mathsf{S}\ n' \Rightarrow \texttt{fun \_} \Rightarrow n'$
$\quad \texttt{end.}$

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$\qquad \mid \mathsf{S}\ n' \Rightarrow \texttt{fun \_} \Rightarrow n' \quad : \mathsf{S}\ \mathsf{n'}> 0 \to \mathsf{nat}$
$\quad \texttt{end.}$

Typing: append of `ilist`

Fixpoint app' $n1$ ($ls1$: **ilist** $n1$) $n2$ ($ls2$: **ilist** $n2$) : **ilist** ($n1$ +$n2$) :=
match $ls1$ in (**ilist** k) return (**ilist** (k +$n2$)) with
| lNil ⇒ $ls2$
| lCons _ x $ls1'$ ⇒ lCons x (app' $ls1'$ $ls2$)
end.

Fixpoint app' $n1$ ($ls1$: **ilist** $n1$) $n2$ ($ls2$: **ilist** $n2$) : **ilist** ($n1$ +$n2$) :=
match $ls1$ in (**ilist** k) return (**ilist** (k +$n2$)) with
| lNil ⇒ $ls2$: ilist (0+n2)
| lCons n' x $ls1'$ ⇒ lCons x (app' $ls1'$ $ls2$) : ilist (S n'+n2)
end.

Typing: append of ilist

```
Fixpoint app' n1 (ls1 : ilist n1) n2 (ls2 : ilist n2) : ilist (n1 +n2) :=
    match ls1 in (ilist k) return (ilist (k +n2)) with
      | lNil ⇒ ls2
      | lCons _ x ls1' ⇒ lCons x (app' ls1' ls2)
    end.
```

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    end.
```

Typing: strong elimination

```
Definition sel (n:nat) := match n with
| O => False
| S _=> True
```

Elimination from nat to Type is needed to show that $0 \neq 1$:

```
Goal 0=1 -> False.
intro H.
change (sel 1).
rewrite <- H.
red.
constructor.
Qed.
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Typing: eq elimination

```
Definition rowne (n,m:nat)(h:n=m)(l:ilist n): ilist m :=
match h in _=m with return (ilist m)
| eq_refl => l
```

```
Definition rowne (n,m:nat)(h:n=m)(l:ilist n): ilist m :=
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```

$$P = fun\ (m : nat)(h : n = m) \Rightarrow ilist\ m$$

- That is an elimination from Prop to Set for a singleton type
- That is how tactic rewrite works

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- That is an elimination from `Prop` to `Set` for a singleton type
- That is how tactic `rewrite` works

For nonsingleton types in Prop elimination to Prop only

```
Inductive or (A B:Prop) : Prop :=
lintro : A -> or A B | rintro : B -> or A B.
```

Incorrect:

```
Definition choice (A B: Prop) (x:or A B): bool :=
match x with lintro a => true | rintro b => false end.
```

```
Error: Incorrect elimination of "x" in the inductive type "or":
the return type has sort "Set" while it should be "Prop".
Elimination of an inductive object of sort Prop is not allowed
on a predicate in sort Set because proofs can be eliminated
only to build proofs.
```

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