# Computer aided verification

lecture 12

## Abstract interpretation I

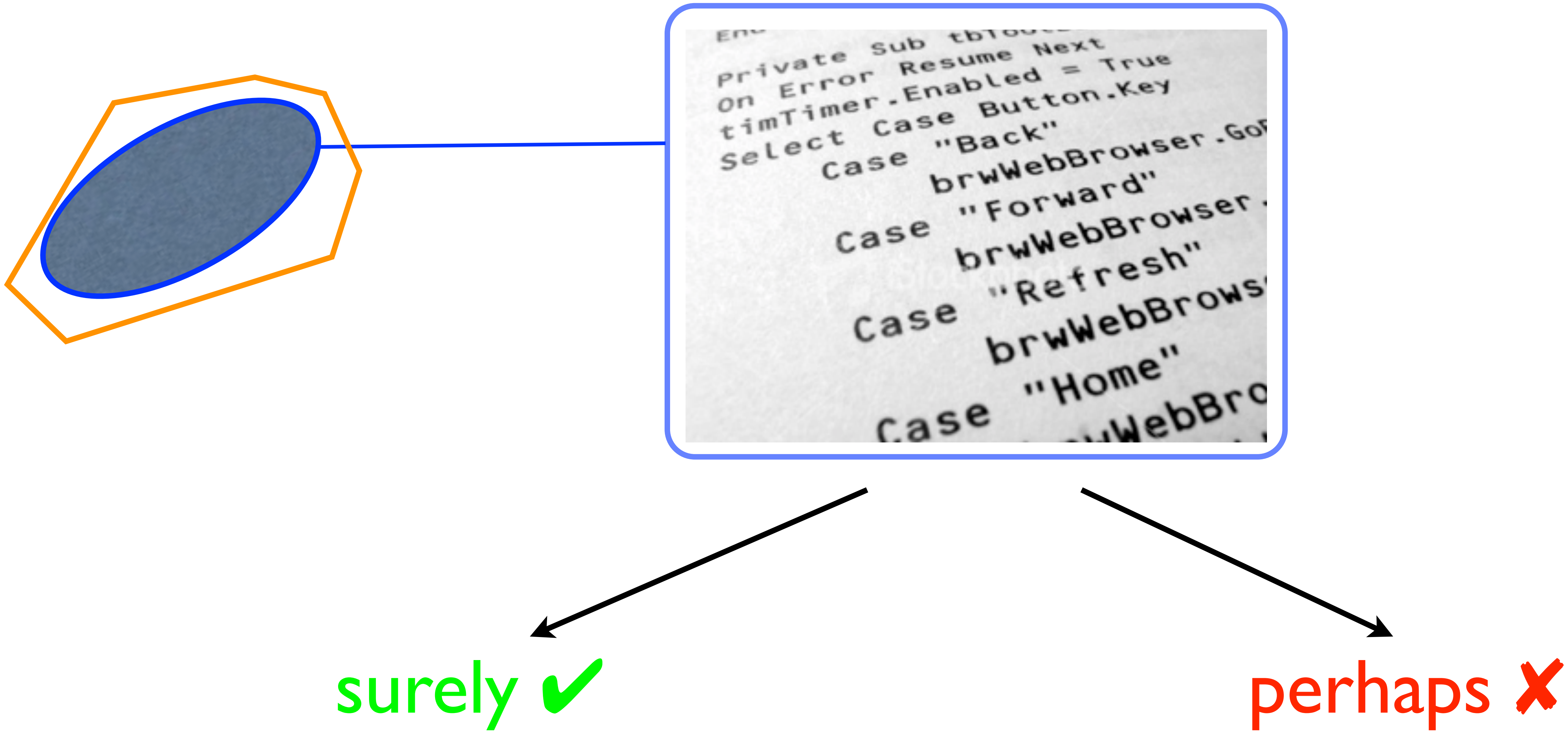
## Literature

- F. Nielson, H.R. Nielson, C. Hankin, Principles of Program Analysis, Springer, 2005.
- http://www.imm.dtu.dk/~riis/PPA/slides2.pdf
- V. D'Silva, D. Kroening, G. Weissenbacher, A Survey of Automated Techniques for Formal Software Verification. IEEE Trans. on CAD of Integrated Circuits and Systems 27 (7):1165-1178, 2008.

## Pionieers

- P. Naur 1965
- P. Cousot, R. Cousot 1977

# Approximate analysis

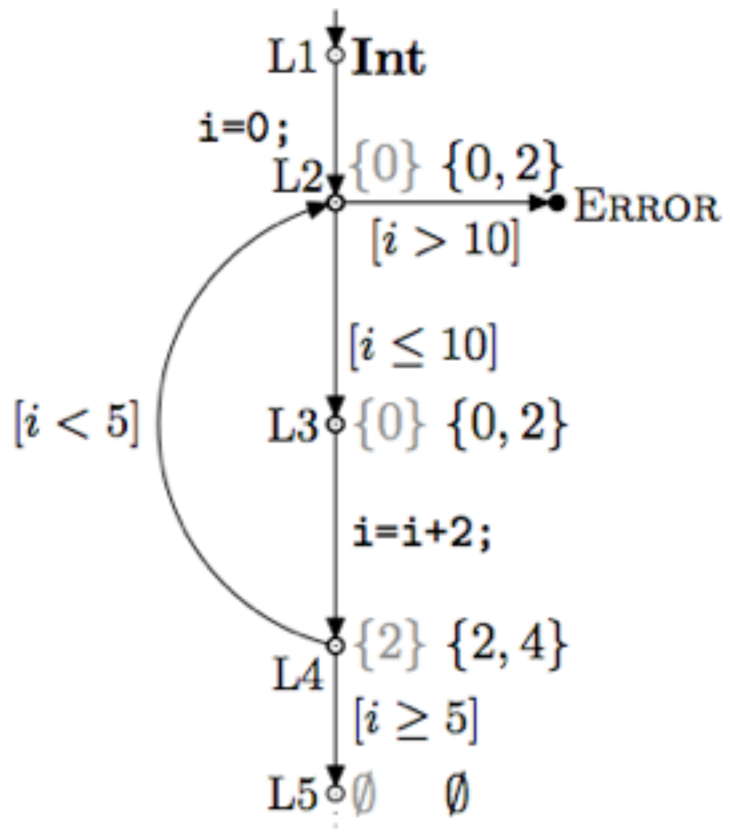


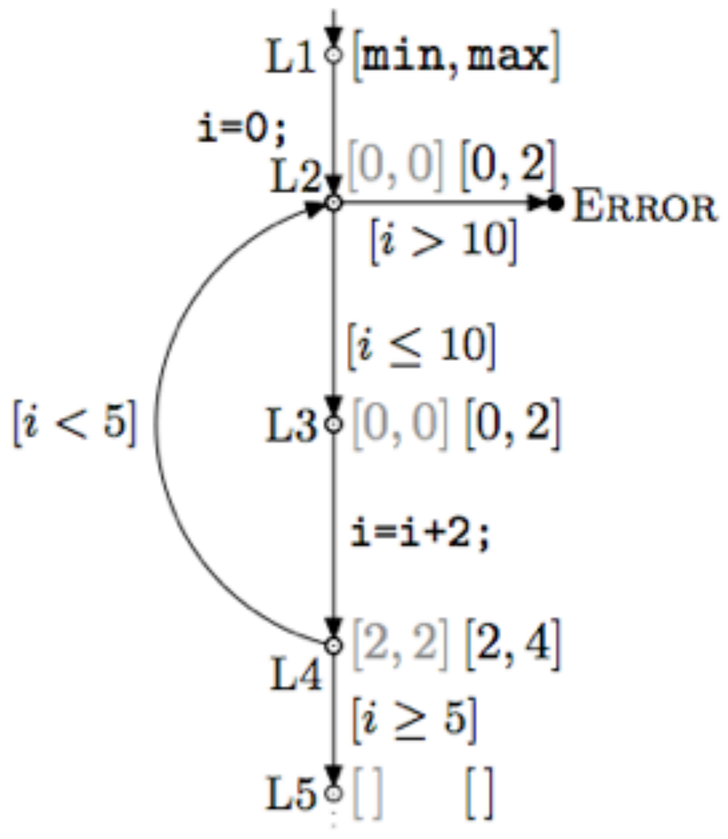
```
En
Private Sub tbTo
On Error Resume Next
timTimer.Enabled = True
Select Case Button.Key
    Case "Back"
        brwWebBrowser.GoB
    Case "Forward"
        brwWebBrowser.
    Case "Refresh"
        brwWebBrows
    Case "Home"
        wWebBro
```

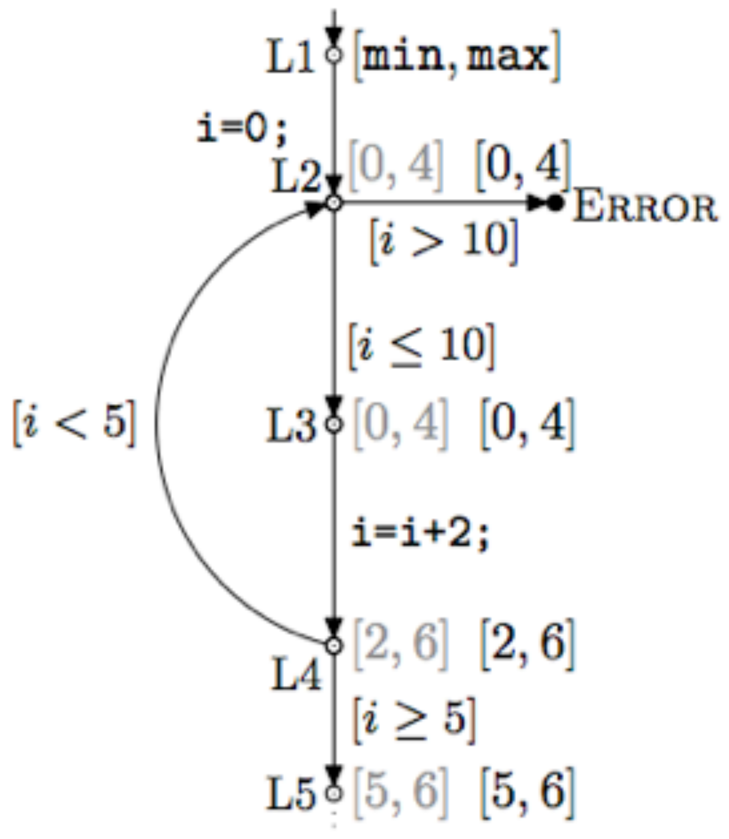
$$123 \cdot 457 + 76543 \quad \overset{?}{=} \quad 132654$$

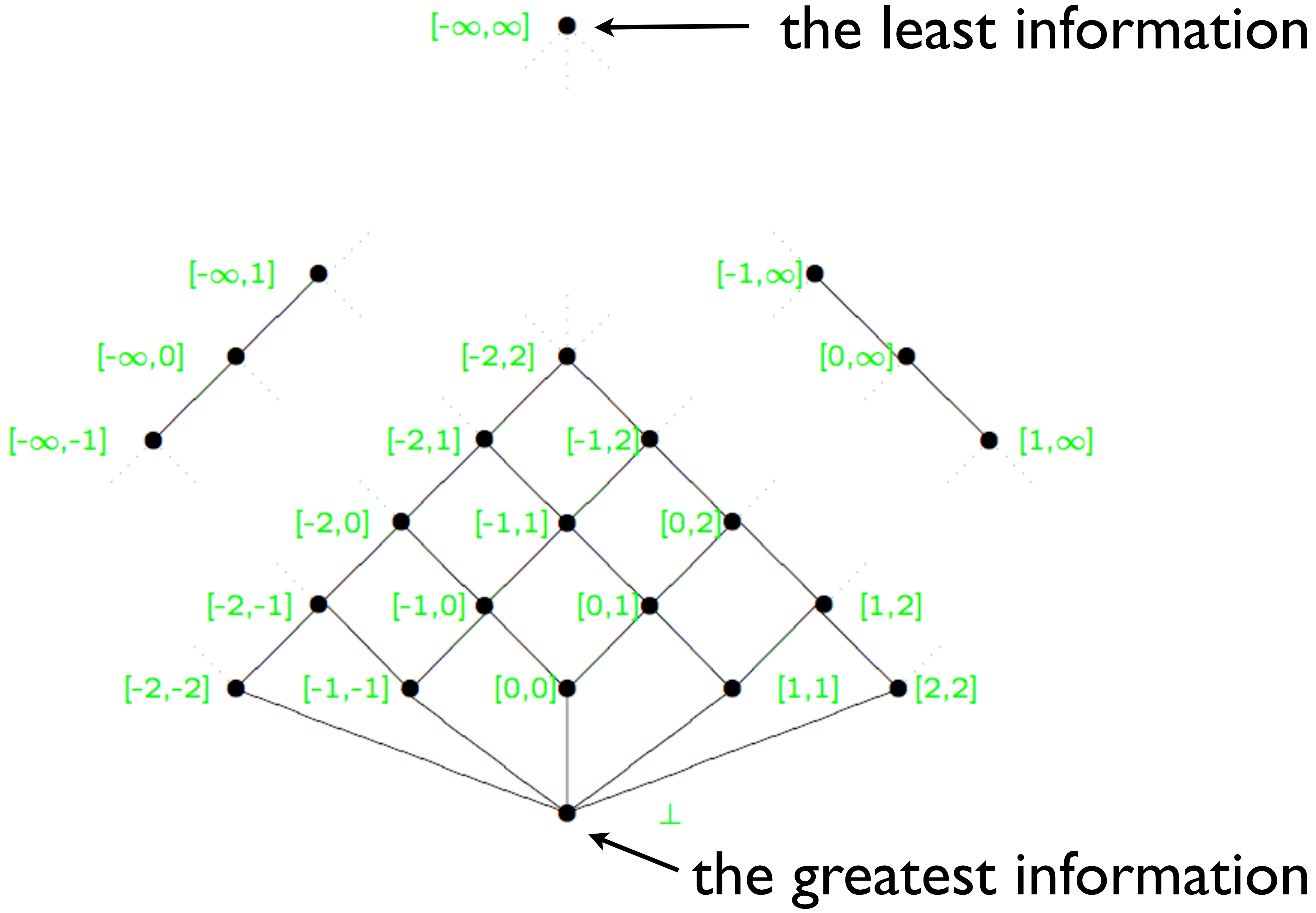
$$\begin{aligned}
123 \cdot 457 + 76543 &\overset{?}{=} 132654 \\
6 \;\cdot\; 7 \quad + \quad 7 &\overset{?}{=} \quad 3 \pmod 9 \\
6 \quad + \quad 7 &\overset{?}{=} \quad 3 \pmod 9
\end{aligned}$$

```
int i = 0;
do {
  assert(i <= 10);
  i = i+2;
} while (i < 5);
```





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[2,2]
⊥
the greatest information

# Approximate analysis

- static analysis
- source code is analyzed (control-flow diagram)
- false alarms (false positives)
- typically oriented towards specific properties
- fully automatic
- scalable
- a diagnostic information if error

## Approximate analysis - methods

- data-flow analysis
- control-flow analysis
- type analysis
- WCET analysis
- ...
- abstract interpretation

## Approximate analysis - applications

- compilation, code optimization, program transformations
- program verification (static analysis)
- verification of quality of code
- abstract interpretation - systematization:
  - ...
  - abstraction in model checking

# Code optimization

- constants propagation (at compile time)
- copies propagation
- available expressions analysis (elimination of comp.)
- live variables analysis (elimination of dead code)
- definition-use and use-definition analysis
- strictness analysis
- array bounds analysis
- ...

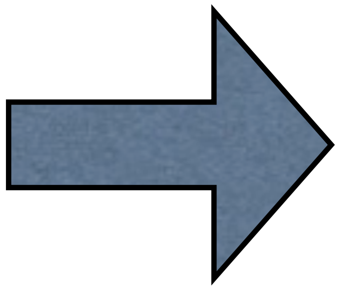
# Program verification

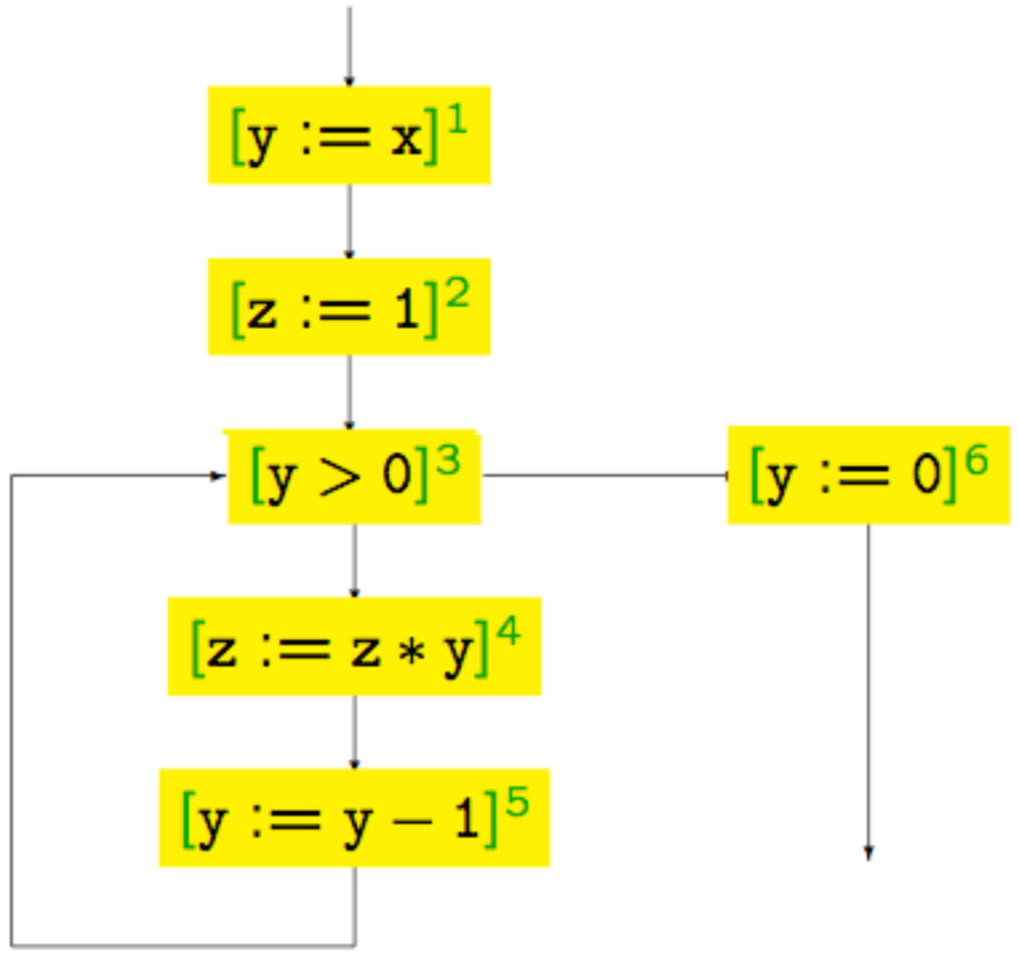
- division by 0, exceptions
- pointers
  - NULL dereferences
  - static and dynamic data (stack and heap)
  - shape analysis
- aliasing analysis
- array bounds
- detection of invariants
- ...

## Data-flow analyses

# Data-flow analysis

```
[y := x]^1;
[z := 1]^2;
while [y > 0]^3 do
    [z := z * y]^4;
    [y := y - 1]^5
od;
[y := 0]^6
```





(while-programs)

## finite set of control locations

$S = \{1, \ldots, n\}, \quad \rightsquigarrow \subseteq S \times S$

$\mathrm{init}(S) \ \subseteq S$

$$\mathrm{State} \ = \ S \times \mathrm{Store}$$

$$\mathrm{Store} \ = \ \mathrm{Var} \to \mathrm{Val}$$

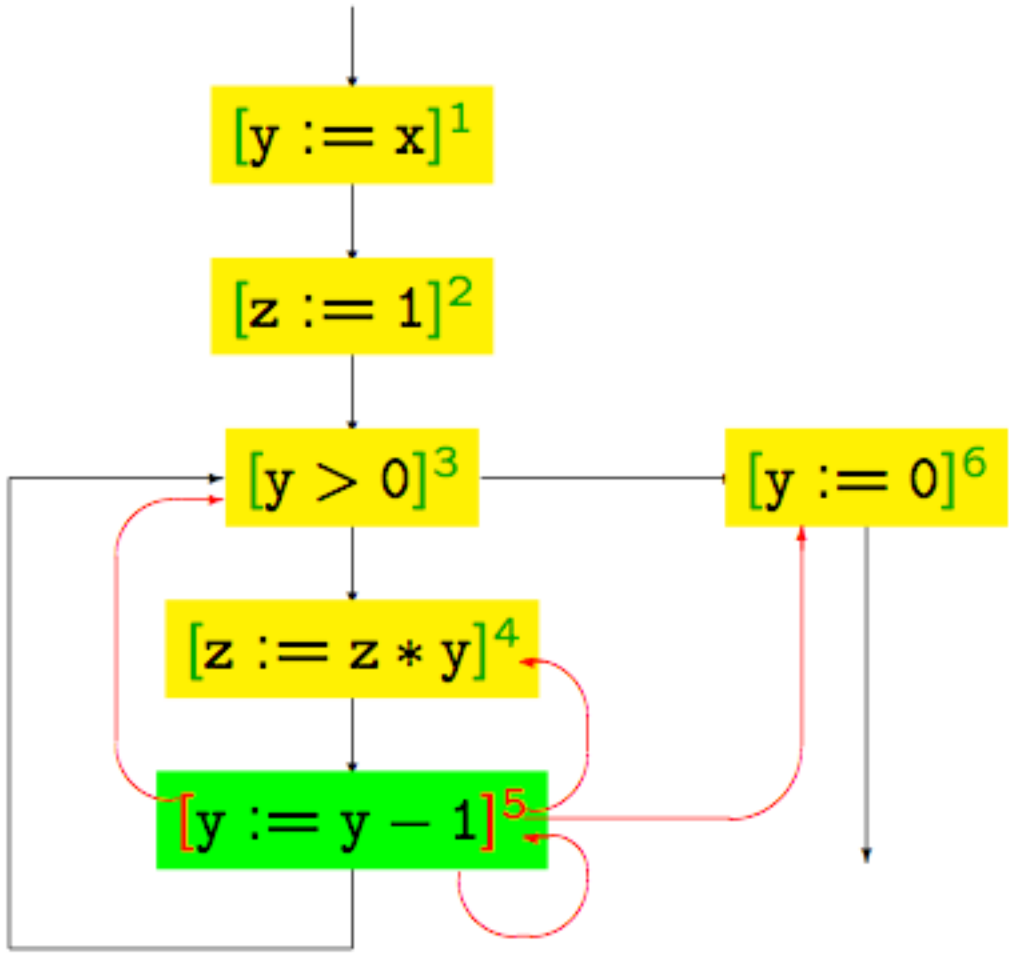
## Data-flow analyses

- reaching definitions analysis
- available expressions analysis
- live variables analysis
- very busy expressions analysis
- constants propagation
- ...

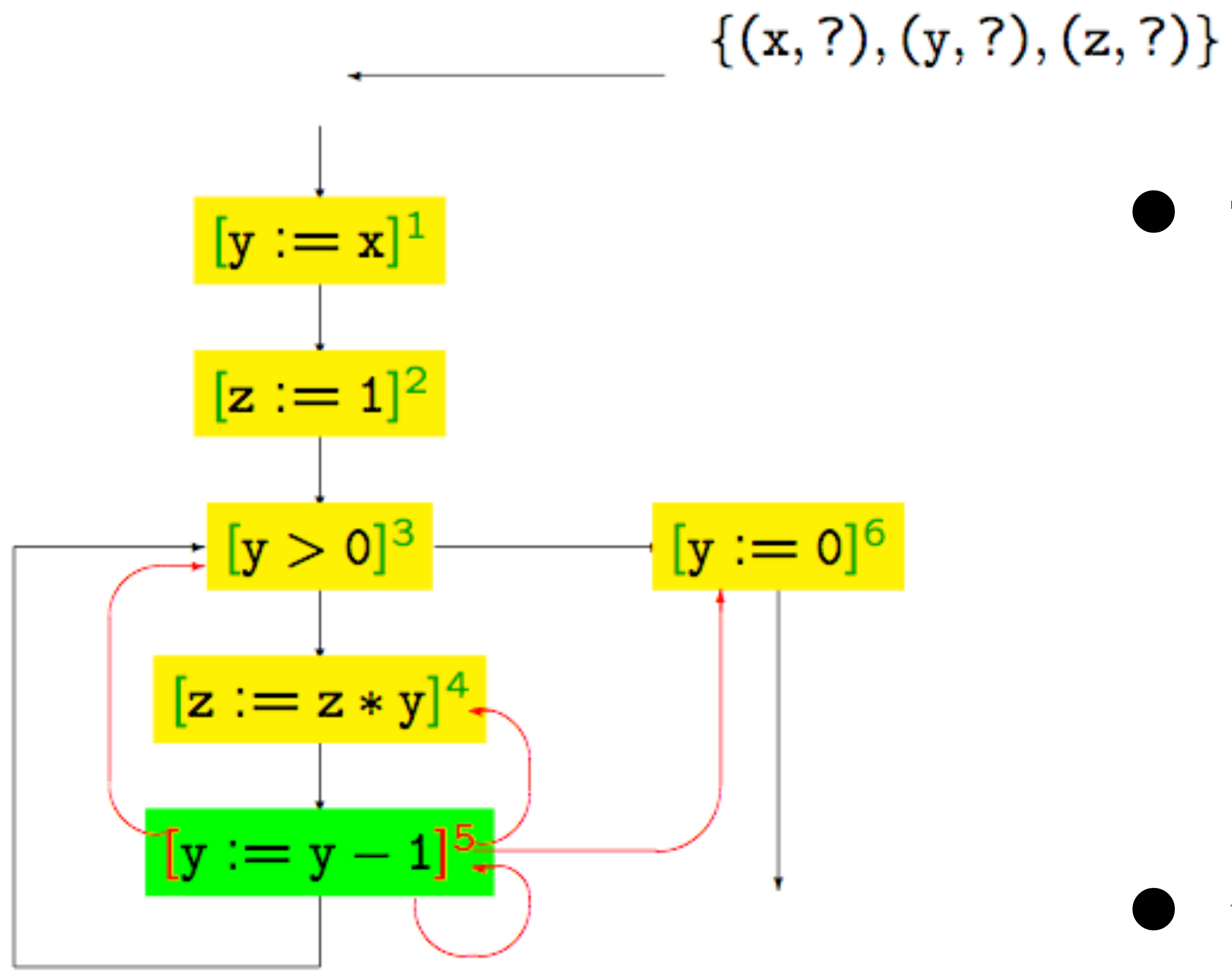
## Reaching definitions analysis

In each control location compute the set of assignments that possibly have been executed (and not „overwritten”) prior to reaching this location.

# Reaching definitions analysis



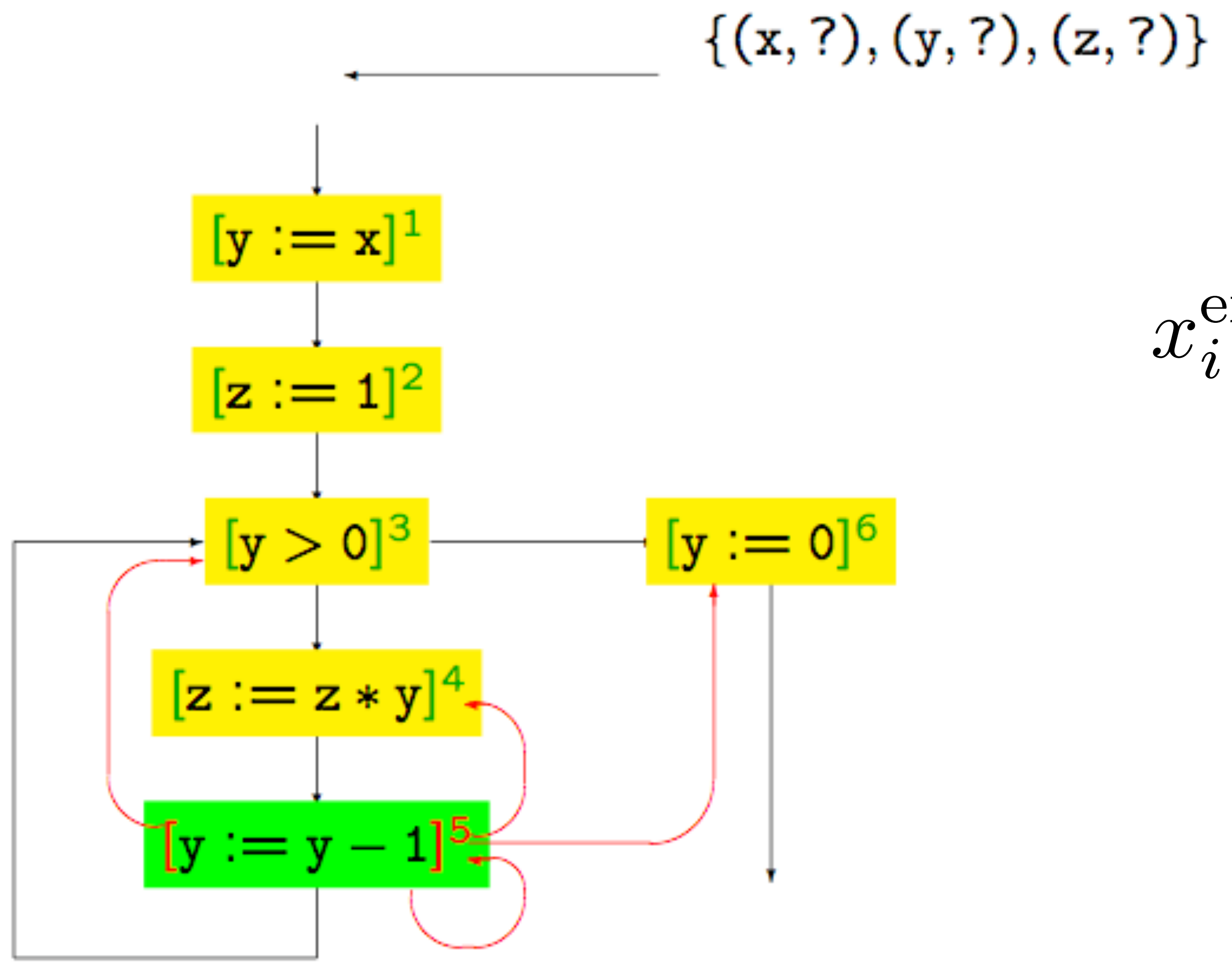
# Reaching definitions analysis



- formalize the problem as a set of equations
  - variables represent information before and after each instruction
- the lest solution
- iterative algorithm

kill(i) gen(i) U

# Reaching definitions analysis



$$x_i^{\text{exit}} = x_i^{\text{entry}} \setminus \text{kill}(i) \cup \text{gen}(i)$$

$$x_i^{\text{entry}} = \bigcup_{j \leadsto i} x_j^{\text{exit}}$$

$$x_1^{\text{entry}} = \{(\text{x}, ?), (\text{y}, ?), (\text{z}, ?)\}$$

we are interested in the least solution

# Reaching definitions analysis

$$\text{kill}(\text{z} := \text{e}) = \{(\text{z}, ?)\} \cup \{(\text{z}, j) \mid j \in S\}$$

$$\text{kill}(\text{b}) = \emptyset$$

$$\text{kill}(\text{skip}) = \emptyset$$

$$\text{gen}([\text{z} := \text{e}]^i) = \{(\text{z}, i)\}$$

$$\text{gen}(\text{b}) = \emptyset$$

$$\text{gen}(\text{skip}) = \emptyset$$

$$x_i^{\text{exit}} = x_i^{\text{entry}} \setminus \ \text{kill}(i) \ \cup \ \text{gen}(i)$$

$$x_i^{\text{entry}} = \bigcup_{j \rightsquigarrow i} x_j^{\text{exit}}$$

$$x_1^{\text{entry}} = \{(\mathtt{x}, ?), (\mathtt{y}, ?), (\mathtt{z}, ?)\}$$

we are interested in the least solution

# Reaching definitions analysis

$\longleftarrow \{(x,?),(y,?),(z,?)\}$

$[y := x]^1;$

$\longleftarrow \{(x,?),(y,1),(z,?)\}$

$[z := 1]^2;$

$\longleftarrow \{(x,?),(y,1),(y,5),(z,2),(z,4)\}$

$\texttt{while}\ [y > 0]^3\ \texttt{do}$

$\longleftarrow \{(x,?),(y,1),(y,5),(z,2),(z,4)\}$

$\quad [z := z * y]^4;$

$\longleftarrow \{(x,?),(y,1),(y,5),\boxed{(z,2)},(z,4)\}$

$\quad [y := y - 1]^5$

$\longleftarrow \{(x,?),\boxed{(y,1)},(y,5),\boxed{(z,2)},(z,4)\}$

$\texttt{od};$

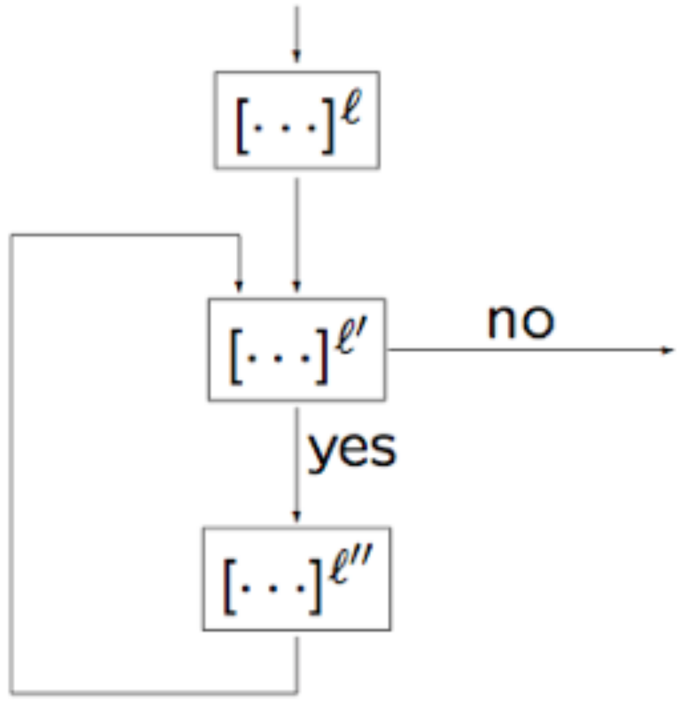
$\longleftarrow \{(x,?),(y,1),(y,5),(z,2),(z,4)\}$

$[y := 0]^6$

$\longleftarrow \{(x,?),(y,6),(z,2),(z,4)\}$

# Reaching definitions analysis

$[\texttt{z:=x+y}]^{\ell}; \texttt{while } [\texttt{true}]^{\ell'} \texttt{ do } [\texttt{skip}]^{\ell''}$



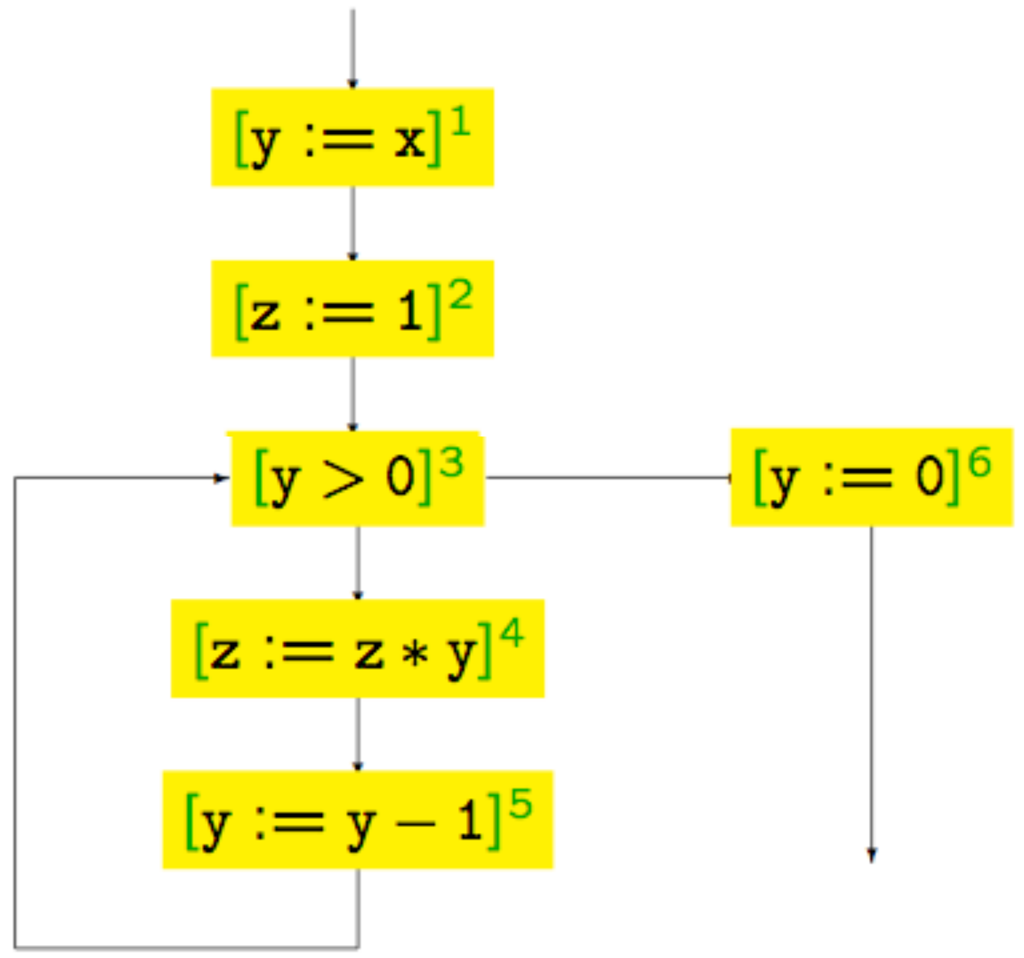
solutions: $x_{l'}^{\text{entry}} \supseteq \{(\mathtt{x}, ?), (\mathtt{y}, ?), (\mathtt{z}, 1)\}$

we are interested in the least solution

## Available expressions analysis

In each control location compute the set of expressions whose value is surely computed whenever this location is entered.

# Available expressions analysis



$$x_i^{\text{exit}} = x_i^{\text{entry}} \setminus \; \text{kill}(i) \; \cup \; \text{gen}(i)$$

$$x_i^{\text{entry}} = \bigcap_{j \rightsquigarrow i} x_j^{\text{exit}}$$

$$x_1^{\text{entry}} = \emptyset$$

we are interested in the greatest solution

# Available expressions analysis

```
[x:=a+b]1; [y:=a*b]2; while [y> a+b ]3 do ([a:=a+1]4; [x:=a+b]5)
```

| $i$ | $x_i^{\text{entry}}$ | $x_i^{\text{exit}}$ |
|---|---|---|
| 1 | $\emptyset$ | {a+b} |
| 2 | {a+b} | {a+b, a*b} |
| 3 | {a+b} | {a+b} |
| 4 | {a+b} | $\emptyset$ |
| 5 | $\emptyset$ | {a+b} |

$$x_i^{\text{exit}} = x_i^{\text{entry}} \setminus \ \text{kill}(i) \ \cup \ \text{gen}(i)$$

$$x_i^{\text{entry}} = \bigcap_{j \rightsquigarrow i} x_j^{\text{exit}}$$

$$x_1^{\text{entry}} = \emptyset$$

we are interested in the greatest solution

# Available expressions analysis

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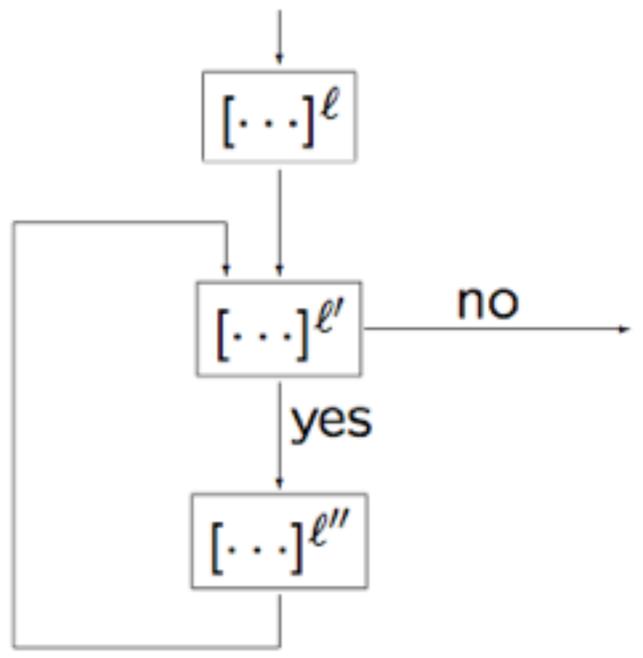


$$x_i^{\text{exit}} = x_i^{\text{entry}} \setminus \text{kill}(i) \cup \text{gen}(i)$$

$$x_i^{\text{entry}} = \bigcap_{j \rightsquigarrow i} x_j^{\text{exit}}$$

$$x_1^{\text{entry}} = \emptyset$$

two solutions:

$$x_{l'}^{\text{entry}} = \{x+y\},\ \emptyset$$

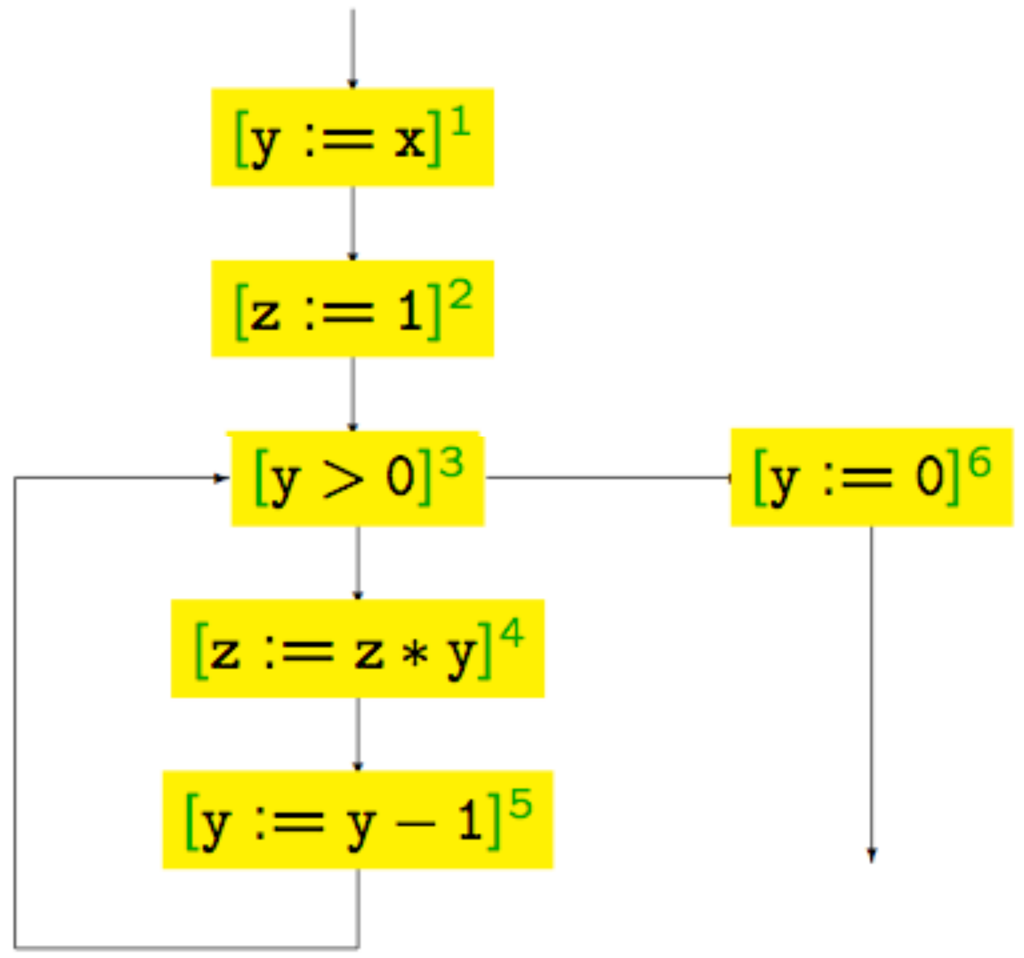
we are interested in the greatest solution

# Live variables analysis

In each program location compute the set of variables that are live (possibly used in future before being redefined) when exiting this location.

backwards analysis

# Live variables analysis



$$x_i^{\text{entry}} = x_i^{\text{exit}} \setminus \text{kill}(i) \cup \text{gen}(i)$$

$$x_i^{\text{exit}} = \bigcup_{i \rightsquigarrow j} x_j^{\text{entry}}$$

$$x_6^{\text{exit}} = \emptyset$$

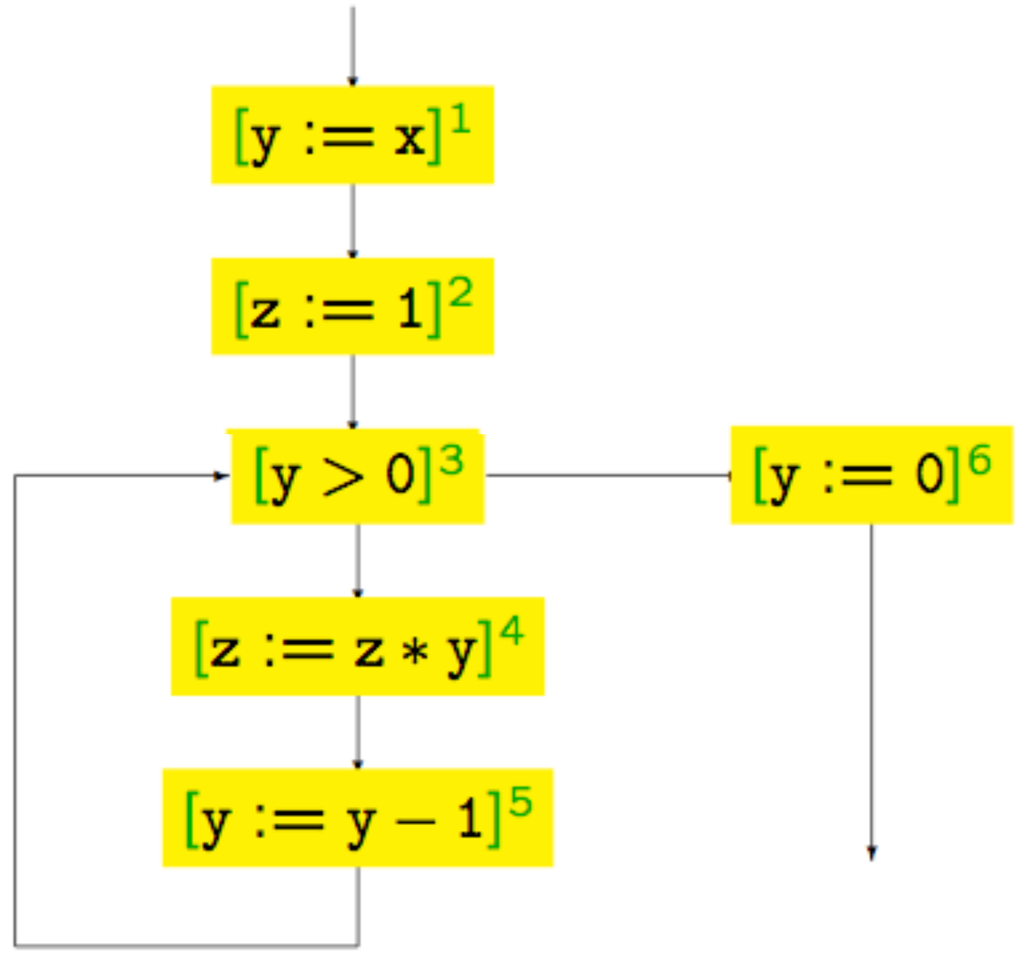
we are interested in the least solution

## Very busy expressions analysis

In each control location compute the set of expressions that will surely be computed in future before redefinition of any of variables appearing in the expression.

backwards analysis

# Very busy expressions analysis



$$x_i^{\text{entry}} = x_i^{\text{exit}} \setminus \text{kill}(i) \cup \text{gen}(i)$$

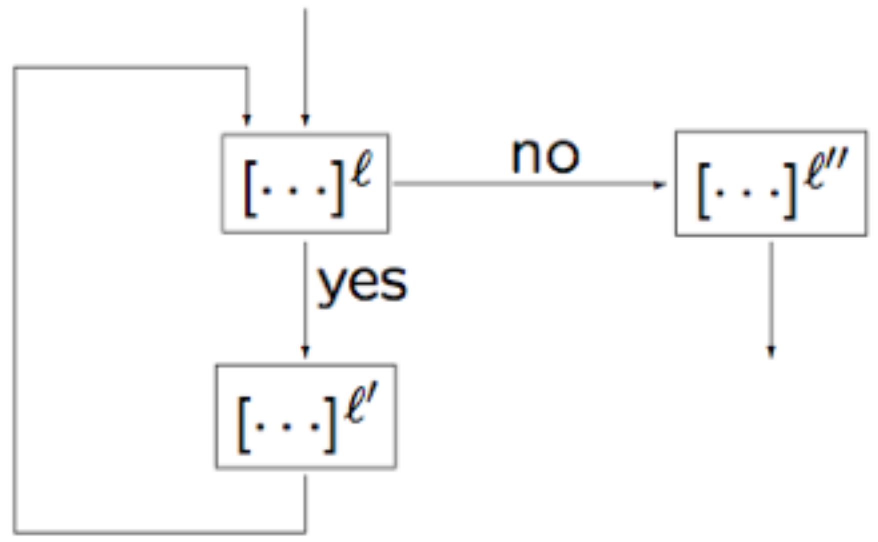
$$x_i^{\text{entry}} = \bigcap_{i \rightsquigarrow j} x_j^{\text{entry}}$$

$$x_6^{\text{exit}} = \emptyset$$

we are interested in the greatest solution

# Very busy expressions analysis

(while [x>1]$^\ell$ do [skip]$^{\ell'}$); [x:=x+1]$^{\ell''}$



$$x_i^{\text{entry}} = x_i^{\text{exit}} \setminus \text{kill}(i) \;\cup\; \text{gen}(i)$$

$$x_i^{\text{entry}} = \bigcap_{i \rightsquigarrow j} x_j^{\text{entry}}$$

two solutions:

$$x_l^{\text{exit}} \;=\; \{x+1\},\; \emptyset$$

$$x_6^{\text{exit}} = \emptyset$$

we are interested in the greatest solution

## Abstract interpretation

# Generalization

$L$ - abstract space

$(L, \sqsubseteq)$ - complete lattice

$\sqcup, \sqcap$ - bounds

$f : S \to \mathrm{Mon}(L \to L)$

$f_{\mathrm{init}} : \mathrm{init}(S) \to L$

$S = \{1, \ldots, n\}, \quad \rightsquigarrow \subseteq S \times S$

$\mathrm{init}(S) \ \subseteq S$

$$x_i^{\mathrm{entry}} = \bigsqcup_{j \rightsquigarrow i} x_j^{\mathrm{exit}} \quad \sqcup \quad f_{\mathrm{init}}(x)$$

$$x_i^{\mathrm{exit}} = f(x)(x_i^{\mathrm{entry}})$$

## Generalization

abstract interpretation

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$$x_i^{\mathrm{exit}} = f(x)(x_i^{\mathrm{entry}})$$

## Data-flow analyses

- reaching definitions analysis
- available expressions analysis
- live variables analysis
- very busy expressions analysis
- constants propagation
- ...

## Data-flow analyses

- reaching definitions analysis $\mathcal{P}(\mathrm{Var} \times (\mathcal{S} \cup \{?\}))$
- available expressions analysis
- live variables analysis
- very busy expressions analysis
- constants propagation
- ...

# Data-flow analyses

- reaching definitions analysis $\mathcal{P}(\mathrm{Var} \times (\mathcal{S} \cup \{?\}))$
- available expressions analysis $\mathcal{P}(\mathrm{Expr})$
- live variables analysis
- very busy expressions analysis
- constants propagation
- ...

# Data-flow analyses

- reaching definitions analysis $\mathcal{P}(\mathrm{Var} \times (\mathcal{S} \cup \{?\}))$
- available expressions analysis $\mathcal{P}(\mathrm{Expr})$
- live variables analysis $\mathcal{P}(\mathrm{Var})$
- very busy expressions analysis
- constants propagation
- ...

# Data-flow analyses

| | |
|---|---|
| • reaching definitions analysis | $\mathcal{P}(\mathrm{Var} \times (\mathcal{S} \cup \{?\}))$ |
| • available expressions analysis | $\mathcal{P}(\mathrm{Expr})$ |
| • live variables analysis | $\mathcal{P}(\mathrm{Var})$ |
| • very busy expressions analysis | $\mathcal{P}(\mathrm{Expr})$ |
| • constants propagation | |
| • ... | |

# Data-flow analyses

| | |
|---|---|
| • reaching definitions analysis | $\mathcal{P}(\mathrm{Var} \times (\mathcal{S} \cup \{?\}))$ |
| • available expressions analysis | $\mathcal{P}(\mathrm{Expr})$ |
| • live variables analysis | $\mathcal{P}(\mathrm{Var})$ |
| • very busy expressions analysis | $\mathcal{P}(\mathrm{Expr})$ |
| • constants propagation | $\mathrm{Var} \to \mathbb{Z}^{\top}$ |
| • ... | |

# Distributivity

$$f(s)(l_1 \sqcup l_2) \quad = \quad f(s)(l_1) \ \sqcup \ f(s)(l_2)$$

holds whenever:

$$L = \mathcal{P}(\mathcal{D}) \quad \mathcal{D} - finite$$

$$f(s)(l) = l \setminus l_1 \cup l_2$$

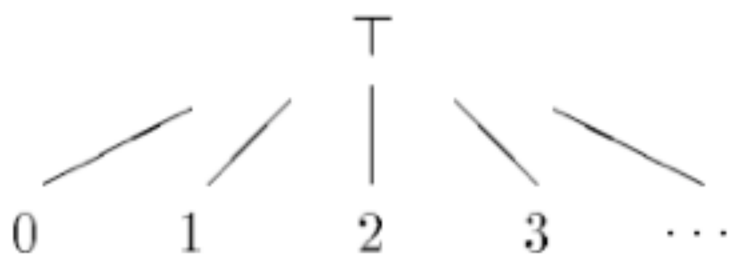
may not hold

## Constants propagation

In each control location compute the set of variables that have a constant value independent from the history.

# Constants propagation

$$L = \mathrm{Var} \to \mathbb{Z}^{\top}$$



$[\mathtt{x}{:=}6]^1; [\mathtt{y}{:=}3]^2; \mathbf{while}\ [\mathtt{x} > \mathtt{y}]^3\ \mathbf{do}\ ([\mathtt{x}{:=}\mathtt{x} - 1]^4; [\mathtt{z}{:=}\mathtt{y} * \mathtt{y}]^6)$

$$\Downarrow$$

$[\mathtt{x}{:=}6]^1; [\mathtt{y}{:=}3]^2; \mathbf{while}\ [\mathtt{x} > 3]^3\ \mathbf{do}\ ([\mathtt{x}{:=}\mathtt{x} - 1]^4; [\mathtt{z}{:=}9]^6)$

$$l_1(\mathrm{y}) = 5 \qquad l_2(\mathrm{y}) = -5$$

$$f(6)(l_1 \sqcup l_2)(\mathtt{z}) = \top$$

$$f(6)(l_1)(\mathtt{z}) = f(6)(l_2)(\mathtt{z}) = 25$$

# Algorithm

$$x_i^{\text{entry}} = \bigsqcup_{j \leadsto i} x_j^{\text{exit}} \quad \sqcup \quad f_{\text{init}}(x)$$

$$x_i^{\text{exit}} = f(x)(x_i^{\text{entry}})$$

$$\Longrightarrow$$

$$\vec{x} = \vec{f}(\vec{x})$$

complete lattice $L^{S \times \{\text{entry},\text{exit}\}}$ with the coordinate-wise order

the lest fix-point of the monotonic function $\vec{f}$

iterative algorithm

we assume that L has only finite chains

# LFP

$L$ - abstract space

$(L, \sqsubseteq)$ - complete lattice

$\sqcup, \sqcap$ - bounds

$f : S \to \mathrm{Mon}(L \to L)$

$f_{\mathrm{init}} : \mathrm{init}(S) \to L$

$S = \{1, \ldots, n\}, \quad \rightsquigarrow \subseteq S \times S$

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$$x_i^{\mathrm{entry}} = \bigsqcup_{j \rightsquigarrow i} x_j^{\mathrm{exit}} \quad \sqcup \quad f_{\mathrm{init}}(x)$$

$$x_i^{\mathrm{exit}} = f(x)(x_i^{\mathrm{entry}})$$

# MOP

$L$ - abstract space

$(L, \sqsubseteq)$ - complete lattice

$\sqcup, \sqcap$ - bounds

$$f : S \to \mathrm{Mon}(L \to L)$$

$$f_{\mathrm{init}} : \mathrm{init}(S) \to L$$

$$S = \{1, \ldots, n\}, \quad \rightsquigarrow \subseteq S \times S$$

$$\mathrm{init}(S) \subseteq S$$

$$y_i^{\mathrm{entry}} = \bigsqcup \{ f(p) \mid p \in \mathrm{paths}^{\mathrm{entry}}(i) \}$$

$$y_i^{\mathrm{exit}} = \bigsqcup \{ f(p) \mid p \in \mathrm{paths}^{\mathrm{exit}}(i) \}$$

$$\text{MOP} \sqsubseteq \text{LFP}$$

$$\vec{y} \sqsubseteq \vec{x}$$

$$\text{MOP} \sqsubseteq \text{LFP}$$

$\boxed{\text{not always computable}} \nearrow$

$$\vec{y} \sqsubseteq \vec{x}$$

MOP = LFP

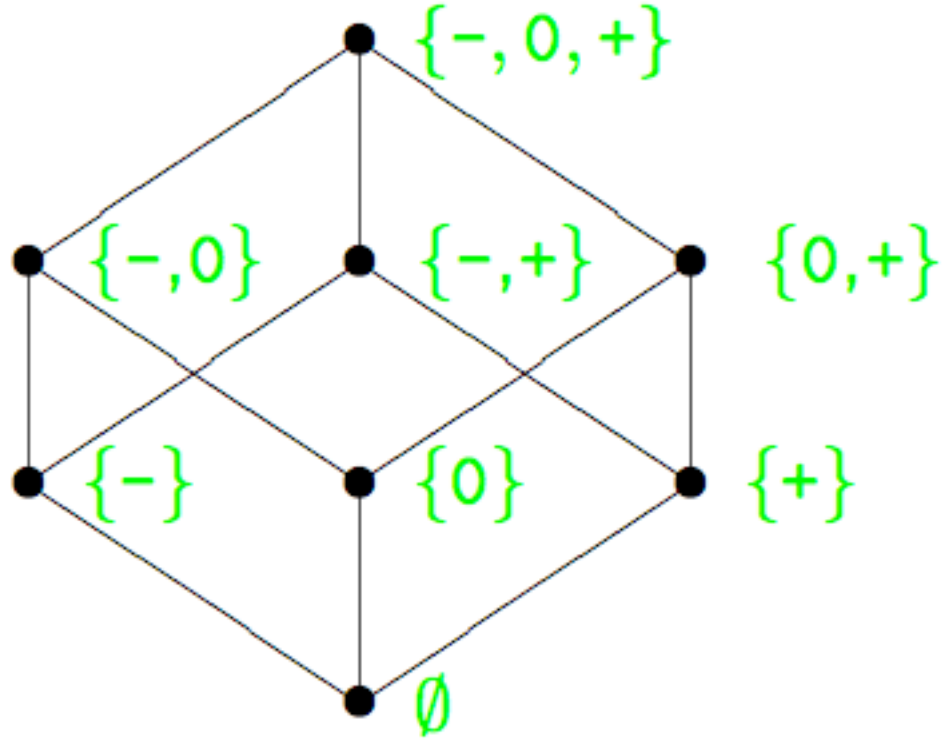
$$\vec{y} = \vec{x}$$

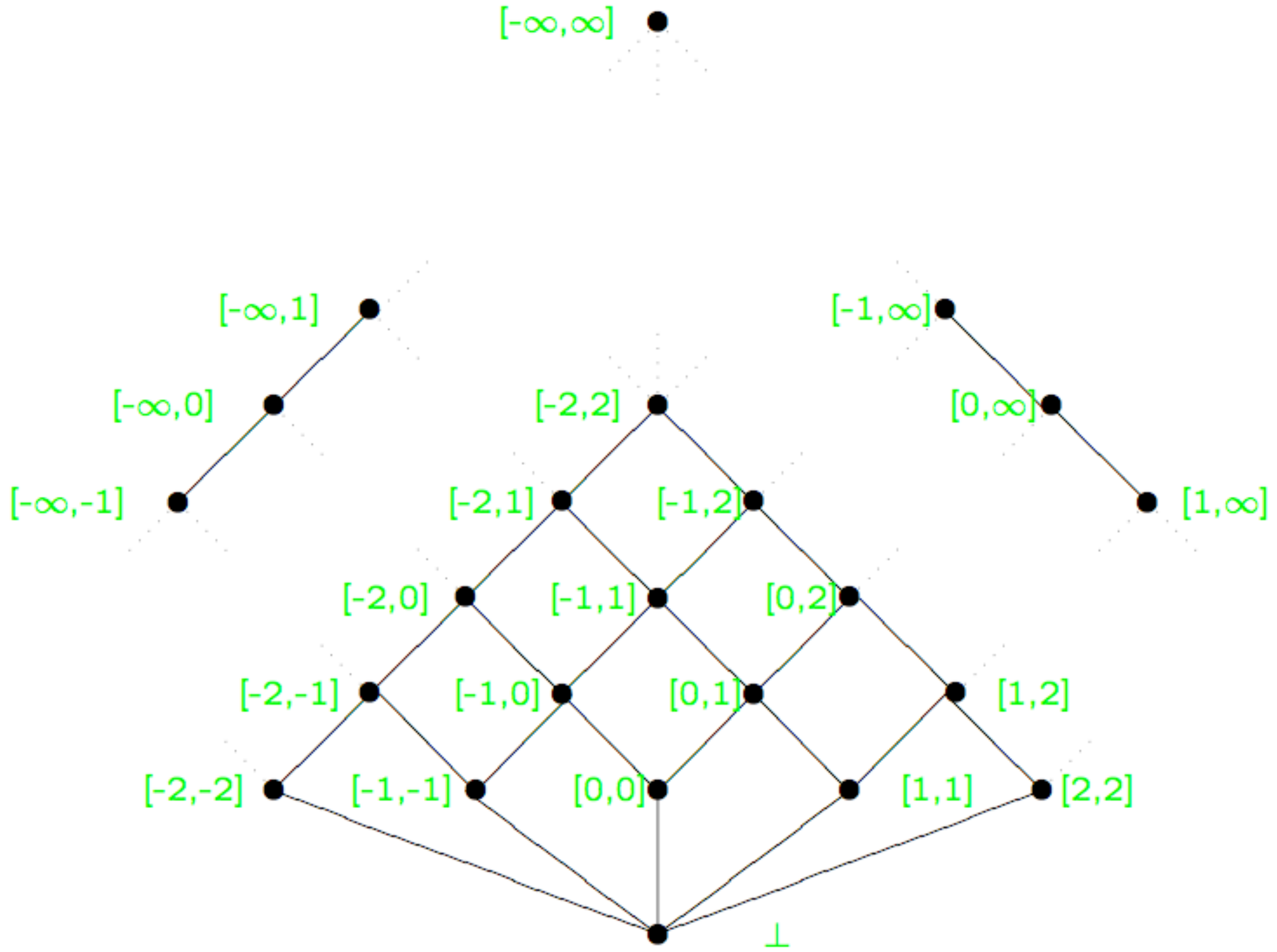
when distributivity holds

## Abstract domains

# Non-relational domains

- signs $\mathcal{P}(-, 0, +)$
- intervals $[n, m]$
- parity
- congruence modulo k

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[-1,-1]
[0,0]
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[2,2]
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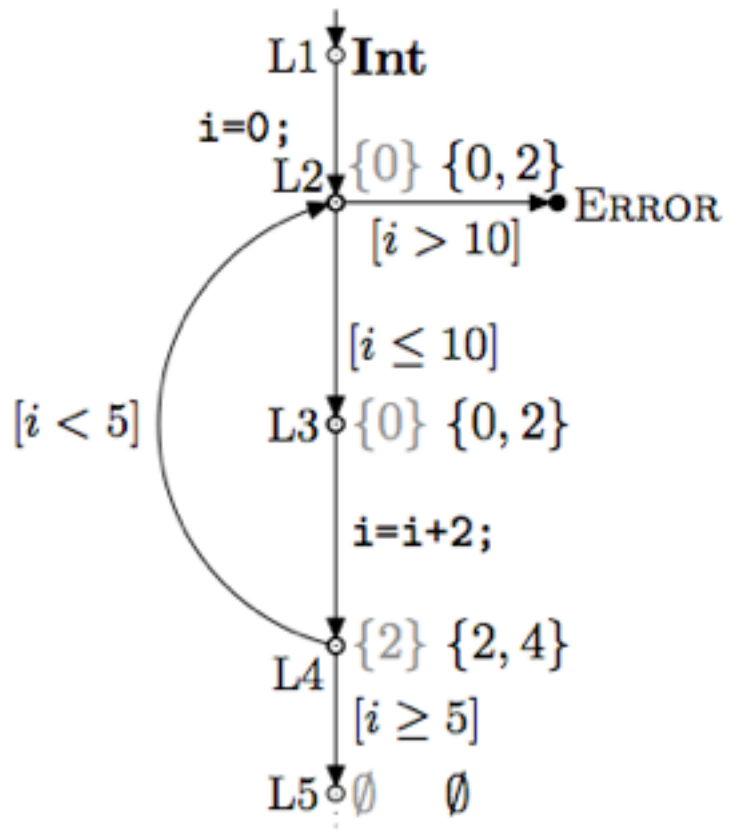
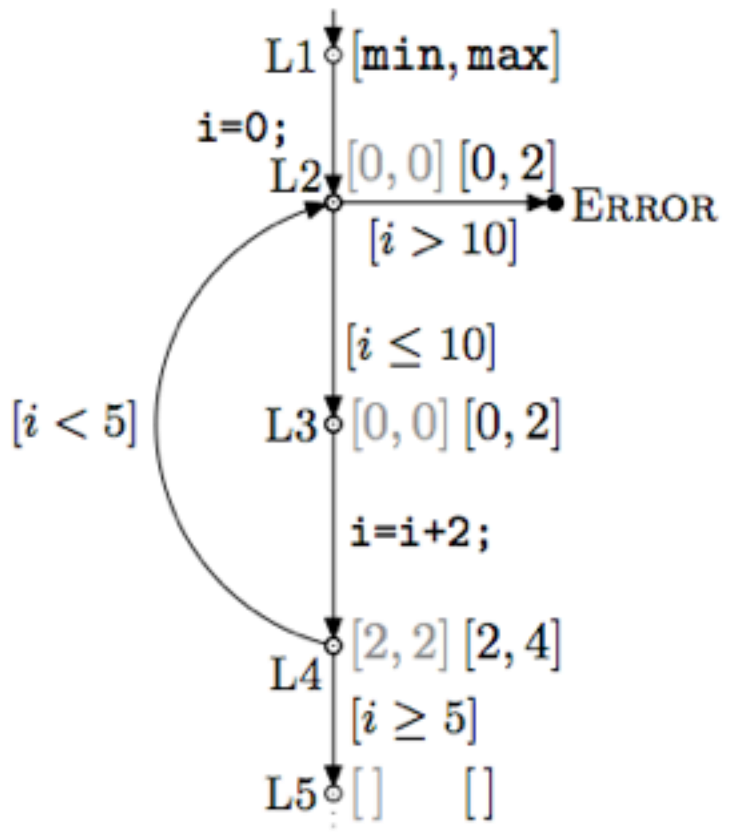
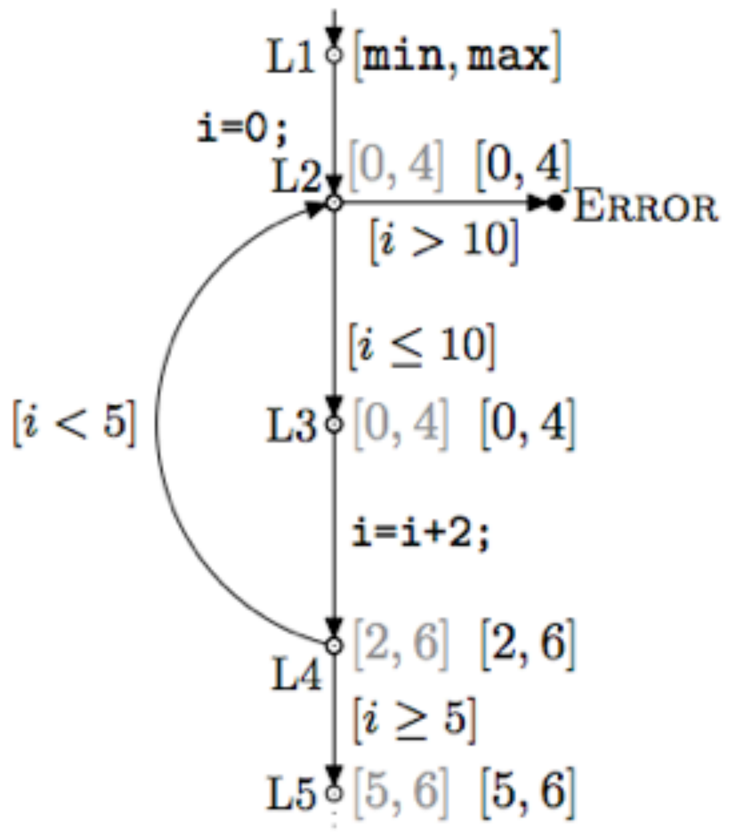
```
int i = 0;
do {
  assert(i <= 10);
  i = i+2;
} while (i < 5);
```







# Relational domains

| | |
|---|---|
| • DBM (difference bounds matrices) | $x - y \leq c$ |
| • octagon | $\overset{+}{-}\, x \overset{+}{-}\, y \leq c$ |
| • octahedra | $\overset{+}{-}\, x_1 \ldots \overset{+}{-}\, x_n \leq c$ |
| • polyhedra | $a_1x_1 + \ \ldots \ + a_nx_n \leq c$ |
| • ellipsoid | $ax^2 + bxy + cy^2 \leq n$ |
| • linear congruence | $ax + by = c \bmod k$ |

# Expressive power

precision $\downarrow$

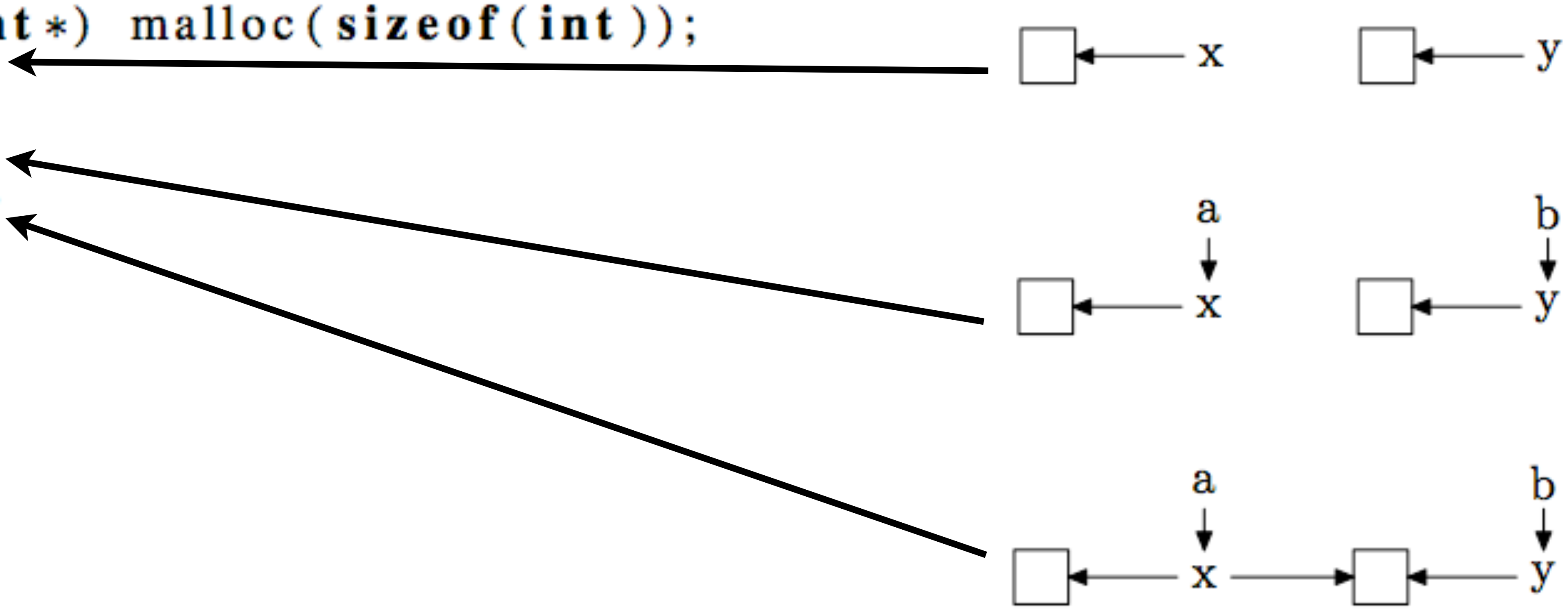
- signs
- intervals
- DBM (difference bounds matrices) $x - y \leq c$
- octagon $\pm x \pm y \leq c$
- octahedra $\pm x_1 \ldots \pm x_n \leq c$
- polyhedra $a_1 x_1 + \ \ldots \ + a_n x_n \leq c$

## Pointer analyses domains

- points-to graphs

# Example: alias analysis

```
int **a, **b, *x, *y;
x = (int*) malloc(sizeof(int));
y = (int*) malloc(sizeof(int));
a = &x;
b = &y;
*a = y;
```



a and b do not point to the same location

x and y may point to the same location

## Correctness ?