1. Let $n$ be a positive integer and $M = 1, 2, \ldots, n$. Find the number of ordered 6-tuples $(A_1, A_2, A_3, A_4, A_5, A_6)$ which satisfy the following two conditions:
   a) sets $A_1, A_2, A_3, A_4, A_5, A_6$ (not necessarily different) are subsets of $M$
   b) each element of $M$ belongs either to exactly three subsets or to exactly six subsets or does not belong to any subset $A_1, A_2, A_3, A_4, A_5, A_6$.

2. Find the largest real number $C_1$ and the smallest real number $C_2$ such that for all real numbers $a, b, c, d, e$ the following inequalities hold
   \[ C_1 < \frac{a}{a + b} + \frac{b}{b + c} + \frac{c}{c + d} + \frac{d}{d + e} + \frac{e}{e + a} < C_2. \]

3. Let $n \geq 2$ be a given integer. Determine all systems of $n$ functions $(f_1, \ldots, f_n)$ where $f_i : R \rightarrow R$ for $i = 1, \ldots, n$ such that for all $x, y \in R$ the following equalities hold
   \[ f_1(x) - f_2(x)f_2(y) + f_1(y) = 0 \]
   \[ f_2(x^2) - f_3(x)f_3(y) + f_2(y^2) = 0 \]
   \[
   \begin{align*}
   \cdots \cdots \\
   f_k(x^k) - f_{k+1}(x)f_{k+1}(y) + f_k(y^k) &= 0 \\
   \cdots \cdots \\
   f_n(x^n) - f_1(x)f_1(y) + f_n(y^n) &= 0.
   \end{align*}
   \]

4. Through a point $P$, which lies inside the triangle $ABC$, are drawn three straight lines $k, l, m$ in such a way that:
   a) $k$ meets the lines $AB$ and $AC$ in $A_1$ and in $A_2$ ($A_1 \neq A_2$) respectively and $PA_1 = PA_2$,
   b) similarly $l$ meets the lines $BC$ and $BA$ in $B_1$ and in $B_2$ ($A_1 \neq A_2$) respectively and $PB_1 = PB_2$,
   c) and similarly $m$ meets the lines $CA$ and $CB$ in $C_1$ and in $C_2$ ($C_1 \neq C_2$) respectively and $PC_1 = PC_2$.
   Prove that the lines $k, l, m$ are uniquely determined by the conditions a), b), c). Find the point $P$ (and prove that there exists exactly one such point) for which the triangles $AA_1A_2, BB_1B_2,$ and $CC_1C_2$ have the same area.

5. A sequence of integers $(a_n)$ satisfies the following recursive equation
   \[ a_{n+1} = a_n^3 + 1999 \quad \text{for } n = 1, 2, \ldots. \]
   Prove that there exists at most one such $n$ for which $a_n$ is the square of an integer.

6. Solve the following system of equations
   \[ x_n^2 + x_nx_{n-1} + x_{n-1}^4 = 1 \quad \text{for } n = 1, 2, \ldots, 1999 \]
   \[ x_0 = x_{1999} \]
   in the set of nonnegative real numbers.

7. Find all pairs $(x, y)$ of positive integers such that
   \[ x^{x+y} = y^{y-x}. \]

8. Let $g$ be a given straight line and let the points $P, Q, R$ all lie on the same side of the line $g$. The points $M, N$ lie on the line $g$ and satisfy $PM \perp g$ and $QN \perp g$. The point $S$ lies between the lines $PM$ and $QN$ and additionally satisfies $PM = PS$ and $QN = QS$. The bisectors of $SM$ and $SN$ meet in the point $R$. The line $RS$ intersects the circumcircle of the triangle $PQR$ in $T \neq R$. Prove that $S$ is the midpoint of the segment $RT$. 

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9. A point in the plane with both integer cartesian coordinates is called a lattice point. Consider the following one player game. A finite set of selected lattice points and finite set of selected segments is called a position in this game if the following hold:
   a) the endpoints of each selected segment are lattice points,
   b) each selected segment is parallel to a coordinate axis, or to the line $y = x$, or to the line $y = -x$,
   c) each selected segment contains exactly five lattice points and all of them are selected,
   d) each two selected segments have at most one common point.
A move in this game consists of selecting a lattice point and a segment such that the new set of selected lattice points and selected segment is a position. Prove or disprove that there exists an initial position such that the game has infinitely many moves.