A discrete model of $O(2)$-homotopy theory

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Previous Work

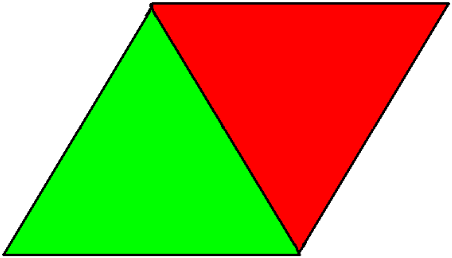
- 1983 Alain Connes introduces cyclic sets.
- 1985 W.G. Dwyer, M. Hopkins and D.M. Kan show that cyclic sets are models for spaces with a circle action.
- 1993 M. Bökstedt, W.C. Hsiang and I. Madsen show that the cyclic set representing an S^1-space keeps track of the fixed point subspaces of finite subgroups of S^1.
- 1995 J.S. The above result is given a precise interpretation in terms of model categories.
- 2007 Andrew Blumberg shows that triples consisting of a cyclic set, a simplicial set and compatibility data encode the entire homotopy type of an S^1-space.

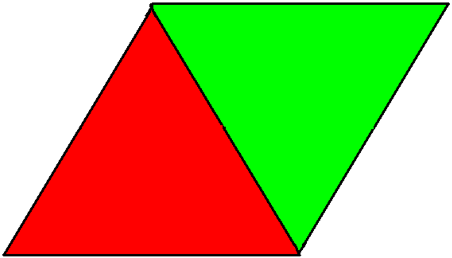
Cyclic set

A cyclic set is analogous to a simplicial set. The category Δ of totally ordered finite sets is replaced by a category Λ, containing both Δ and all finite cyclic groups.

$$\dots [3] \overset{z_4}{\curvearrowleft} \leftrightarrows [2] \overset{z_3}{\curvearrowleft} \leftrightarrows [1] \overset{z_2}{\curvearrowleft} \leftrightarrows [0]$$

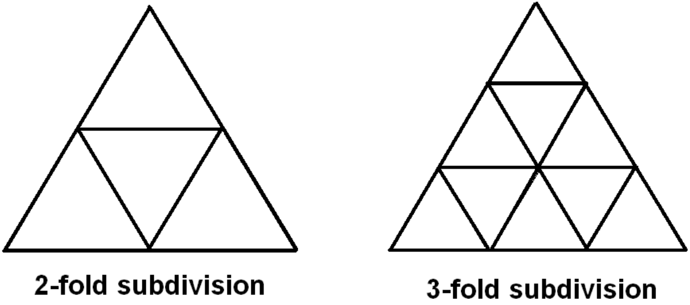
There are standard cyclic sets given by $\Lambda[n] = \hom_{\Lambda^{op}}([n], -)$. The realization of $\Lambda[n]$ is $S^1 \times \Delta^n$, i.e. the product of the circle and the standard topological n-simplex. The cyclic group of order $n+1$ acts on $\Lambda[n]$. For $n = 1$ this action is given by the formula: $t_2(\theta, u_0, u_1) = (\theta - u_0, u_1, u_0)$. It can be displayed as follows:





Edgewise subdivision

In order to gain access to the fixed point data contained in a cyclic set, Bökstedt, Hsiang and Madsen introduced the r-fold edgewise subdivision of a simplicial set, which for small r looks as follows:



Model structure for cyclic sets

For $r \geq 1$ there is a functor $\Phi_r : \mathbf{S}^c \to \mathbf{S}, X \mapsto \mathrm{sd}_r\,(X)^{Z_r}$.
In fact the image is again a cyclic set. Moreover, there is a natural isomorphism of functors

$$|\Phi_r(?)| \sim |?|^{Z_r} : \mathbf{S}^c \to \mathbf{Top}$$

Cyclic sets have a model category structure in which a map $f : X \to Y$ is

- a weak equivalence (fibration) if for all $r \geq 1$, the map $\Phi_r(f) : \Phi_r(X) \to \Phi_r(Y)$ is a weak equivalence (fibration) of simplicial sets,
- a cofibration if it has the left lifting property with respect to acyclic fibrations.

Model structure for S^1-spaces

The category of S^1-spaces has a model structure such that a map $f : X \to Y$ is

- a weak equivalence if $f^H : X^H \to Y^H$ is a weak equivalence of spaces for all finite subgroups H in S^1,
- a fibration if $f^H : X^H \to Y^H$ is a Serre fibration for all finite subgroups H in S^1,
- a cofibration if it has the left lifting property with respect to acyclic fibrations.

Equivalence of homotopy categories

It is shown in the paper “Strong homotopy theory of cyclic sets" (JPAA 1995) that the two model categories above have equivalent homotopy categories.
The fixed point set of the entire circle is not taken into account, as the fixed point set of S^1 acting on the realization of a cyclic set is always discrete.

Blumberg's contribution

A. Blumberg overcomes this deficiency of cyclic sets by introducing a category consisting of triples: a simplicial set, a cyclic set, and appropriate compatibility data. A model structure is established whose homotopy category is equivalent to the homotopy category of S^1 spaces taking into account all closed subgroups.

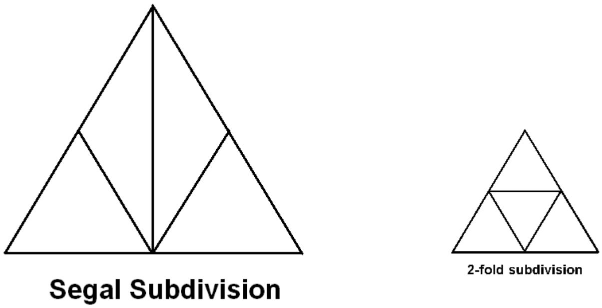
dihedral set

A dihedral set is analogous to a cyclic set. The category Λ is replaced by a category Λ^d, containing both Δ and all finite dihedral groups.

$$\dots [3] \overset{D_4}{\circlearrowleft} \leftrightarrows [2] \overset{D_3}{\circlearrowleft} \leftrightarrows [1] \overset{D_2}{\circlearrowleft} \leftrightarrows [0] \overset{D_1}{\circlearrowleft}$$

There are standard dihedral sets given by $\Lambda^d[n] = \hom_{\Lambda^{d\,op}}([n], -)$. One can check that the realization of $\Lambda^d[n]$ is $O(2) \times \Delta^n$, i.e. the product of $O(2)$ and the standard topological n-simplex.
The dihedral group of order $2m$, where $m \geq 1$, has the presentation $D_m = \langle x, y | x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$.

Segal subdivision



Model structure on dihedral sets

For $r \geq 1$ there is a functor $\Gamma_r : \mathbf{S}^d \to \mathbf{S}, X \mapsto \mathrm{sq}\,(\mathrm{sd}_r\,(X))^{D_r}$.
There is a natural isomorphism of functors

$$|\Gamma_r(?)| \sim |?|^{D_r} : \mathbf{S}^d \to \mathbf{Top}$$

Dihedral sets have a model category structure in which a map $f : X \to Y$ is

- a weak equivalence (fibration) if for all $r \geq 1$, the maps $\Phi_r(f) : \Phi_r(X) \to \Phi_r(Y)$ and $\Gamma_r(f) : \Gamma_r(X) \to \Gamma_r(Y)$ are weak equivalences (fibrations) of simplicial sets,
- a cofibration if it has the left lifting property with respect to acyclic fibrations.

model structure using *finite* subgroups of $O(2)$

The category $\mathbf{Top}^{\mathbf{O}(2)}$ has a model structure such that a map $f : X \to Y$ is

- a weak equivalence if $f^H : X^H \to Y^H$ is a weak equivalence of spaces for all finite subgroups H in $O(2)$,
- a fibration if $f^H : X^H \to Y^H$ is a Serre fibration for all finite subgroups H in $O(2)$,
- a cofibration if it has the left lifting property with respect to acyclic fibrations.

The two model categories above have equivalent homotopy categories (Topology 2000).

Coupled model category structures

Definition (Blumberg) Let **C** and **D** be categories, $F : \mathbf{C} \to \mathbf{D}$ a functor. The objects of $\mathbf{C}_F\mathbf{D}$ are triples $(A, B, FA \to B)$, where $A \in \mathbf{C}$, $B \in \mathbf{D}$ and morphisms are pairs of maps $\alpha : A \to A'$ and $\beta : B \to B'$ such that the two possible maps $FA \to B'$ are equal.

Proposition (Blumberg) Let **C** and **D** be model categories and $F : \mathbf{C} \to \mathbf{D}$ be a Reedy admissible functor. Then $\mathbf{C}_F\mathbf{D}$ admits a model structure such that $(A, B, FA \to B) \to (A', B', FA' \to B')$ is

- a weak equivalence if $A \to A'$ is a weak equivalence in **C** and $B \to B'$ is a weak equivalence in **D**,
- a fibration if $A \to A'$ is a fibration in **C** and $B \to B'$ is a fibration in **D**.
- a cofibration if $A \to A'$ is a cofibration in **C** and $FA' \cup_{FA} B \to B'$ is a cofibration in **D**.

$\mathbb{Z}/2$-sets

A discrete model for $\mathbf{Top}^{\mathbb{Z}/\mathbf{2}}$ is provided by the $\Delta(\mathbb{Z}/2)$-sets of Fiedorowicz and Loday. The category $(\Delta(\mathbb{Z}/2))^{op}$ is the subcategory of $\Lambda^{d\,op}$ of dihedral operators generated by Δ^{op} and the morphisms $w_{n+1} : [n] \to [n]$, $n \geq 0$. A $\Delta(\mathbb{Z}/2)$-set is a functor $(\Delta(\mathbb{Z}/2))^{op} \to$ **Sets** and a morphism is a natural transformation of such functors. We denote this category by $\mathbf{S}^{\mathbb{Z}/2}$.

Hence we can define our model of $O(2)$-spaces as the category $\mathbf{S}^{\mathbb{Z}/2}{}_{\nabla}\mathbf{S}^{d}$ with the above model structure.

model structure using *all closed* subgroups of $O(2)$

The category $\mathbf{Top}^{\mathbf{O(2)}}$ has a model structure such that a map $f : X \to Y$ is

- a weak equivalence if $f^H : X^H \to Y^H$ is a weak equivalence of spaces for all closed subgroups H in $O(2)$,
- a fibration if $f^H : X^H \to Y^H$ is a Serre fibration for all closed subgroups H in $O(2)$,
- a cofibration if it has the left lifting property with respect to acyclic fibrations.

Quillen's Equivalence Theorem

Theorem

Let **C** *and* **D** *be model categories and let* $F : C \leftrightarrows D : G$ *be a pair of adjoint functors.*

- *If* F *preserves cofibrations and* G *preserves fibrations, then the derived functors* $LF : C \leftrightarrows D : RG$ *exist and form an adjoint pair.*
- *If, in addition, for each cofibrant object* A *of* **C** *and each fibrant object* X *of* **D** *a map* $f : A \to G(X)$ *is a weak equivalence in* **C** *if and only if the corresponding map* $f^\flat : F(A) \to X$ *is a weak equivalence in* **D**, *then* LF *and* RG *are inverse equivalences of categories.*

Main result

Theorem

- *There is a pair of adjoint functors*

$$L : \mathbf{S}^{\mathbb{Z}/2}\nabla\mathbf{S}^{d} \leftrightarrow \mathbf{Top}^{\mathbf{O(2)}} : R$$

- *The functors L and R satisfy the assumptions of Quillen's equivalence theorem, hence induce adjoint equivalences of homotopy categories.*

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Reedy admissible functor

Let **C** and **D** be model categories. A functor $F : C \to D$ is *Reedy admissible* if F preserves colimits (e.g. F is a left adjoint) and F has the property that given a morphism $(A, B, FA \to B) \to (A', B', FA' \to B')$ in $\mathbf{C}_F\mathbf{D}$ such that $A \to A'$ is a trivial cofibration in **C** and $FA' \cup_{FA} B \to B'$ is a trivial cofibration in **D** then $B \to B'$ is a weak equivalence in **D**.

model category I

A *model category* is a category **C** with three distinguished classes of maps called *weak equivalences, fibrations and cofibrations* each of which is closed under composition and contains all indentity maps. A map which is both a (co)fibration and a weak equivalence is called an *acyclic (co)fibration*. One requires the following aximos:

MC1 Finite limits and colimits exist in **C**.

MC2 If f, g are maps in **C** such that gf is defined, then if two out of f, g and gf are weak equivalences, that so is the third.

MC3 If f is a retract of g, and g is a fibration, cofibration or a weak equivalence, than so is f.

model category II

MC4 Suppose that in the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow i & & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

the map i is a cofibration, p is a fibration, and either i or p is a weak equivalence. Then a lifting exists (i.e. a map $h : B \to X$, such that $ph = g$ and $hi = f$).

MC5 Each map $f : X \to Y$ can be factored as

$$X \xrightarrow{i} Z \xrightarrow{p} Y$$

where i is a cofibration, p is a fibration, and, more over, we can choose either i or p to be a weak equivalence.

derived functor

Suppose that **C** is a model category and that $F : \mathbf{C} \to \mathbf{D}$ is a functor. Consider pairs (G, s) consisting of a functor $G : \mathrm{Ho}(C) \to \mathbf{D}$ and a natural transformation $s : G\gamma \to F$. *A left derived functor* for F is a pair (LF, t) of this type which is universal from the left, in the sense that if (G, s) is any such pair, then there exists a unique natural transformation $s' : G \to LF$ such that the composite natural transformation

$$G\gamma \xrightarrow{s' \circ \gamma} (LF)\gamma \xrightarrow{t} F$$

is the natural transformation s.

A right derived functor for F is a pair (RF, t), where $RF : \mathrm{Ho}(C) \to \mathbf{D}$ is a functor and $t : F \to (RF)\gamma$ is a natural transformation with the analogous property of being “universal from the right".