Controlled coarse homology and isoperimetric inequalities

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CAT 09

Joint work with Ján Špakula (Universität Münster)

Coarse geometry

In coarse geometry we are interested in large scale phenomena, which do not depend on the local, infinitesimal structure.

Definition

X, Y are quasi-isometric if there is a map $f : X \to Y$ such that

$$\frac{1}{C} d_X(x, y) - L \leq d_Y(f(x), f(y)) \leq C d_X(x, y) + L.$$

and $f(X)$ is a discrete κ-net in X for some $\kappa > 0$.

Example : $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$

Example : $\pi_1(M)$ is quasi-isometric to the universal cover $\widetilde{M}$.

Thus in coarse geometry we consider only discrete metric spaces.

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Amenability

X - discrete metric space, e.g. G with the word length metric, graph.

X is amenable if there is a sequence $\{F_n\}$ of finite sets with small boundaries:

$$\frac{\#\partial F_n}{\#F_n} \to 0.$$

In other words, X is non-amenable if the inequality

$$\#F \leq C\#\partial F$$

holds for every finite $F \subset X$.

Amenable groups: finite, abelian, their extensions, quotients, etc.
Non-amenable groups: the free group $\mathbb{F}_n$, groups with free subgroups.

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Isoperimetric inequalities

The isoperimetric inequality we will study first appeared in the work of Andrzej Żuk:

$$\#A \leq C \sum_{x \in \partial A} f\big(d(x,e)\big)$$

If $f \equiv const$ then this is means non-amenability. On the other hand:

Theorem (Żuk 2000)

For every finitely generated group G and every finite subset $A \subset G$ the inequality holds with f linear.

Żuk asked for which G can one have f other than linear, constant.

Theorem (A. Erschler 2003)

- *For $G = F \wr \mathbb{Z}^d$ the inequality holds with $f(t) = Ct^{1/d}$,*
- *For $G = F \wr (F \wr \mathbb{Z})$ the inequality holds with $f(t) = C \ln t$.*

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Coarse homology

Coarse homology was first introduced and studied around 1991 by John Roe under the name “exotic homology”. He found a way to “coarsen” a homology theory via a homomorphism

$$c : h_*(X) \to hx_*(X).$$

The purpose was to apply this to K-homology which allowed to assign a “coarse index” to a differential operator on an open Riemannian manifold:

$$K_*(M) \xrightarrow[c]{} KX_*(M) \xrightarrow[\text{coarse index}]{} K_*(C^*(M)).$$

This led to the formulation of the coarse Baum-Connes conjecture and eventually to the recent progress on Novikov-type conjectures in which coarse geometry played a major role.

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Uniformly finite homology

J.Block and S.Weinberger in 1992 defined a version of Roe's homology, which they called uniformly finite homology.

The chain groups $C_n^{uf}(X)$ consist of chains $c = \sum_{s \in X^{n+1}} c_s[s]$ (s=simplex) such that:

1. each chain is supported on simplices of uniformly bounded diameter,
2. $\sup |c_s| < \infty$.

The differential:

$$\partial[x_0, \ldots, x_n] = \sum_{i=0}^{n} (-1)^i [x_0, \ldots, \hat{x}_i, \ldots, x_n].$$

The associated homology theory is the uniformly finite homology $H_n^{uf}(X)$.

Theorem (Block and Weinberger)

$$X \text{ is not amenable} \iff H_0^{uf}(X) = 0$$

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Controlled coarse homology

We will study a similar theory to Block and Weinberger's, where instead of bounding the coefficients in a chain we will allow these coefficients to grow in a controlled way.

Let $f : [1, \infty) \to [0, \infty)$ be non-decreasing and $e \in X$ fixed. The chain groups $C_n^f(X)$ consist of chains $c = \sum_{s \in X^{n+1}} c_s[s]$ such that:

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The associated homology theory is denoted denoted $H_n^f(X)$.

Remark : If $f \equiv const$ then $H_*^f(X)$= uniformly finite homology.

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The philosophy

Consider the fundamental class

$$[X] = \sum_{x \in X} [x] \in H_0^f(X).$$

The main idea :

we are interested in determining f for which $[X] = 0$ in $H_0^f(X)$.

Such f corresponds to "how amenable" X is: the slower f grows the less amenable X is.

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How can one kill the fundamental class?

The Eilenberg swindle

$$1 = 1 + \underbrace{(-1+1)}_{0} + \underbrace{(-1+1)}_{0} + \cdots = \underbrace{(1-1)}_{0} + (1-1) + \cdots = 0$$

This does not work with numbers but works in homology!

$$\begin{aligned}
[x_0] &= [x_0] + \big(-[x_1] + [x_1]\big) + \big(-[x_2] + [x_2]\big) + \ldots \\
&= \big([x_0] - [x_1]\big) + \big([x_1] - [x_2]\big) + \ldots \\
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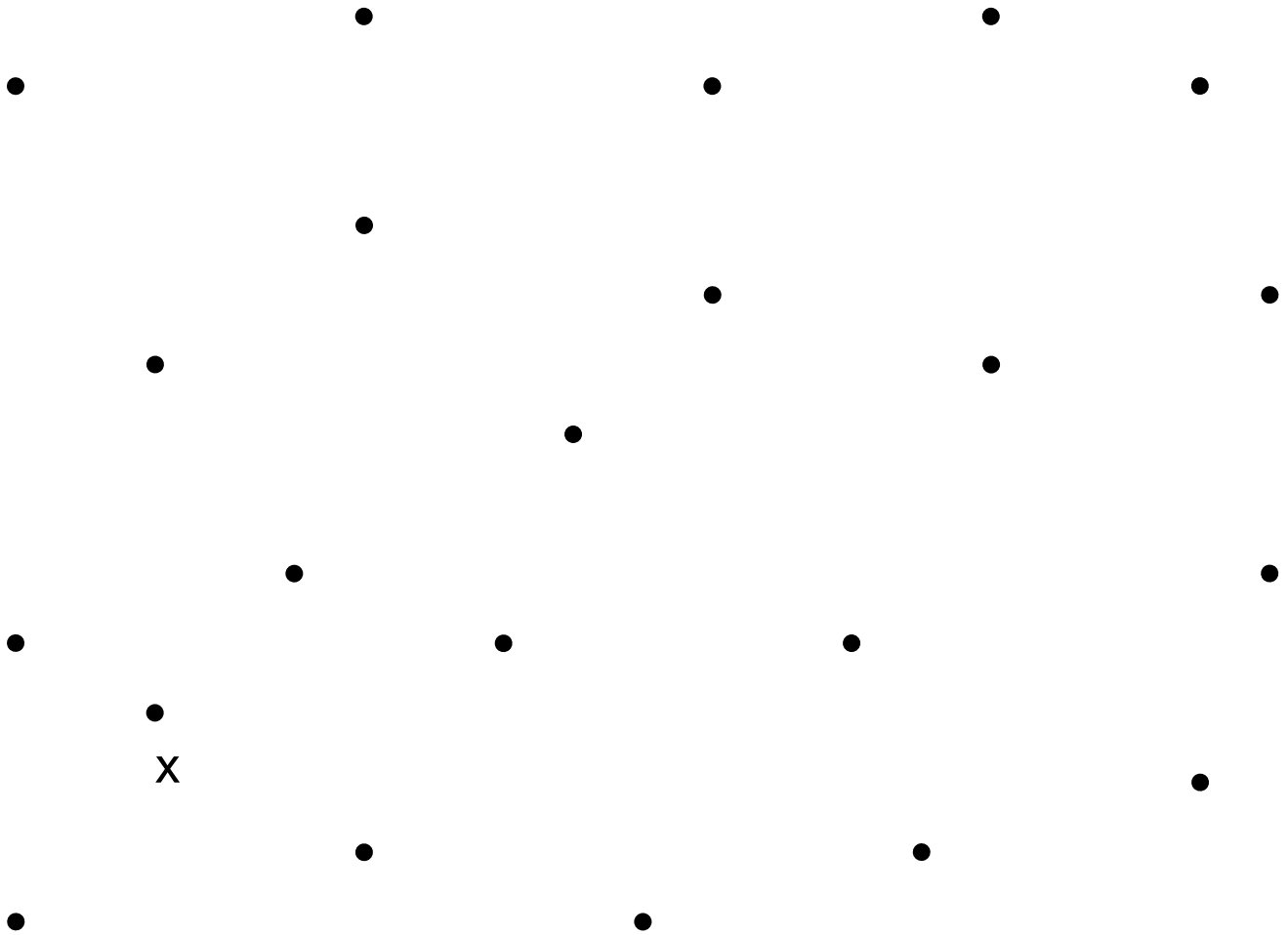

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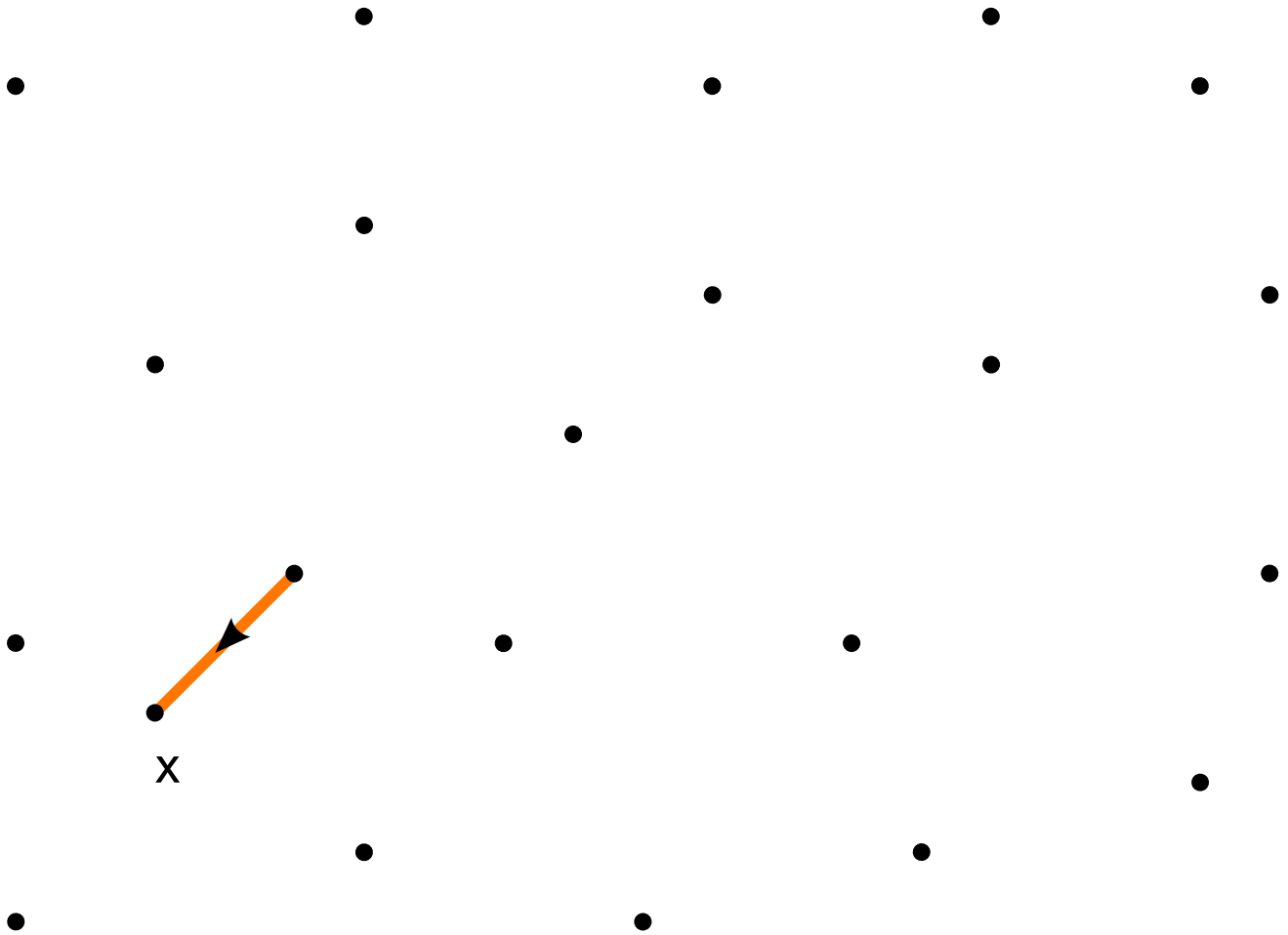

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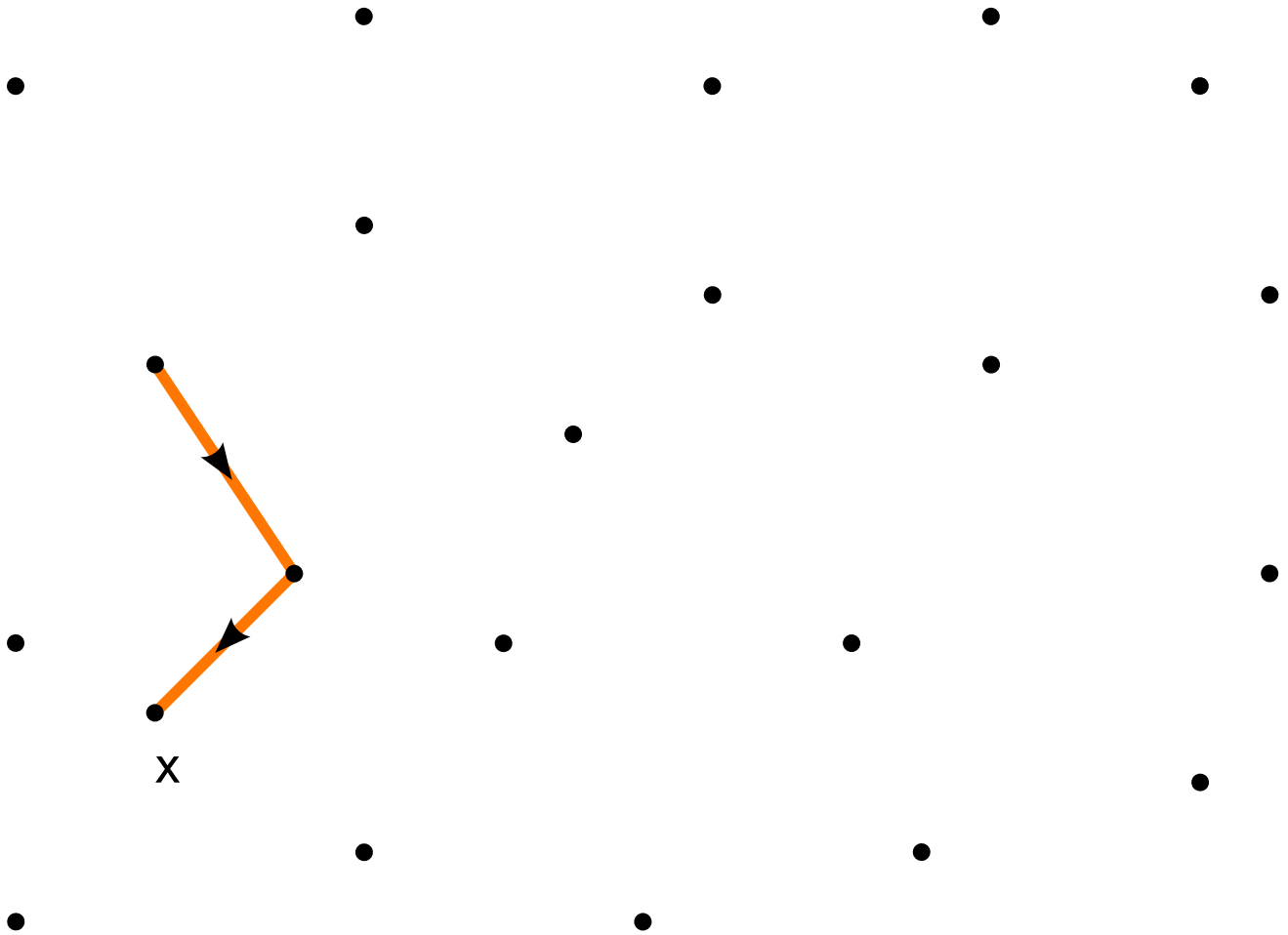

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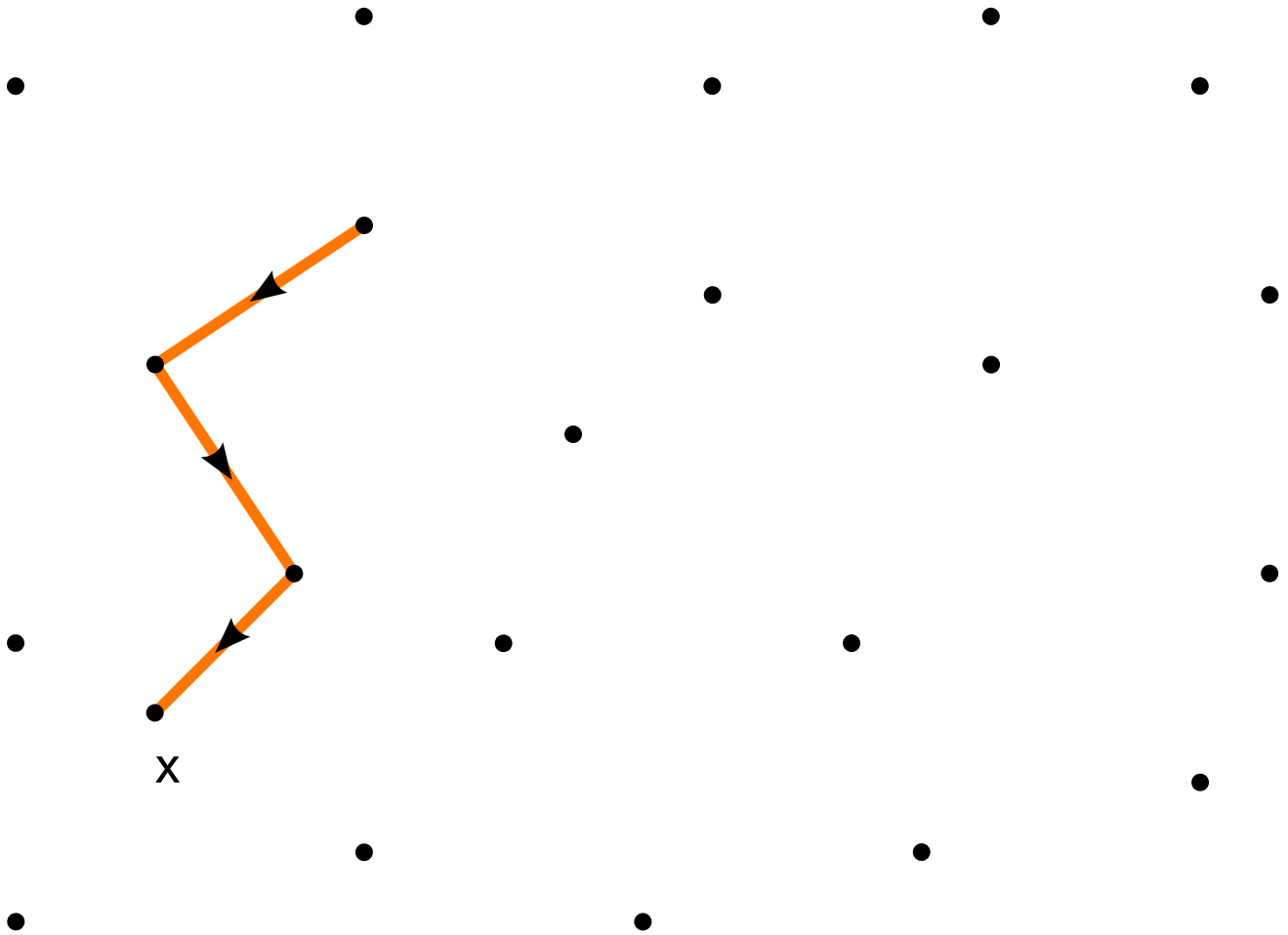

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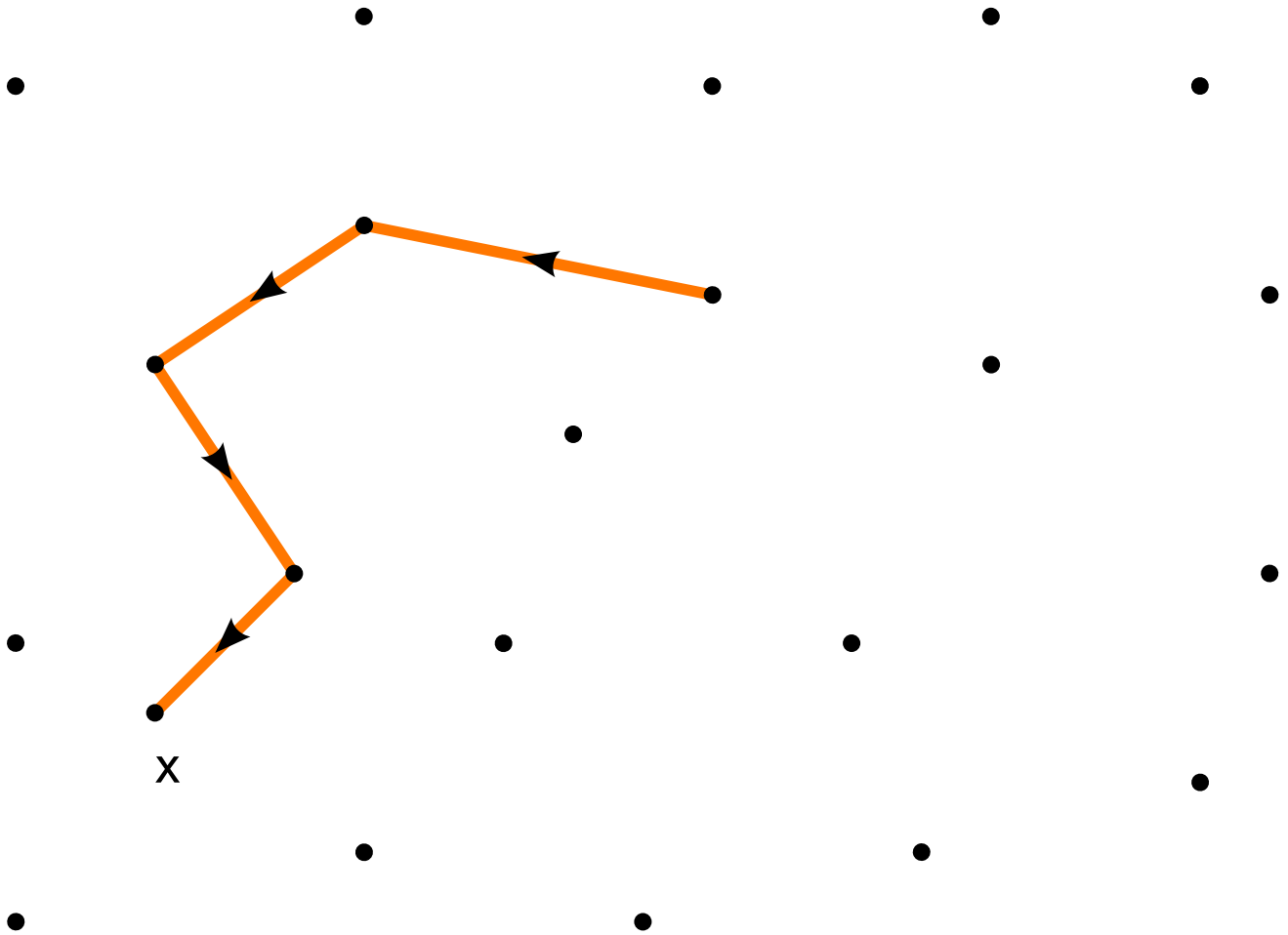

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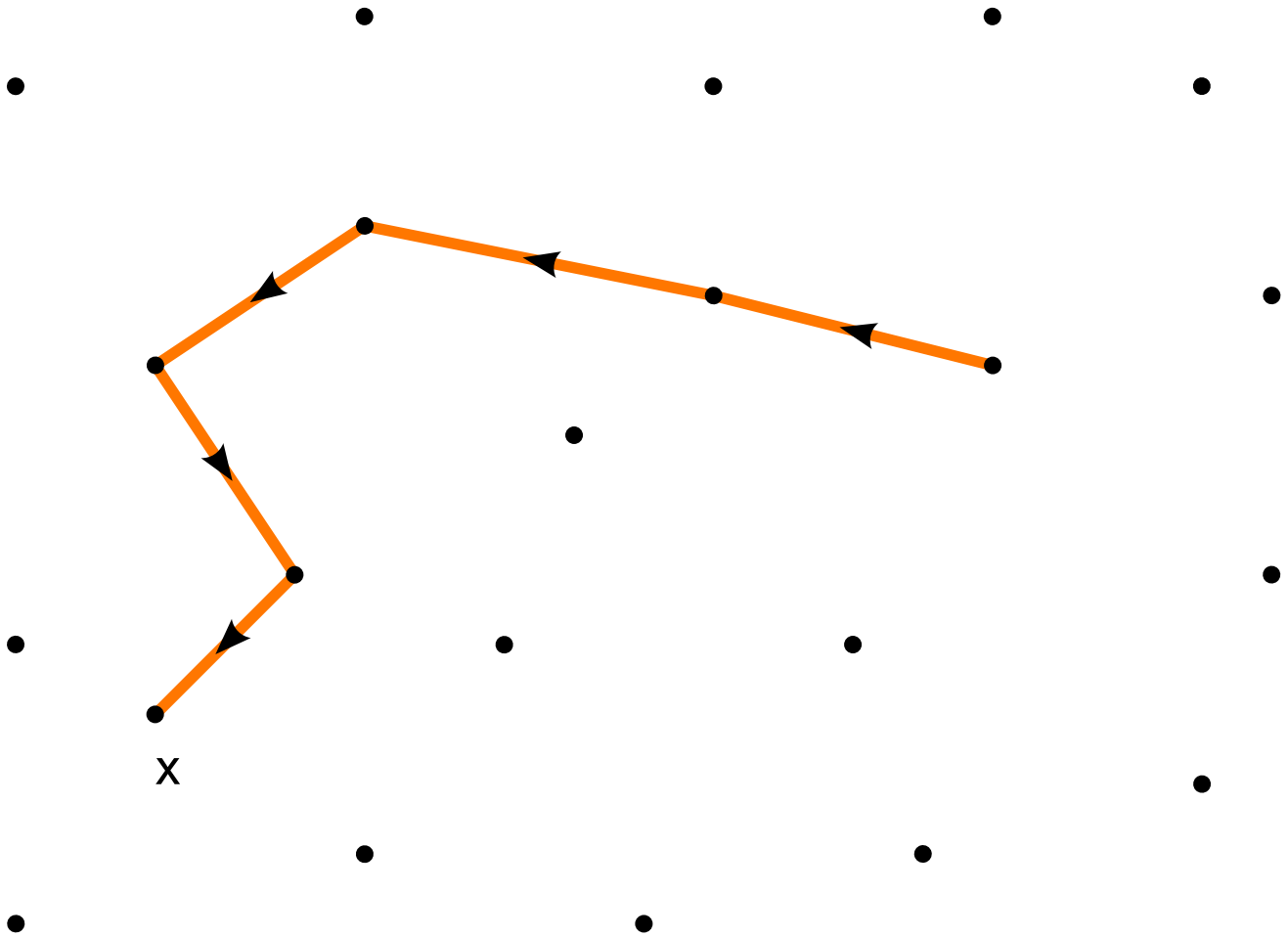

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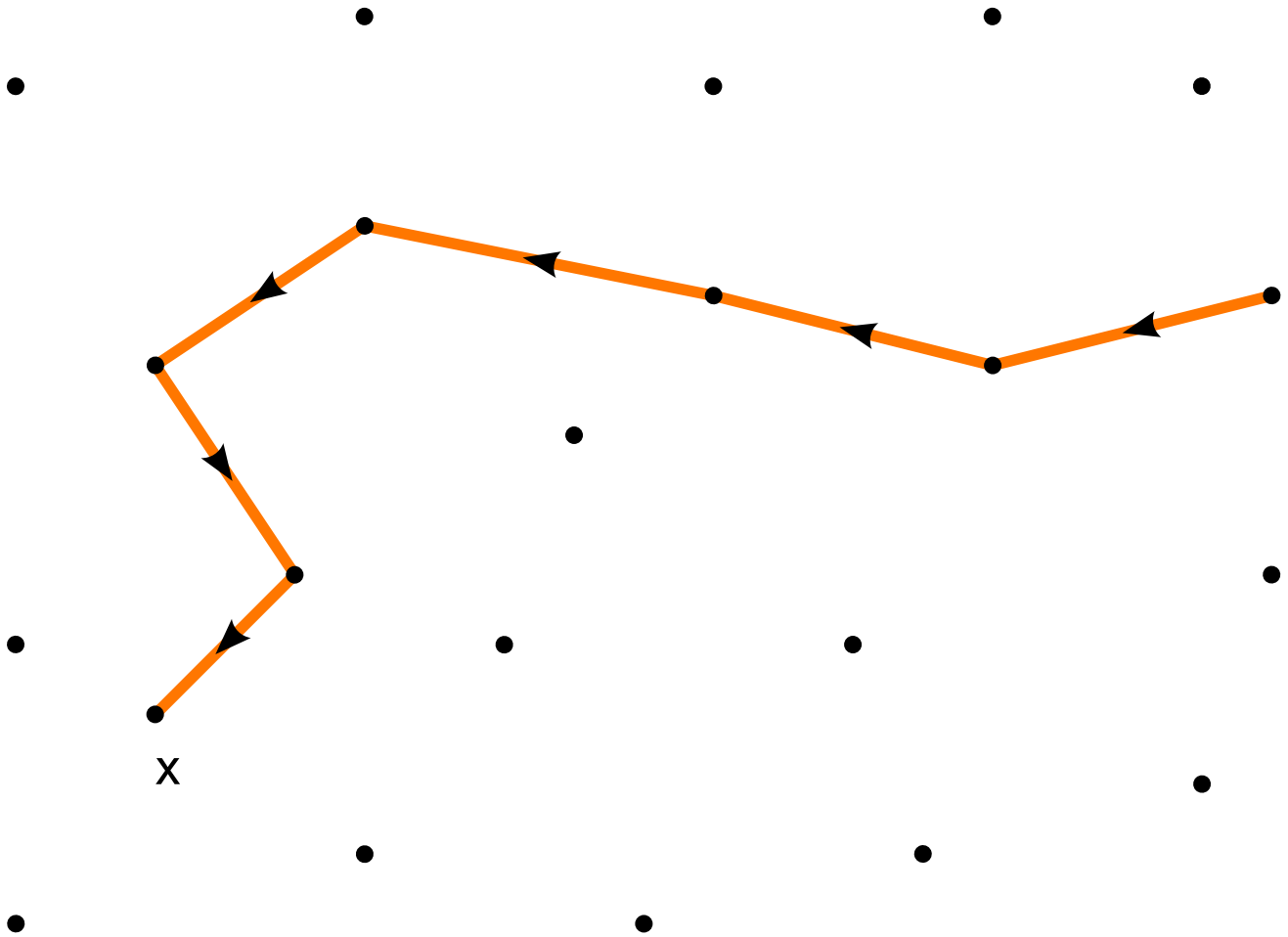

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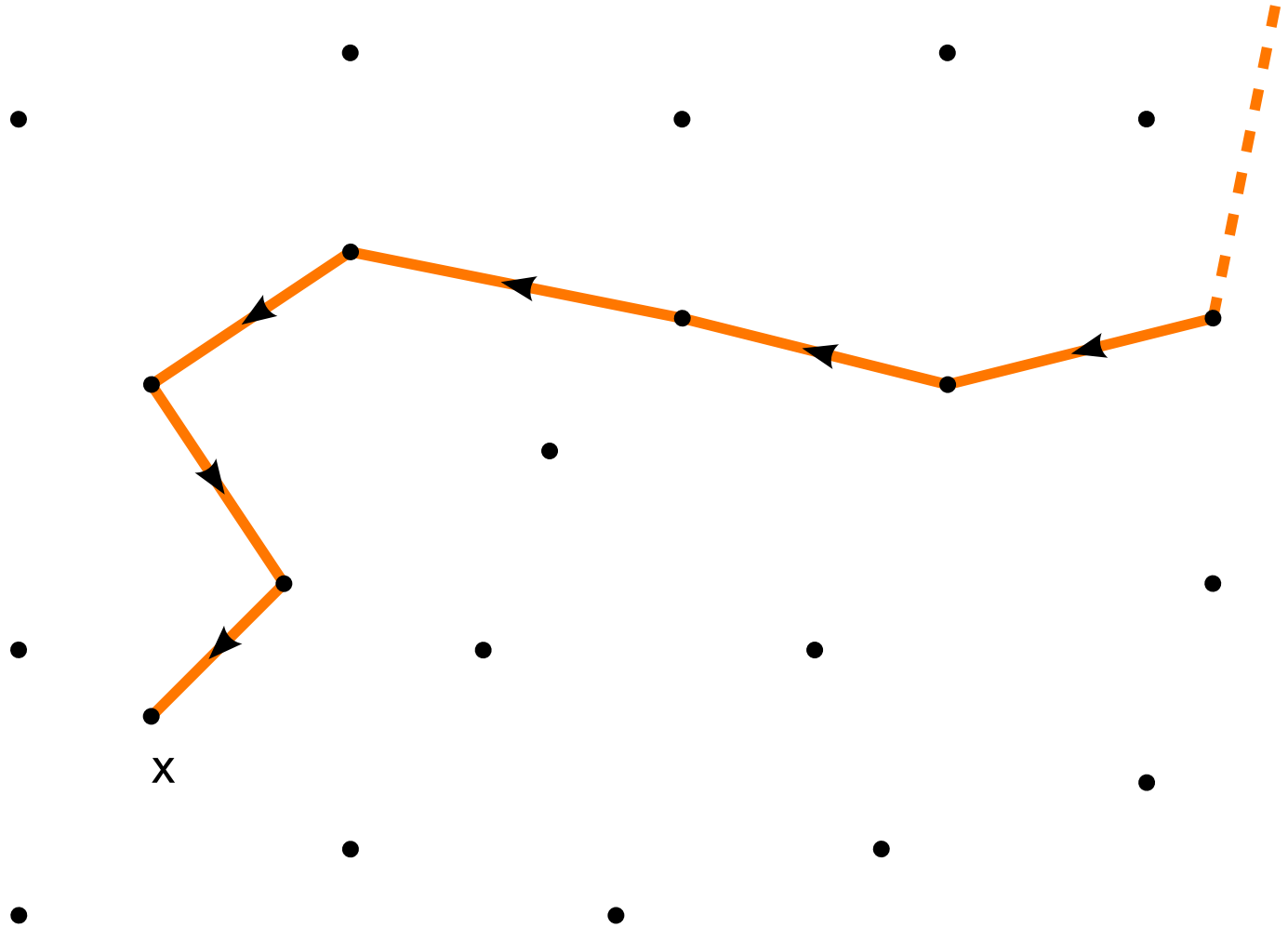

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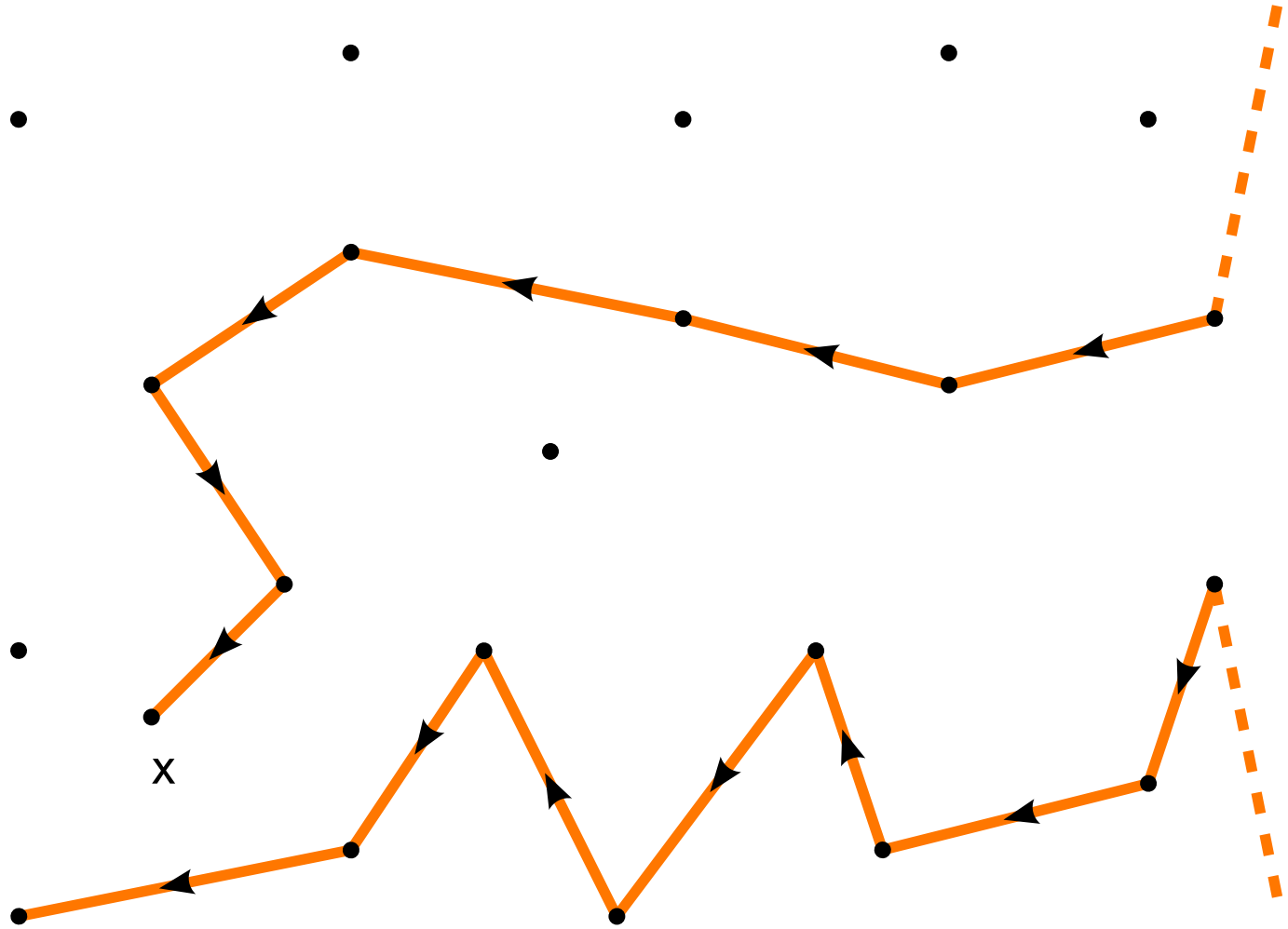

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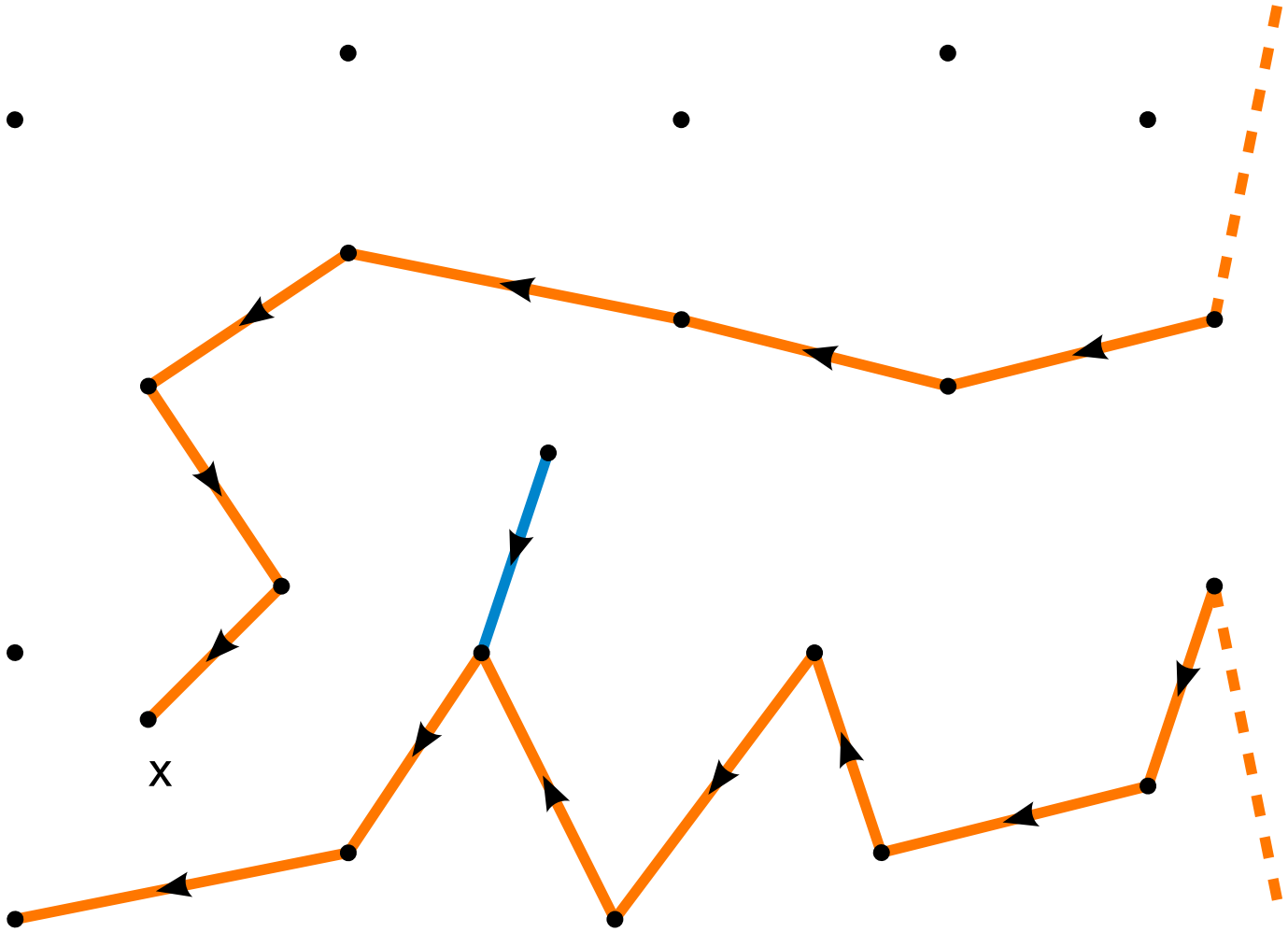

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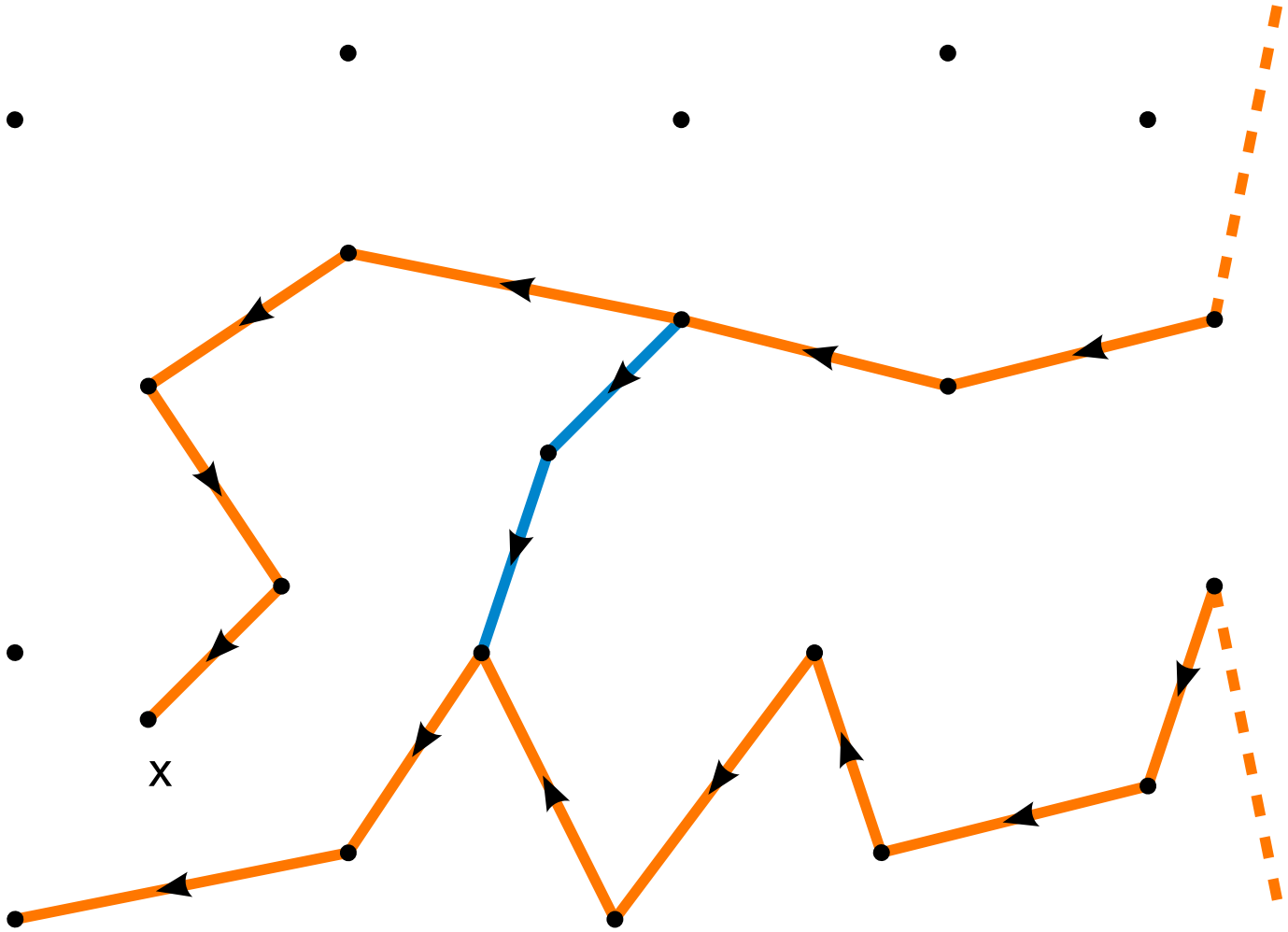

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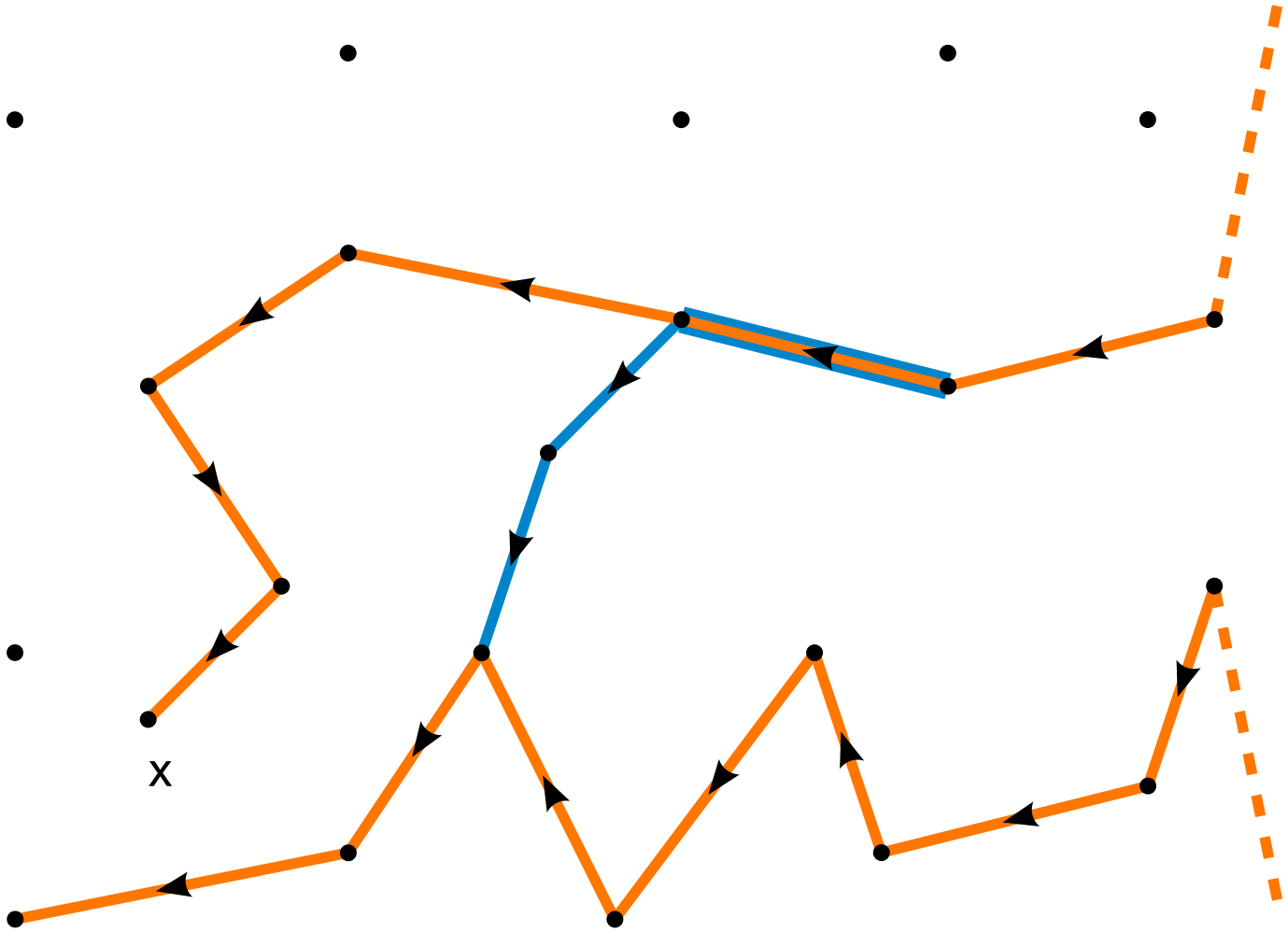

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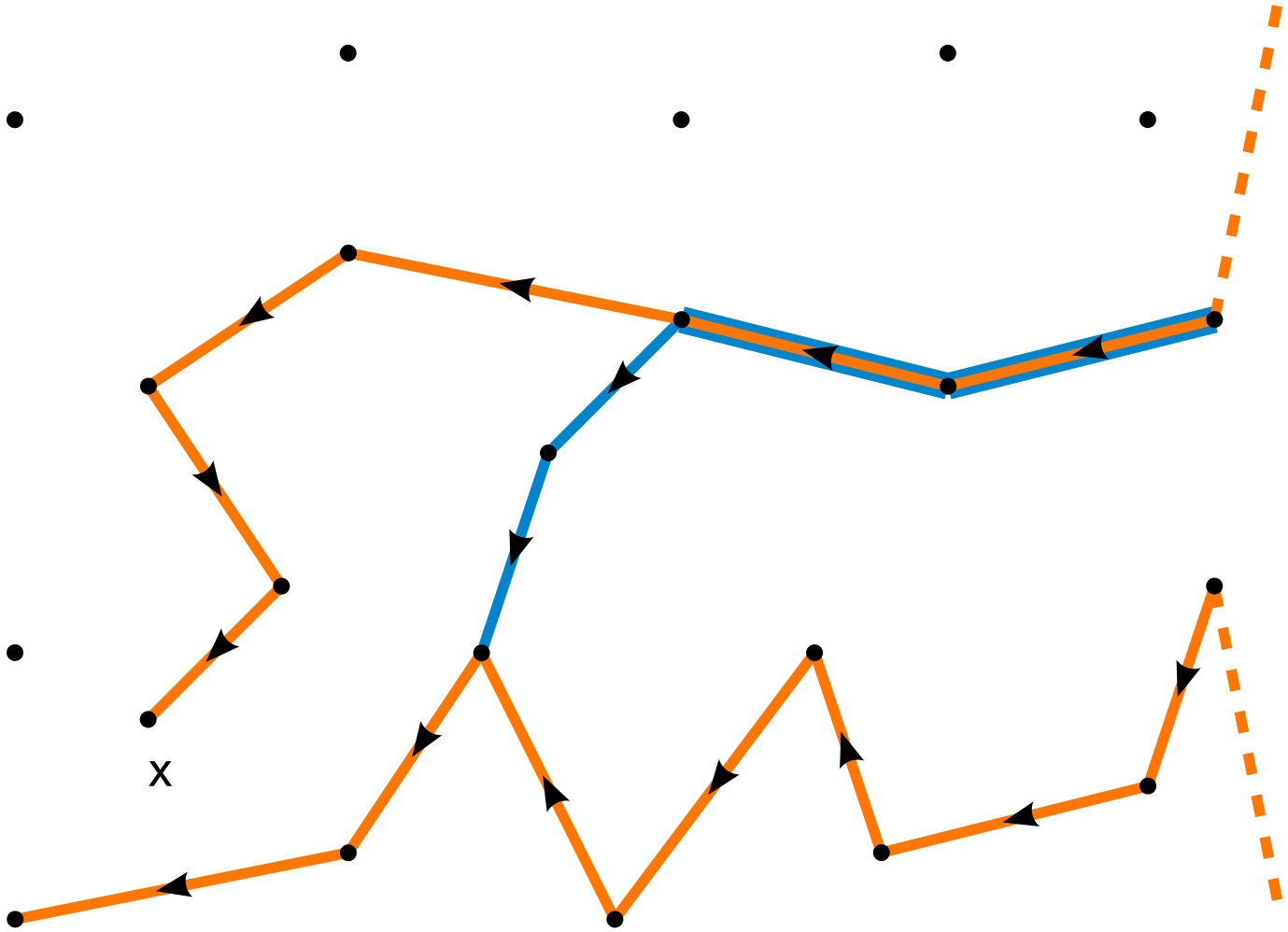

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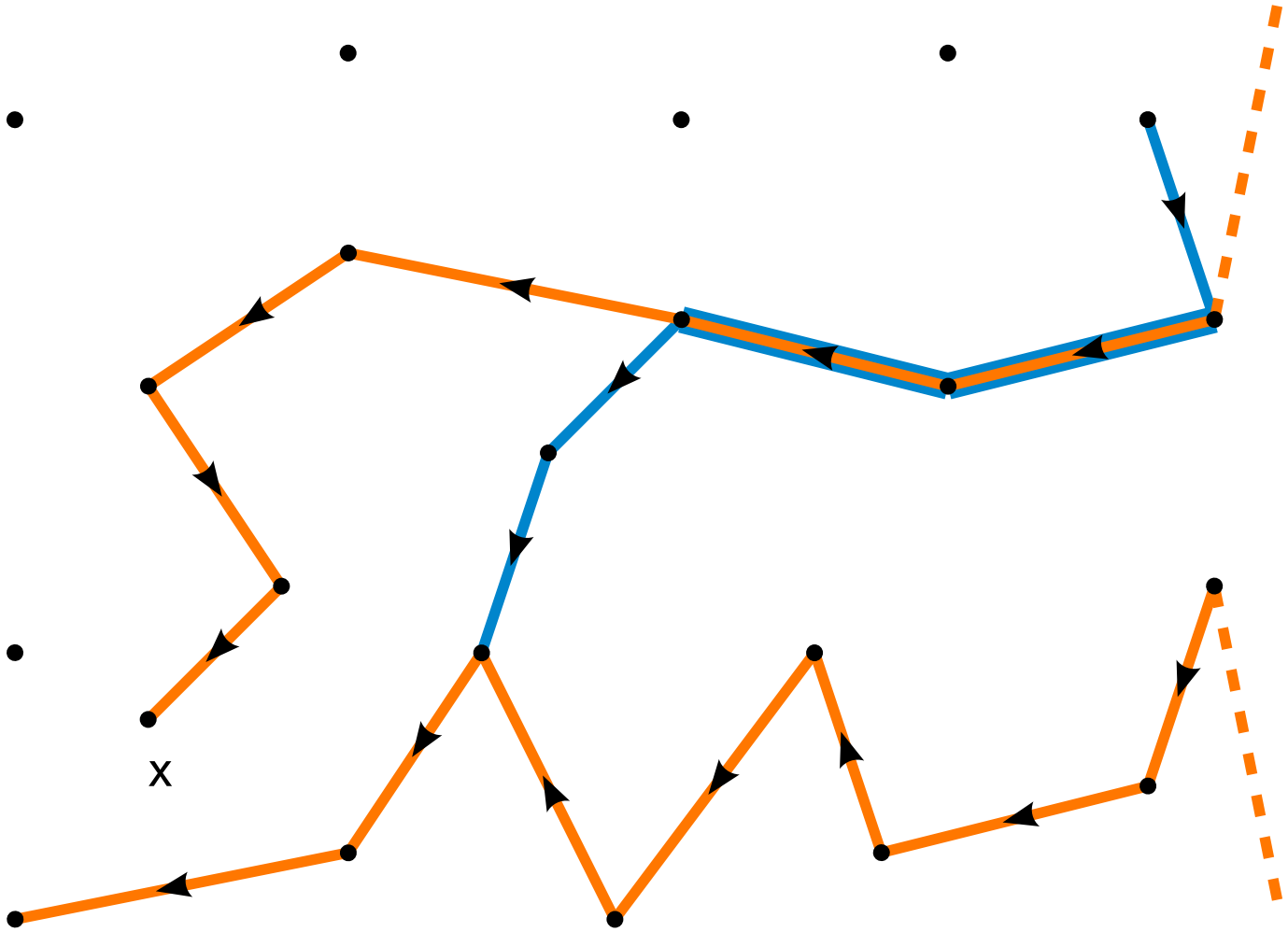

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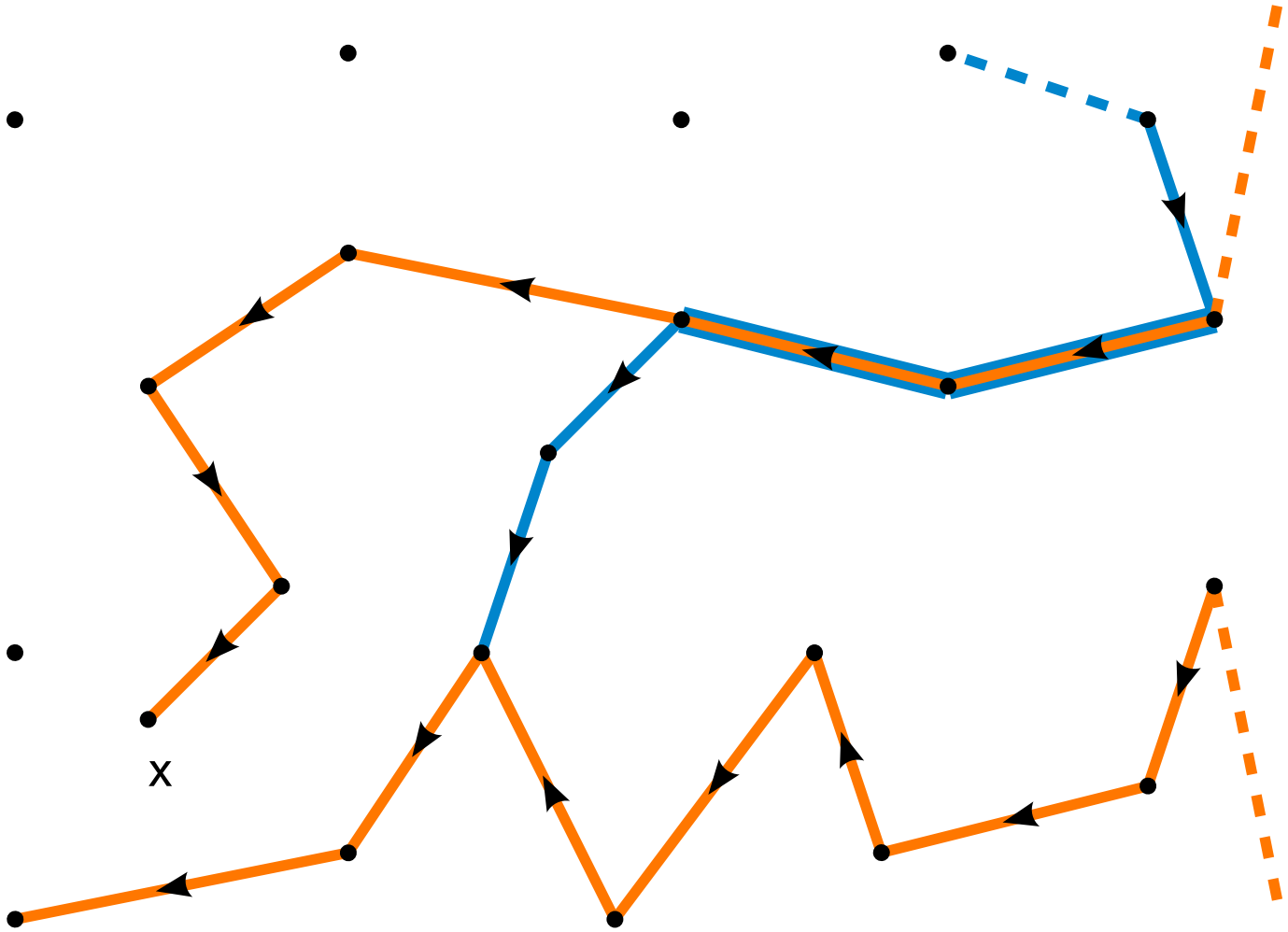

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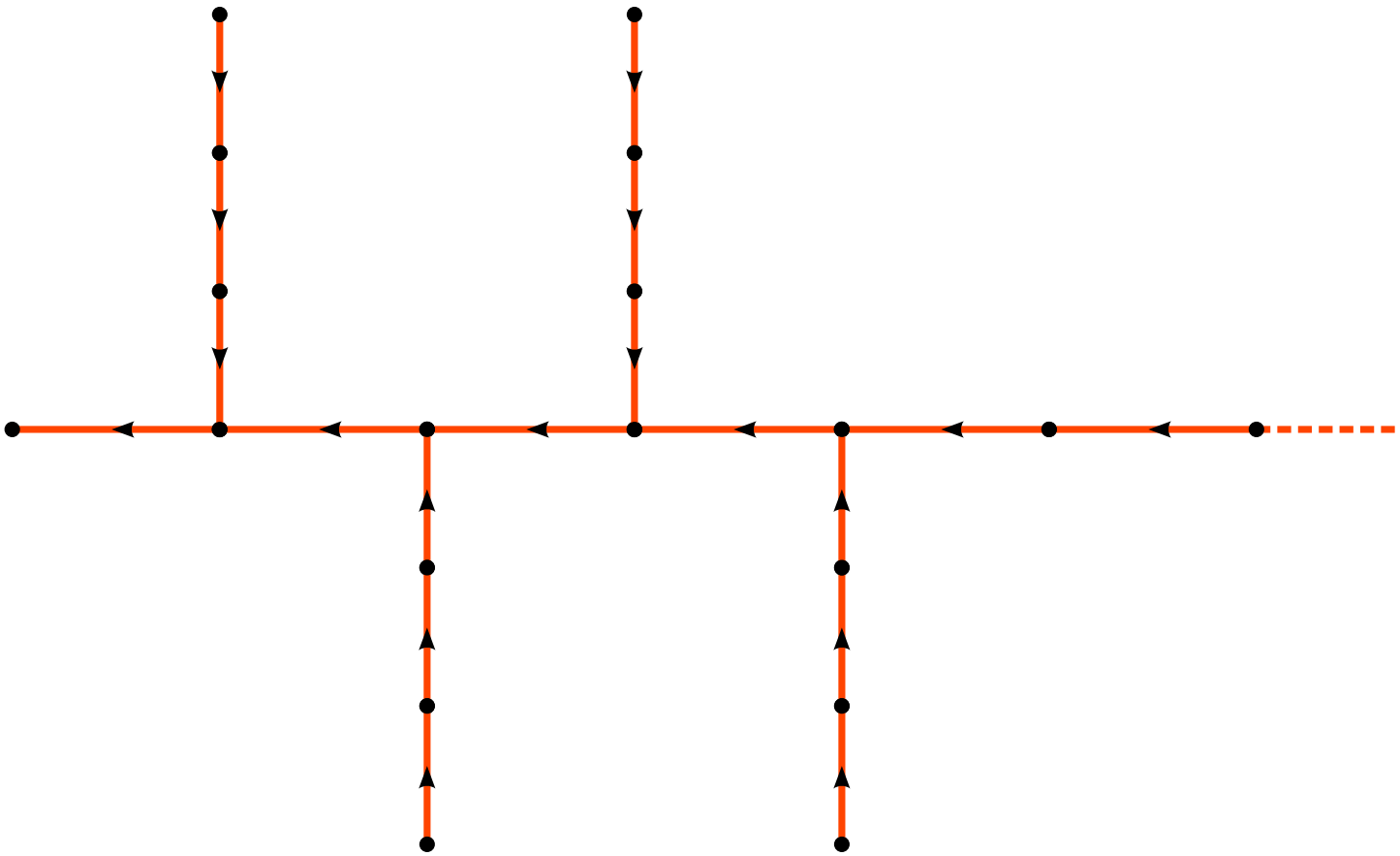

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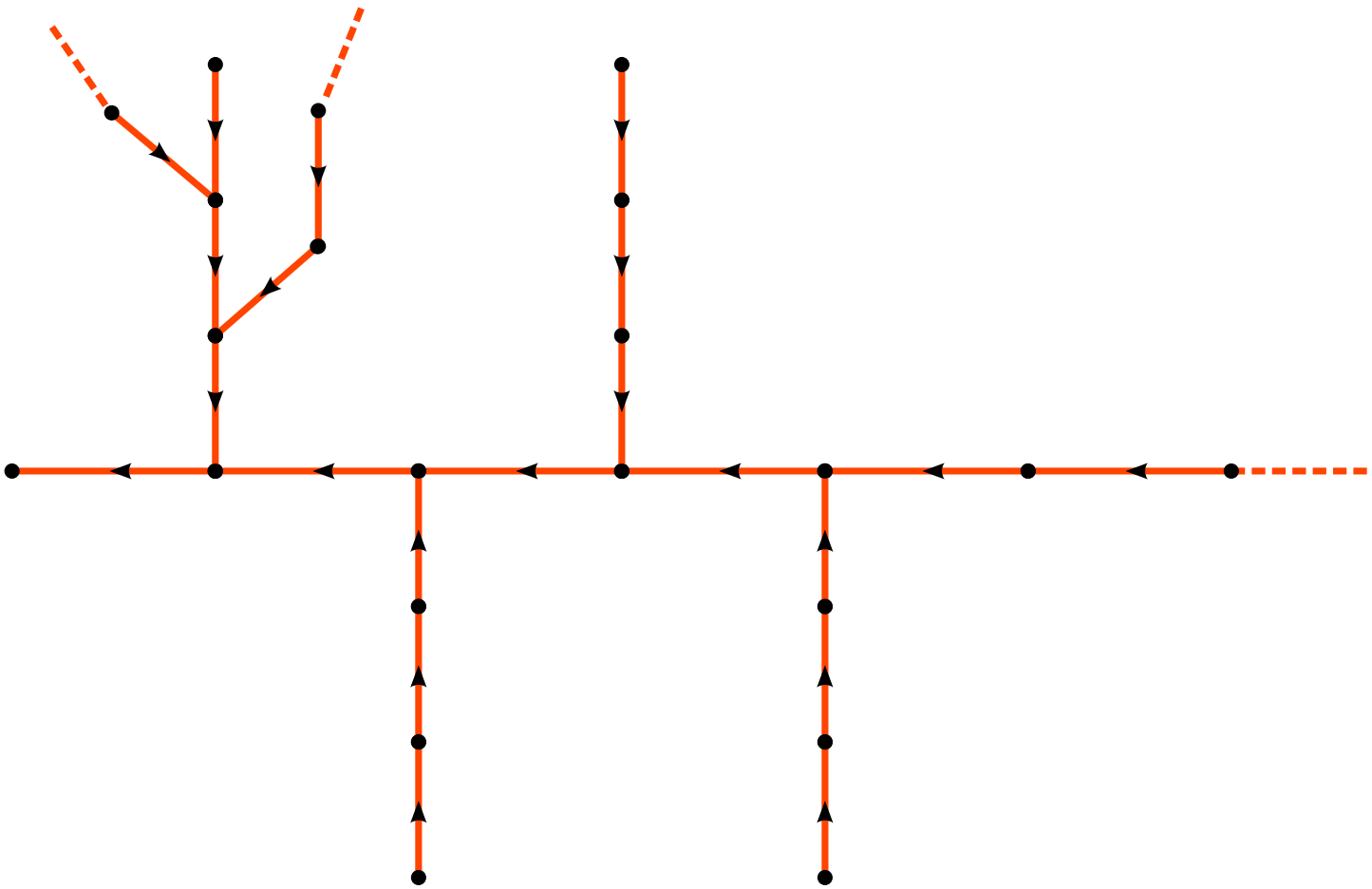

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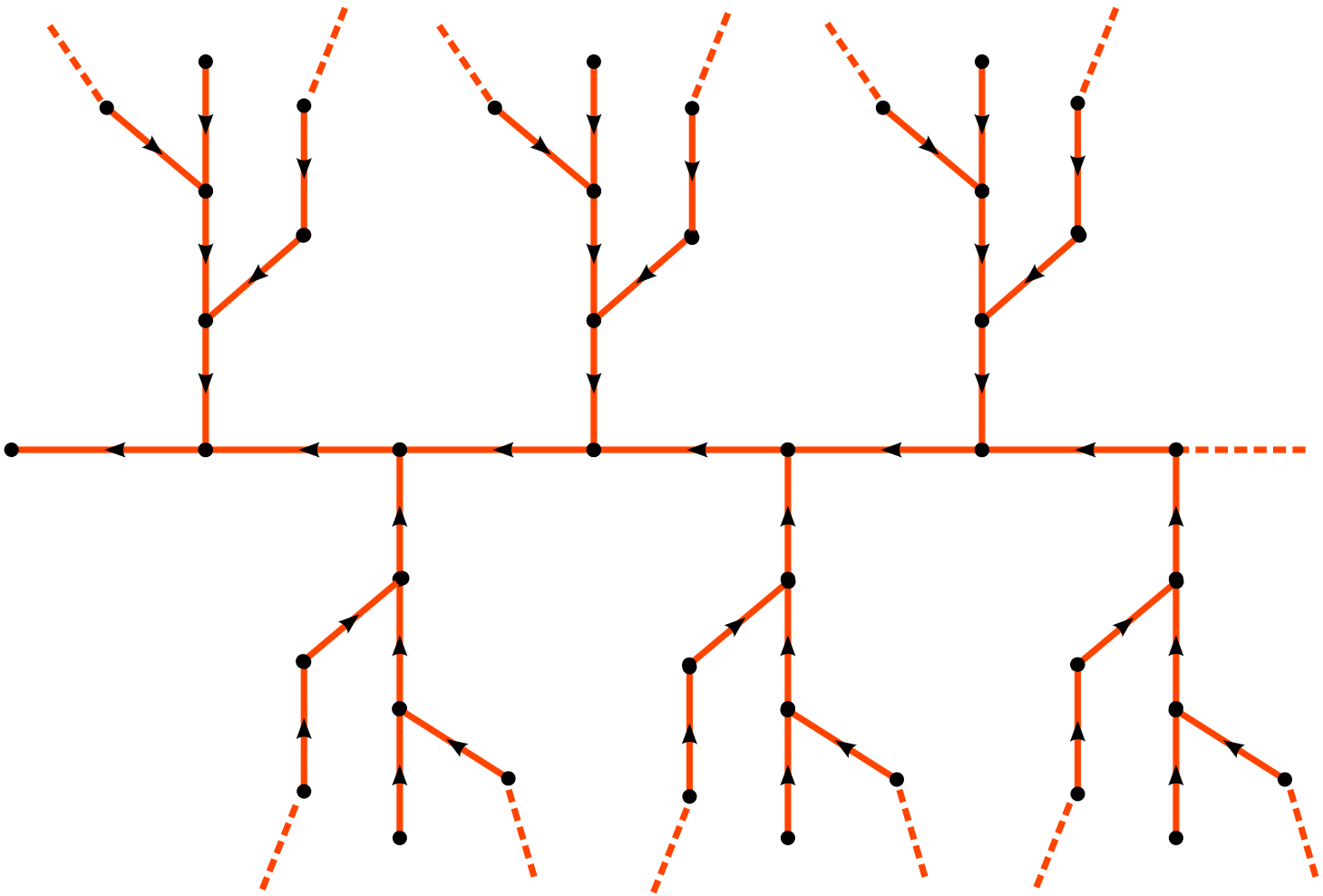

There is very little room on the integers

Example : $\mathbb{Z}$. Then

$$[\mathbb{Z}] = \partial\left(\sum_{n\in\mathbb{Z}} \underbrace{n}_{=c_s} \underbrace{[n, n-1]}_{simplices}\right)$$

So coefficients grow

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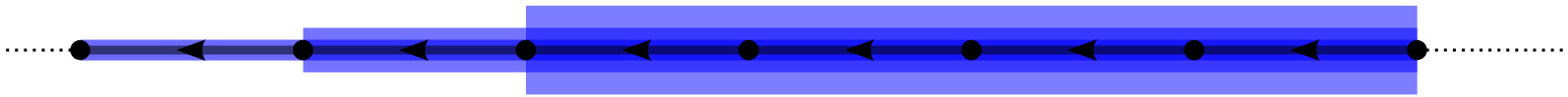

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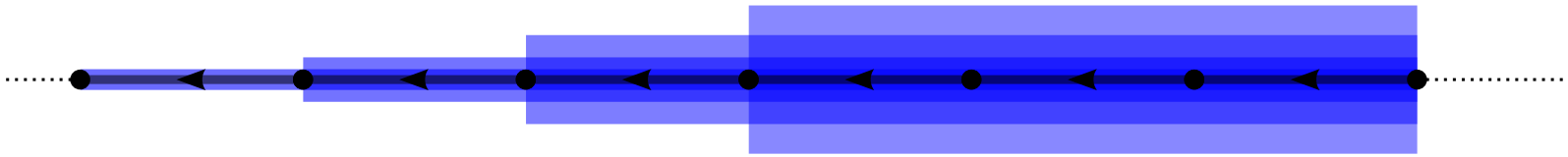

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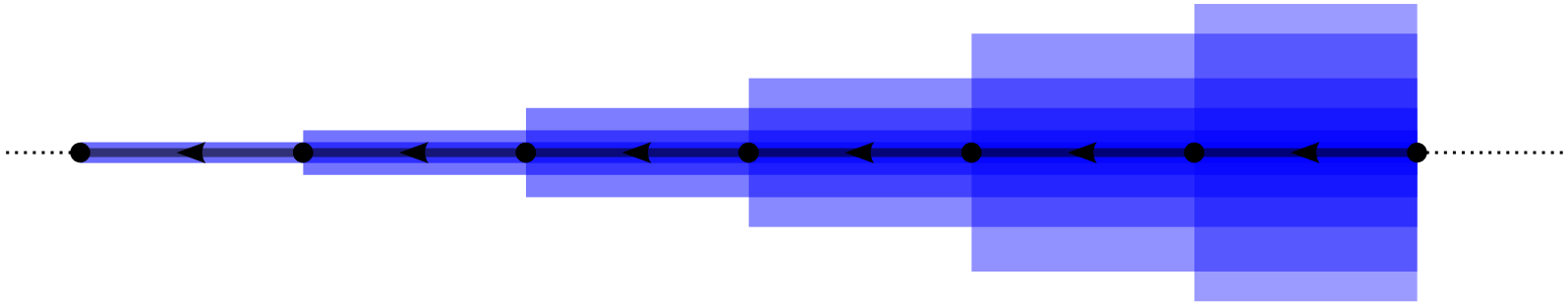

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Growth of tails

Thus: f measures how many tails pass through each edge depending on how this edge is from a fixed point.

Actually the picture with the tails is complete:

Lemma

If $[X] = 0$ in $H_0^f(X)$ then one can always reconstruct tails t_x for every $x \in X$ such that

- $\sum_{x \in X} t_x \in C_1^f(X)$,
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Linearly controlled homology

Theorem (A)

Let G be an infinite group. Then $[G] = 0$ in $H_0^{lin}(G)$.

Two classical problems in group theory of the form

If G is _____ then it has a subgroup which is _____.

Both false but weaker versions in coarse homology are true.

1) The von Neumann conjecture: nonamenable ,free

$\mathbb{F}_2$ is a subgroup in $G \Longrightarrow$ copy the chain which kills $[\mathbb{F}_2]$ to every coset $\Longrightarrow$ this kills $[G]$ in $H_0^{uf}(G)$.

If G non-amenable then it might not have a free subgroup, but $[G]$ always bounds a uniformly bounded 1-chain.

Thus: Block-Weinberger thm = "homological von Neumann theorem"

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2) The Burnside problem : infinite, infinite cyclic.

Assume G is infinite and $\mathbb{Z}$ is an undistorted subgroup in G

$\Longrightarrow$ copy the linearly growing 1-chain which kills $[\mathbb{Z}]$ onto every coset

$\Longrightarrow$ this gives a linearly growing 1-chain on G which kills $[G]$ in $H_0^{lin}(G)$.

Not every infinite group has an infinite cyclic subgroup (not to mention undistorted), but $[G]$ always bounds a linearly growing 1-chain.

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Sketch of proof: Fix a bi-infinite geodesic g_0 passing through e

We will assume for simplicty that $N = \#\{g_0\gamma \ : \ \gamma \in g_0\}$ is finite

Denote $\mathcal{G} = \{g_0\gamma \ : \ \gamma \in \Gamma\}$ and let $\mathcal{G}_e$ be the set of all geodesics of $\mathcal{G}$ which pass through e.

Measurable subsets : $\mathcal{G}_e\gamma_1 \cup_{...} \cup \mathcal{G}_e\gamma_n$ for some $\gamma_i \in \Gamma$.

Collection of all measurable subsets : $\mathcal{F}$

Let $\mu\left(g_0\gamma\right) = 1/N$

μ is a finitely additive right-invariant measure on $\mathcal{F}$ such that $\mu(\mathcal{G}_e) = 1$.

On each $h \in \mathcal{G}$ pick a point $p(h) \in h$ closest to e.

Then for each geodesic $h \in \mathcal{G}$ such that $\gamma \in h$ choose the subray $\overrightarrow{s}_\gamma(h)$ which does not contain $p(h)$.

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Sketch of proof: Fix a bi-infinite geodesic g_0 passing through e

We will assume for simplicty that $N = \#\{g_0\gamma \ : \ \gamma \in g_0\}$ is finite

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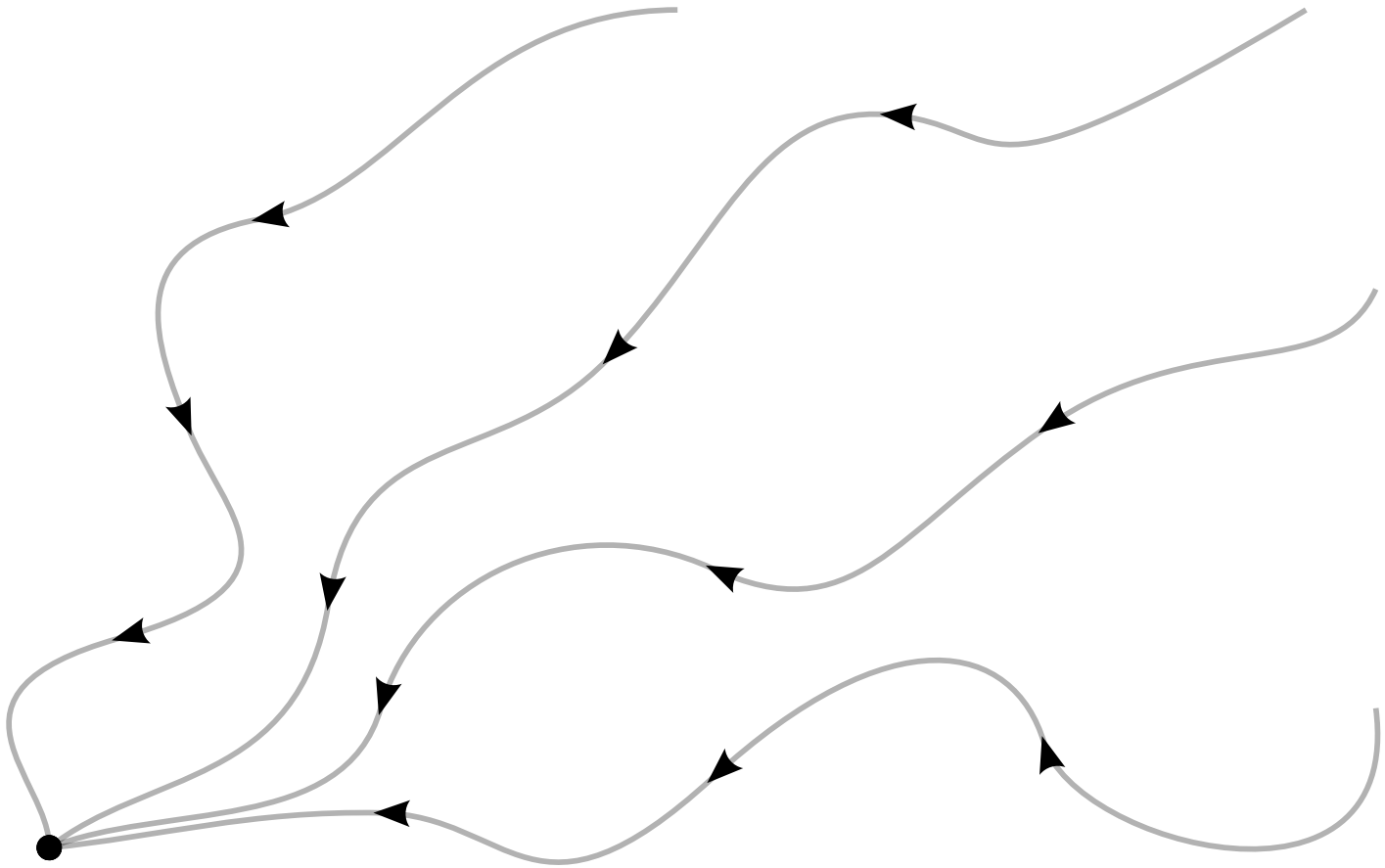
A “spread tail”



Then

$$\partial t_\gamma = [\gamma] \quad \text{and} \quad \sum_\gamma t_\gamma \in C_1^{lin}(G)$$

as needed.

Things get more complicated when $\#\{g_0\gamma \ : \ \gamma \in g_0\}$ is infinite. . . then we can use an invariant mean on the set of indexed geodesics - such a measure was constructed by Żuk. There are some hurdles on the way but eventually everything works and Theorem (A) is proved. □

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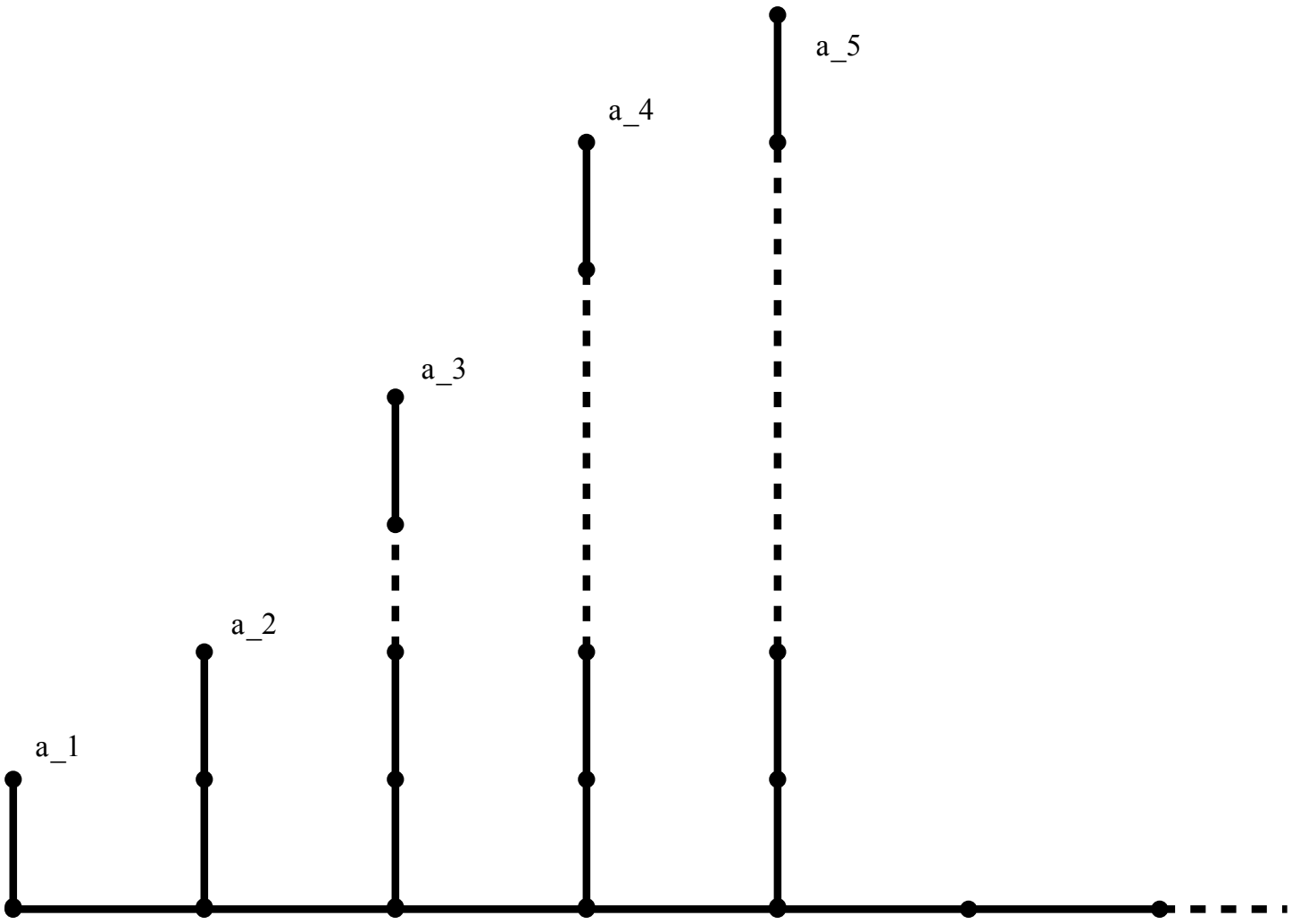
Remark : Theorem (A) is not true for general metric spaces: for the example below one needs $f(n) \geq (\sum_{i=1}^{n} a_i)$ to kill the fundamental class.



$H_0^f(X)$ and isoperimetric inequalities

Theorem (B)

$$\#A \leq C \sum_{x \in \partial A} f(d(x,e)) \quad \Longleftrightarrow \quad [X] = 0 \text{ in } H_0^f(X).$$

Sketch for B-W theorem (then inequality $\Leftrightarrow H_0^{uf}(X) = 0$):
$X = (V, E)$ - a graph.

$$\begin{array}{ccc} C_1^{uf}(X) & \xrightarrow[\partial]{} & C_0^{uf}(X) \\ \| & & \| \\ \ell_\infty(E) & \xrightarrow[\partial]{} & \ell_\infty(V) \end{array}$$

∂ is surjective iff its predual map has a closed image:

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$$\ell_\infty^f(E) = \{\psi : E \to \mathbb{R} \ : \ \psi(x,y) \le Cf(|(x,y)|)\}$$

with the norm

$$\|c\|_\infty^f = \sup_{x \in X} \frac{|c_x|}{f(|x|)}.$$

We view $C_0^f(X) = \ell_\infty^f(V)$, $C_1^f(X) \subseteq \ell_\infty(E)$.

The differential gives rise to a linear operator $d : \ell_\infty^f(E) \to \ell_\infty^f(V)$,

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Inequality $\Longrightarrow$ vanishing :

Lemma

The isoperimetric inequality holds on X if and only if

$$\sum_{x\in X} |\eta(x)| \leq C \sum_{d(x,y)\leq 1} |\eta(y)-\eta(x)| f(|(x,y)|)$$

for every finitely supported function $\eta \in \mathbb{F}$.

We view $\delta : \mathbb{F} \to \delta\mathbb{F}$, where $\mathbb{F}$=finitely supported functions on X.

There is an inverse $\delta^{-1} : \delta\mathbb{F} \to \mathbb{F}$.

We can interpret this inequality as

$$||\delta^{-1}(\delta\eta)||_1 = ||\eta|| \ \leq \ C||\delta\eta||_{1,f}$$

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$$\begin{tikzcd}
\ell^f_\infty(E) \arrow[d, "i^*"'] & \\
\overline{\delta\mathbb{F}}^* & C^{uf}_0(X) \arrow[l, "(\delta^{-1})^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\delta\mathbb{F}} \arrow[r, shift left, "\delta^{-1}"] \arrow[d, hook, "i"'] & \ell_1(V) \arrow[l, dotted, shift left, "\delta"] \\
\ell_1(E;f) &
\end{tikzcd}$$

Since $1_X \in C^{uf}_0(X)$ we take

$$(\delta^{-1})^* 1_X \quad \in \quad \overline{\delta\mathbb{F}}^*$$

and extend it to a functional

$$\phi \quad \in \quad \ell^f_\infty(E)$$

on the whole $\ell_1(E;f)$ via Hahn-Banach.

Then anti-symmetrize:

$$\widetilde{\phi}(x,y) = \phi(x,y) - \phi(y,x).$$

We obtain $\widetilde{\phi} \in C^f_1(X)$ and $\partial\widetilde{\phi} = 1_X$. $\square$

$$\begin{tikzcd}
\ell^f_\infty(E) \arrow[d, "i^*"'] & \\
\overline{\delta\mathbb{F}}^* & C^{uf}_0(X) \arrow[l, "(\delta^{-1})^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\delta\mathbb{F}} \arrow[r, shift left, "\delta^{-1}"] \arrow[d, hook, "i"'] & \ell_1(V) \arrow[l, dotted, shift left, "\delta"] \\
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\end{tikzcd}$$

Since $1_X \in C^{uf}_0(X)$ we take

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Actually even more is true...

Note that we proved the following:

Corollary

The following are equivalent:

- $[X] = 0$ *in* $H_0^f(X)$
- *the natural homomorphism* $i : H_0^{uf}(X) \to H_0^f(X)$ *is trivial.*

There are several other invariants which quantify amenability of a group:

- isoperimetric/isodiametric profiles,
- rate of return to the origin of the simple random walk
- type of asymptotic dimension

We can estimate f for which $[G] = 0$ in $H_0^f(G)$ using all of the above.

Theorem

G-polycyclic then $[G] = 0$ in $H_0^f(G) \iff f$ is exactly linear.

Theorem

Let G be a group with finite asymptotic dimension of linear type (e.g. Baumslag-Solitar groups). Then $[G] = 0$ in $H_0^f(G)$ for f

- *exactly linear if G is amenable,*
- *bounded if G is not amenable.*

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1. Growth of differential forms

Sullivan, 1976: *if on an open Riemmanian manifold* $Vol(\Omega) \leq C Area(\partial\Omega)$, *then does the volume form have a bounded primitive?*

YES : Gromov (1984). Later studied by Sikorav for unbounded growth.

Theorem (C)

M - open Riemannian manifold of bounded geometry.
The volume form has a primitive of growth $\leq Cf \Longleftrightarrow [M] = 0$ *in* $H_0^f(M)$.

Proof : Let $[M] = \partial\psi$, take a cover by appropriately small balls and the associated partition of unity $\{\varphi_x\}$ and take "bump forms" $v_x = \varphi_x$ Vol. Then

$$v_x - v_y = d\omega_{(x,y)}$$

Define $\omega = \sum_{(x,y)} \psi(x,y)\omega_{(x,y)}$. Then $\partial\omega =$ Vol.

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2. Weighted Poincaré inequalities

P.Li and J.Wang studied inequalities of the form

$$\int_M \eta(x)^2 \rho(x)\, dv \leq C \int_M |\nabla \eta|^2(x)\, dv. \qquad (WP_\rho)$$

The applications include rigidity statements for open manifolds.

Theorem (D)

Let M be a compact manifold with universal cover $\widetilde{M}$. Let f be a function which is slowly growing (convex with derivative vanishing at ∞) and $\rho(x) = 1/f(d(x,e))$. Then

$$(WP_\rho) \quad \Longrightarrow \quad [G] = 0 \text{ in } H_0^f(G)$$

where $G = \pi_1(M)$.

$(WP_\rho) \Longrightarrow$ there is $\psi \in C_1^{uf}(G)$, $\partial\psi = \rho = 1/f$. Take $\phi = f \cdot \psi$. Then

$$\partial\phi = \partial(f \cdot \psi) = (\partial f)\psi + f(\partial\psi) = 0 + 1_G.$$

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3. Pontryagin classes

One of the most useful features of vanishing in $H_0^f(X)$ is that we can do surgery along the 1-chain which kills the fundamental class.

Theorem (E)

Let G be a group such that $[G] = 0$ in $H_0^f(G)$. Then there are M and N compact manifolds with $\pi_1(M) = \pi_1(N) = G$ and a smooth homotopy equivalence

$$g : M \to N$$

which does not pull back some rational Pontryagin class such that the lift $\widetilde{g} : \widetilde{M} \to \widetilde{N}$ is bounded distortion homotopic to a diffeomorphism whose distortion is controlled by f.
In particular for any G one needs at most linear distortion.

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In particular for any G one needs at most linear distortion.

Sketch of proof : K = any closed manifold, $\pi_1(K) = G$

L = simply connected with some additional homological properties

Take $M = L \times K$.

Surgery theory then produces a manifold W and a homotopy equivalence $W \to L \times D^m$ which restricts to a diffeo on the boundary and is detected by some Pontryagin class. Then

$$N = \Big(M \setminus \text{interior}(L \times D^m)\Big) \cup W$$

and there is a map

$$f : M \to N.$$

Then we "unfold" the grafting in the universal cover of M along $\psi \in C_1^f(G)$ such that $\partial\psi = [G]$, while controlling distortion.
(this actually gets quite involved . . .)

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4. Positive scalar curvature

Similarly we can do surgery in positive scalar curvature a'la Gromov-Lawson along the 1-chain that kills the fundamental class and obtain

Conjecture

Let $[G] = 0$ in $H_0^f(G)$. Then there is a G-covering space $\widetilde{M} \to M$ where M is a compact spin manifold, $\hat{A}(M) \neq 0$ and $\widetilde{M}$ has a metric of positive scalar curvature which is diffeomorphic to the lifted metric via a diffeomorphism with distortion controlled by f.

Remarks and questions

1. There is a corresponding controlled de Rham cohomology which is Poincare dual to $H_0^f(M)$.
2. There should be a certain index theory associated to this kind of controlled homology, however to make things work one would need to take a homology controlled by a class of functions instead of just f.
3. Whyte proved a "geometric von Neumann conjecture": *every non-amenable group/space can be partitioned into nice copies of a tree.*

Question

Is there a "geometric Burnside theorem"?

E.g. can one partition any infinite group into copies of $\mathbb{Z}$? If so should they be uniformly quasi-isometric to $\mathbb{Z}$? Uniformly coarsely equivalent to $\mathbb{Z}$?

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