KK-equivalences and T^{2n}-duality type isomorphisms

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Outline of the talk:

- Topological T-duality.
- KK-equivalence of C^*-algebras.
- KK-equivalence $\Rightarrow$ T^{2n}-duality type isomorphisms.

- Topological T-duality.

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In open string theory the boundaries of open strings lie in some specific submanifolds of the target space. These submanifolds come with special Chan-Paton vector bundles and together they form D-branes. The 'charges of the D-branes' are classified by the topological K-theory classes of the target space.

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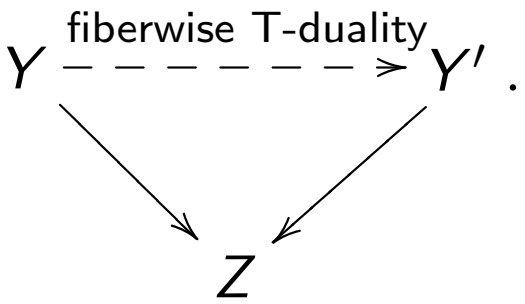
Conjecture (Homological Mirror Symmetry - Kontsevich)

Y and Y' mirror dual CYs $\Rightarrow D^b(Coh(Y)) \cong Fuk(Y')$.

Strominger-Zaslow-Yau said that whenever Y and Y' are mirror dual one can find a (generically) torus-fibration over a common base Z



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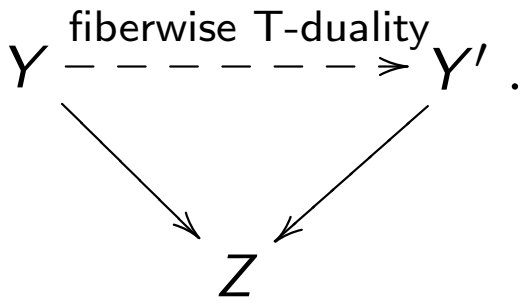

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Restricting to the topological sector we say that a topological homological T-duality transform is a DG functor between DG categories generalizing $D^b(Coh(Y))$ and $Fuk(Y')$ which changes (actually shifts) 'D-brane charges in a suitable manner'.

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Corr. between spaces $\rightsquigarrow$ Equiv. of DG categories.

- KK-equivalence of C^*-algebras.

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Definition

$A, B \in \mathrm{Sep}_{C^*}$ are KK-equivalent iff they are isomorphic in $\mathtt{KK}_{C^*}$.

A functor $F : \mathrm{Sep}_{C^*} \longrightarrow \mathcal{C}$ is a called $\underline{C^*\text{-stable}}$ (Morita invariant) if $F(\iota)$ is an isomorphism, where $\iota : A \to A \otimes \mathbb{K}$ is the corner embedding.

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Now let $\mathcal{C}$ be additive. We say F is split exact if whenever

$$0 \longrightarrow A \xrightarrow{i} B \underset{j}{\overset{s}{\rightleftarrows}} C \longrightarrow 0$$

is an exact sequence in Sep_{C^*} one has $F(A) \cong F(I) \oplus F(A/I)$ in $\mathcal{C}$.

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Theorem (Higson)

The canonical functor $\mathrm{Sep}_{C^*} \longrightarrow \mathrm{KK}_{C^*}$ *is the universal* C^**-stable and split exact functor on* Sep_{C^*}*. Such a functor is automatically homotopy invariant.*

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Theorem (Cuntz)

$\mathrm{KK}_{C^*}(A, B) = [qA, B \otimes \mathbb{K}]$, *where [-,?] denotes homotopy classes of maps in* Sep_{C^*}. *Furthermore, Kasparov prod.* $=$ *comp. of* $*$*-homo.*

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where $\alpha \in \mathrm{KK}_1(CT(E,H), CT(\hat{E},\hat{H}))$ is invertible.

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Note: T-duality applied twice gives rise to an invertible KK_0-class.

- KK-equivalence $\Rightarrow$ T^{2n}-duality type isomorphisms.

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This category is enriched over monoids and we perform a naïve group completion of morphisms to obtain an additive category $\mathtt{NCC}^{\mathbf{K}}_{\mathtt{dg}}$ (noncommutative correspondence category).

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The category $\mathrm{Top}^{\mathbf{K}}_{\mathrm{fib}}(A)$ *is independent (up to homotopy) of the way in which one realizes A as a two sided ideal inside a unital* $\mathbb{C}$*-algebra R. Furthermore,* $\pi_0(\mathbf{K}(\mathrm{Top}^{\mathbf{K}}_{\mathrm{fib}}(A))) = \mathrm{K}_0(A)$.

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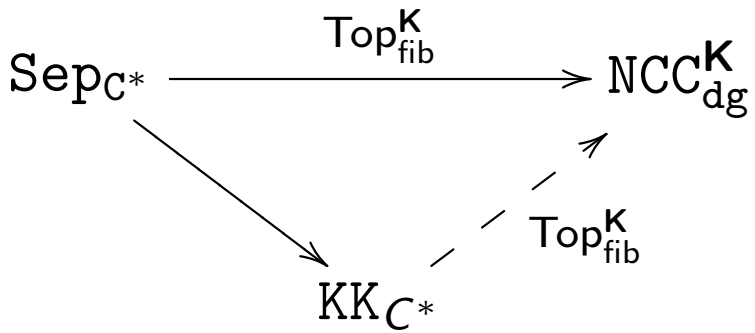
Theorem (-)

The functor $\mathrm{Top}^{\mathbf{K}}_{\mathrm{fib}} : \mathtt{Sep}_{C^*} \longrightarrow \mathtt{NCC}^{\mathbf{K}}_{\mathrm{dg}}$ *is* C^**-stable and split exact, i.e., one has*



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It seems that KK*-equivalences can account for them.*

Reference: Noncommutative correspondence categories, simplicial sets and pro C^*-algebras, *http://arxiv.org/abs/0906.5400*