# Some Stable and Unstable Homotopy of Cell Complexes

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# 1. Exponents (at odd primes)

- Instead of answering the very difficult question "what are the homotopy groups of a space?", we ask for something weaker like "approximately how fast do the homotopy groups grow?".

- The $p$-exponent $exp_p(G)$ of an abelian group $G$ is the smallest power $p^t$ such that
$$p^t \times (p\text{-torsion in } G) = 0.$$

- We measure the growth of homotopy groups of $X$ by finding bounds for the growth of $exp_p(\pi_i(X))$.

- The largest value of $exp_p(\pi_i(X))$ for $i > 1$ (if it exists) is called the $p$-exponent of $X$, and is denoted $exp_p(X)$. In this case we can also say $exp_p(\pi_*(X))$ is bounded.

# 2. Cases of Bounded Exponents

## 2.1. Finite $p$-local $H$-spaces

- Building on work of Toda, for odd primes $p$ Cohen, Moore, and Neisendorfer showed
$$exp_p(S^{2n+1}) = p^n.$$
- Selick proved the special case
$$exp_p(S^3) = p.$$
- For every $n > 0$ the image of the suspension map $X \longrightarrow \Omega^n \Sigma^n X$ on $\pi_*$ has bounded $p$-exponent given $X$ is a finite connected $CW$-complex [Don Stanley] .
- A consequence is that every finite connected $H$-space (or more generally an $H$-space that is the $p$-localization of a finite connect $CW$-complex) has a bounded $p$-exponent [Stanley, Long].

## 2.2. Rationally trivial spaces

- The *Moore space* $P^n(p^r)$ has $p$-exponent $p^{r+1}$ [Cohen, Moore, and Neisendorfer].
- Any finite wedge of Moore spaces has a bounded $p$-exponent.
- A simply connected finite $CW$-complex $X$ is rationally trivial if and only if $\pi_*^s(X)$ has bounded $p$-exponent [Stanley].
- A simply connected finite $CW$-complex $X$ s.t. $\pi_*(X) \otimes \mathbb{Q}$ is a finite dimensional vector space has a bounded $p$-exponent at *almost all* primes $p$ [McGibbon,Wilkerson].

## 2.3. Conjectured Examples

- <u>Moore conjecture:</u> A simply connected finite $CW$-complex $X$ has $\pi_*(X) \otimes \mathbb{Q}$ as a finite dimensional vector space if and only if it has bounded $p$-exponent at all primes $p$.
- <u>Barrats conjecture (weak form):</u> If the identity map on $\Sigma^2 X$ has order $p^r$ then $exp_p(\Sigma^2 X) = p^{r+1}$.

# 3. Unbounded Exponents

- Any wedge of spheres does not have a bounded $p$-exponent (by Hilton-Milnor Theorem).
- More generally, $\Sigma X$ is rationally non-trivial implies $S^m \vee \Sigma X$ does not have a $p$-exponent at any prime $p$ and all $m > 1$ [Stanley].

  note: That $\Sigma X$ is rationally nontrivial is necessary since $S^{2k+1} \vee P^n(p^r)$ has $p$-exponent $\max\{p^{r+1}, p^k\}$.
- Any 2-cell complex $X = S^m \cup_\alpha e^n$ such that the attaching map $S^{n-1} \xrightarrow{\alpha} S^m$ is of finite order has no bounded $p$-exponent when $n, m \geq 2$ [Selick, Neisendorfer].
- More generally any finite cell complex with torsion free integral homology of dimension $> 1$ has no bounded exponent at odd primes [Selick].

**Theorem 3.1 (Hilton-Milnor)** *For any spaces X and Y, there is a functorial decomposition*

$$\Omega\Sigma(X \vee Y) \simeq \Omega\Sigma X \times \Omega\Sigma(Y \vee \bigvee_{i\geq 1}(X^{(i)} \wedge Y)).$$

$\square$

Iterating the Hilton-Milnor Theorem implies $\Omega(S^n \vee S^m)$ is the loopspace of a certain infinite weak product of spheres of increasing dimension. Then using the known exponents of spheres, an upper bound for the growth of $exp_p(\pi_i(S^n \vee S^m))$ can be found.

PROBLEM: For other less trivial spaces, try to obtain bounds for growth of exponents...

MAIN ASSUMPTIONS: From now on assume all spaces are $CW$-complexes localized at an odd prime $p$. Take some such $p$-localized finite cell complex $X$, let

$$V = \widetilde{H}_*(X; \mathbb{Z}_p),$$

and let $M$ be the sum of the degrees of the generators in $V$.
Let $\mathbb{Z}_p[S_k]$ be the group ring of the symmetric group $S_k$ on $k$ letters.

- For the $k$-fold self smash $X^{(k)}$, we have $\widetilde{H}_*(X^{(k)}; \mathbb{Z}_p) \cong V^{\otimes k}$.
- $\mathbb{Z}_p[S_k]$ acts on $V^{\otimes k}$ sending elements to sums of their permutations.
- For each $\sigma \in \mathbb{Z}_p[S_k]$, the action of $\sigma$ on $V^{\otimes k}$ determines a self map

$$V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k}.$$

- When $X$ is a suspension, one can construct a self-map $X^{(k)} \xrightarrow{\bar{\sigma}} X^{(k)}$ inducing $V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k}$ on homology.
- If $\sigma \circ \sigma = \sigma$ (i.e. $\sigma$ is an idempotent), the mapping telescope $T(\sigma)$ of the sequence

$$X^{(k)} \xrightarrow{\bar{\sigma}} X^{(k)} \xrightarrow{\bar{\sigma}} \cdots$$

  is a retract of $X^{(k)}$ with homology isomorphic to the image of $\sigma$.

By the Poincare-Birkhoff-Witt Theorem, there is an isomorphism of $\mathbb{Z}_{(p)}$-modules

$$T(V) \cong \bigotimes_{i=1}^{\infty} S(L_i(V)),$$

where $L_n(V)$ is the submodule of length $n$ Lie brackets in $V^{\otimes n}$. Part of this decomposition can be geometrically realized:

**Theorem 3.2 (Jie Wu)** *Let $X$ be a suspension. There is a decomposition*

$$\Omega\Sigma X \simeq \prod_j \Omega\Sigma L_{k_j}(X) \times (\textit{Some other space}).$$

*for each sequence $1 < k_1 < k_2 < \cdots$ such that*

1. *$k_i$ is prime to $p$;*
2. *$k_i$ is not a multiple of $k_j$ whenever $i \neq j$.*

□

Each space $L_n(X)$ is a retract of $X^{(n)}$, and is isomorphic to $L_n(V)$ on homology. They are geometrically realized as telescopes $T(\beta_n)$ using the *Dynkin-Specht-Wever element*

$$\beta_n \in \mathbb{Z}_p[S_n].$$

FURTHER RESTRICTIONS: From now on assume $X$ is a suspension, and has either only cells of even dimension or cells of odd dimension.

Then the basis of $V = \widetilde{H}_*(X; \mathbb{Z}_p)$ has generators in degrees corresponding to the dimensions of the cells. Also $\widetilde{H}_*(X; \mathbb{Z}_{(p)}) \cong V$ and $\widetilde{H}_*(X^{(i)}; \mathbb{Z}_{(p)}) \cong V^{\otimes i}$ as $\mathbb{Z}_{(p)}$-modules.

For the odd case, let $s_k \in \mathbb{Z}_{(p)}[S_k]$ be

$$s_k = \sum_{\sigma \in S_k} \sigma,$$

and for the even case

$$s_k = \sum_{\sigma \in S_k} sgn(\sigma)\sigma.$$

In either case it is well known that $s_k s_k = k! s_k$.

We obtain the self map $V^{\otimes \ell} \xrightarrow{s_\ell} V^{\otimes \ell}$ and its geometric realization $X^{(\ell)} \xrightarrow{\bar{s}_\ell} X^{(\ell)}$.

**Proposition 3.3** *When $V$ has $\ell < p$ generators,*

(i) *the telescope $T(s_\ell)$ is a retract of $X^{(\ell)}$ and*

(ii) $T(s_\ell) = S^M$ *where $M$ is the sum of the degrees for each generator in $V$.*

$\square$

**Corollary 3.4** *When $V$ has $\ell < p$ generators,*

(i) $\Sigma^M X$ *is a retract of $X^{(\ell+1)}$;*

(ii) *as a submodule of $\widetilde{H}_*(X^{(\ell+1)};\mathbb{Z}_p) \cong V^{\otimes \ell+1}$, $\widetilde{H}_*(\Sigma^M X;\mathbb{Z}_p)$ is the image of $V^{\otimes \ell+1} \xrightarrow{s_\ell \otimes \mathbb{1}} V^{\otimes \ell+1}$.*

$\square$

Take the self maps

$$V^{\otimes k+1} \xrightarrow{\beta_{k+1}} V^{\otimes k+1}$$

and

$$V^{\otimes k} \xrightarrow{s_k} V^{\otimes k}.$$

**Proposition 3.5** *Suppose a free $\mathbb{Z}_{(p)}$-module $V$ has basis of dimension $k > 1$ consisting either of only odd degree generators or of even degree generators. Then*

$$(s_k \otimes \mathbb{1}) \circ \beta_{k+1} \circ (s_k \otimes \mathbb{1}) = (k! + (k-1)!)(s_k \otimes \mathbb{1}).$$

□

Therefore the composition

$$\Sigma^M X \longrightarrow X^{(\ell+1)} \longrightarrow L_{\ell+1}(X) \longrightarrow X^{(\ell+1)} \longrightarrow \Sigma^M X$$

is a $p$-local homotopy equivalence when $1 < \ell < p-1$.

Notice $\beta_{k+1} \circ (s_k \otimes \mathbb{1})$ is an idempotent modulo $p$ when $k = 2, 3, ..., p-2, p$ (excluding $p-1$).

**Theorem 3.6** *Let $X$ be a suspension and $M$ be the sum of degrees of generators in $V = \widetilde{H}_*(X;\mathbb{Z}_p)$. When $V$ has $1 < \ell < p-1$ generators, and all generators are either of even or of odd degree, then*

(i) *$\Omega\Sigma^{M+1}X$ is a retract of $\Omega\Sigma X$;*

(ii) *Therefore $\Omega\Sigma^{c_k+1}X$ is a retract of $\Omega\Sigma X$ for all $k \geq 0$, where (recursively)*

$$c_0 = 0$$

*and*

$$c_k = (\ell + 1)c_{k-1} + M.$$

(iii) *Therefore $\pi_i^s(\Sigma X)$ is a retract of $\pi_{i+c_k}(\Sigma X)$ for all $k$ large enough s.t. $i < c_k$.*

$\square$

$\longrightarrow$ A weaker version of Theorem 3.6 when $\ell = p$: Proposition 3.5 implies $\beta_{\ell+1} \circ (s_\ell \otimes \mathbb{1})$ is an idempotent modulo $p$. Then the telescope $T(\beta_{\ell+1} \circ (s_\ell \otimes \mathbb{1}))$ splits off from $L_{\ell+1}(X)$, and its mod-$p$ homology is isomorphic to that of $\Sigma^M X$.

QUESTION: is $T(\beta_{\ell+1} \circ (s_\ell \otimes \mathbb{1}))$ homotopy equivalent to $\Sigma^M X$ when $\ell = p$?

$\longrightarrow$ Computational evidence indicates that Proposition 3.5 can be generalised so that

$$(s_\ell^{\otimes k} \otimes 1) \circ \beta_{\ell k+1} \circ (s_\ell^{\otimes k} \otimes 1) = s_\ell^{\otimes k} \otimes 1 \mod p,$$

for $1 < \ell < p - 1$ and all $k \geq 1$. So Theorem 3.6 can probably be strengthened considerably.

FURTHER WORK: What happens when $V$ has *both* even and odd degree generators.

# 4. Exponent Growth (Lower bounds)

An immediate consequence of Theorem 3.6:

**Corollary 4.1** *If $X$ has $1 < \ell < p$ cells,*

$$exp_p(\Sigma X) \geq exp_p(\Sigma^{M+1} X).$$

□

<u>QUESTION:</u> When is it true that $exp_p(Y) \geq exp_p(\Sigma Y)$?

**Theorem 4.2 (Cohen, Neisendorfer)** *Let $X$ be any $p$-localized cell complex consisting of $\ell < p-1$ odd cells, $V = \widetilde{H}_*(X)$. Then there exists a functorial decomposition*

$$\Omega\Sigma X \simeq A(X) \times \Omega Q(X)$$

*such that $A(X)$ is a finite $H$-space with*

$$H_*(A(X)) \cong S(V)$$

*as primitively generated algebras, and*

$$H_*(\Omega Q(X)) \cong S([L(V), L(V)]) = \bigotimes_{i=2}^{\infty}(S(L_i(V))).$$

$\square$

Given $X$ consists only of $1 < \ell < p-1$ odd cells, Theorem 3.6 and 4.2 imply $A(\Sigma^{c_k} X)$ is a retract of $\Omega\Sigma X$ for all $k \geq 1$ such that $c_k$ is even. Since

$$T(L_{\ell+1}(V)) \cong \bigotimes_{i=1}^{\infty} S(L_i(L_{\ell+1}(V))) \subseteq S([L(V), L(V)]),$$

this result can be strengthened:

**Theorem 4.3** *If $V$ consists of $1 < \ell < p-1$ odd dimensional generators and $\ell$ is even:*

$$\Omega\Sigma X \simeq \prod_{i=0}^{\infty} A(\Sigma^{c_i} X) \times (\mathit{Some\ other\ space}).$$

□

If $N$ is the dimension of the top cell(s) in $X$, and $\bar{X}$ is $X$ minus an $N$-cell, there exist homotopy fibrations

$$A(\Sigma^{c_i}\bar{X}) \longrightarrow A(\Sigma^{c_i}X) \xrightarrow{q} S^{c_i+N}.$$

If the order of the attaching map for the $N$-cell that we removed (from $X$ to get $\bar{X}$) is known (say $p^t$), there exist degree $p^t$ factorizations

$$S^{c_i+N} \longrightarrow A(\Sigma^{c_i}X) \xrightarrow{q} S^{c_i+N}.$$

In this case, by a result of Theriault we get homotopy fibrations

$$\Omega S^{c_i+N} \times \Omega A(\Sigma^{c_i}\bar{X}) \longrightarrow \Omega A(\Sigma^{c_i}X) \longrightarrow S^{c_i+N}\{p^t\}$$

where $exp_p(S^{c_i+N}\{p^t\}) = p^t$.

## 4.1. The curious case of rank 2

When $X$ is a suspension with only $\ell = 2$ even dimensional cells ($M$ is the sum of their dimensions), from Proposition 3.5

$$L_2(X) = S^M,$$

and

$$L_3(X) = \Sigma^M X.$$

Recall $c_1 = M$, $c_k = (\ell + 1)c_{k-1} + M = 3c_{k-1} + M$.

By iteration we have:

**Theorem 4.4** *For any suspended 2-cell complex $X$ of even cells*

$$\Omega\Sigma X \simeq \prod_{i=1}^{\infty} \Omega(S^{c_i+1}) \times (\mathit{Some\ other\ space}).$$

□

QUESTION: In general, do loop spaces of spheres retract off from $\Omega\Sigma X$ when $\ell > 2$.

## 4.2. Moore Conjecture

Recall simply connected finite $CW$-complex is rationally trivial if and only if its stable homotopy has bounded $p$-exponent.

Since our space $\Sigma X$ has either even or odd cells, it is *not* rationally trivial, and so has unbounded $p$-exponent on stable homotopy.

Since the stable homotopy of $\Sigma X$ retracts off from its unstable homotopy (by Theorem 3.6), $\Sigma X$ has unbounded $p$-exponent.

GUESS 1: For any general finite simply connected cell complex $X$, there exists an infinite sequence $Y_1, Y_2, \ldots$ of finite dimensional rationally nontrivial spaces such that each $\Omega Y_i$ is a retract of $\Omega\Sigma X$ and the mod-$p$ homology of $Y_{i+1}$ is a suspension of the mod-$p$ homology of $Y_i$.

- We saw this happens when when $X$ has $\ell < p-1$ or $\ell = p$ number of odd or even cells
- When $X$ has $\ell = 4$ even cells and $p \neq 2, 5$, apparently $\beta_{\ell k+2} \circ (s_\ell^{\otimes k} \otimes s_2)$ is an idempotent modulo $p$ for all $k \geq 1$. So the guess might be true in this case.
- When $X$ has a combination of both even and odd cells, then the guess is true when $\ell = 2$ and $p > 2$.

Stanely's result (stable homotopy of $X$ has finite exponent if and only if $X$ is rationally trivial) can be restated as: For any infinite sequence of spaces $Y_1, Y_2, \ldots$ such that $Y_{i+1}$ is a suspension of $Y_i$,

$$lim_{i \to \infty}(exp(Y_i)) = \infty$$

if and only if the rational homology of $Y_1$ is non-trivial.

The "sufficency" part of his result might hopefully be strengthened to:

GUESS 2: For any infinite sequence of spaces $Y_1, Y_2, \ldots$ such that the mod-$p$ homology of $Y_{i+1}$ is a suspension of the mod-$p$ homology of $Y_i$, the rational homology of each $Y_i$ is non-trivial $\Rightarrow lim_{i \to \infty}(exp(Y_i)) = \infty$

# 5. THANKS!