Braids, trees, and operads

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§1 The bubbletree operad and quantum cohomology

Work on **conformal field theories** leads physicists to an interest in configuration spaces=20

$$\text{Config}^{n+1}\mathbb{C}P_1 \sim \text{Config}^{n}\mathbb{C}\ .$$

of points on the complex projective line. They are most interested=20 in the **quotients** of these spaces by the action of $\text{PGl}_2(\mathbb{C})$.=20 The points are noncoincident, so both the spaces and the group are noncompact, and taking the quotient is tricky: it leads naturally to a compactification

$$\overline{\mathcal{M}}_{0,n}(\mathbb{C}) \sim \text{Config}^{n}(\mathbb{C}P_1)/\text{PGl}_2(\mathbb{C})\ .$$

The physicists discovered a **repulsive potential** among these points: pushing two together creates a bubble onto which they escape [cf. Parker et al].=20

Thus $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ is the moduli space of marked genus zero=20 **stable** algebraic curves (which have (at worst) double points,=20 and at least three marked points on each irreducible component).=20

Example: $\overline{\mathcal{M}}_{0,4}(\mathbb{C}) \cong \mathbb{C}P_1$

via the classical cross-ratio. Note, $\overline{\mathcal{M}}_{0,3}(\mathbb{C})$ is a point: a configuration of three points on $\mathbb{C}P_1$ is **rigid**.=20

These spaces are very nice in some ways: they are compact manifolds, with cohomology concentrated in even dimension, and no torsion.=20

Operads, by example:

An operad $\mathcal{O}_* = 3D\{\mathcal{O}_k, k \geq 1\}$ is a collection of spaces together with some **composition** maps

$$\mathcal{O}_n \times \mathcal{O}_{i_1} \times \ldots \times \mathcal{O}_{i_n} \to \mathcal{O}_i$$

(where $i = 3D \sum i_k$) satisfying some axioms . . .

ex. i) $\overline{\mathcal{M}}_{0,*+1}(\mathbb{C})$

ex. ii) $\text{End}_n(X) = 3D\text{Maps}(X^n, X)$ is the **endomorphism operad** of an object X in a monoidal category. Composition is defined by=20

$$X^i = 3DX^{i_1} \times \ldots X^{i_n} \to X \times \ldots \times X \to X \ .$$

Def'n a morphism $\mathcal{O}_* \to \text{End}_*(X)$ of operads makes X into an $\mathcal{O}_*$-algebra.

ex. iii) Br_n =3D Artin's braid group on n strings defines the **braid operad**, with **cabling**=20

$$\text{Br}_n \times \text{Br}_{i_1} \times \ldots \text{Br}_{i_n} \to \text{Br}_i$$

as composition.=20

Observation Monoidal functors preserve operads; hence the homology of an operad (in spaces) is an operad in graded modules.

Theorem (WDVV, Kontsevich, ...): The (rational) homology of a smooth projective algebraic variety V is an $H_*(\overline{\mathcal{M}}_{0,*+1}(\mathbb{C}))$-operad algebra.=20

[This led to the solution of the 19th-century enumerative geometry problem of classification of lines with specified incidence in $\mathbb{C}P_2$.]

The **construction** uses Gromov-Witten invariants:

$\exists$ a (compact) moduli spaces $GW_k(V)$ of holomorphic maps from genus zero stable curves with k marked points, to V.

These spaces have many components, indexed by degree $[h : C \to V] \in H_2(V, \mathbb{Z})$. There is also an **evaluation** map

$$GW_k(V) \to \overline{\mathcal{M}}_{0,k}(\mathbb{C}) \times V^k$$

=20 which defines a cycle=20

$$GW_k \in H_*(\overline{\mathcal{M}}_{0,k}(\mathbb{C})) \otimes H_*(V)^{\otimes k}\ .$$

=20 [Actually the coefficients lie in the Novikov ring $\Lambda = 3D = 20\mathbb{Q}[H_2(V,\mathbb{Z})]$, but this will be supressed.] Using Poincaré=20 duality, we can rewrite GW_{k+1} as an element of=20

$$\mathrm{Hom}(H_*(\overline{\mathcal{M}}_{0,k+1}(\mathbb{C})), \mathrm{Hom}(H_*(V)^{\otimes k}, H_*(V)))$$

which then defines a morphism

$$H_*(\overline{\mathcal{M}}_{0,k+1}(\mathbb{C})) \to \mathrm{End}_k(H_*(V))$$

of operads, QED.=20

In particular, the point $\overline{\mathcal{M}}_{0,3}(\mathbb{C})$ defines a **quantum multiplication**

$$H_*(V,\Lambda) \otimes_\Lambda H_*(V,\Lambda) \to H_*(V,\Lambda)$$

which is usually not standard . . .

§2, the mosaic operad and Fukaya's Lagrangian cohomology

The moduli space [cf. Devadoss]

$$\mathrm{Config}^n(\mathbb{R}P_1)/\mathrm{PGl}_2(R) \sim \overline{\mathcal{M}}_{0,n}(\mathbb{R})$$

of configurations of points on the circle can be pictured as a space of **trees** or **mosaics** of hyperbolic polygons.=20

Unlike the complex case, the points have an intrinsic cyclic order, and $\{\overline{\mathcal{M}}_{0,*}(\mathbb{R})\}$ is naturally a **cyclic** operad [Getzler-Kapranov].=20

Fukaya considers a compact symplectic manifold (M, ω) together with an oriented Lagrangian submanifold L (i.e. of half the dimension of M, such that $\omega|_L = 3D0$; some subtle issues involving the Stiefel-Whitney class $w_2(L)$ will be ignored.)

I **conjecture** that the following is a theorem. Something slightly weaker (cf. below) has been proved by Fukaya and his school:

For a generic almost-complex structure compatible with ω, $\exists$ compact oriented moduli spaces FO_k of pseudo-holomorphic hyperbolic polygons=20

$$(P, \partial P) \to (M, L)$$

together with evaluation maps

$$FO_k \to \overline{\mathcal{M}}_{0,k}(\mathbb{R}) \times L^k$$

which define an action of $H_*(\overline{\mathcal{M}}_{0,*+1}(\mathbb{R}))$ on $H_*(L, \Lambda)$ (where now $\Lambda = 3D\mathbb{Q}[H_2(M, \mathbb{Z})]$).

Note, the (co)homology of these spaces is **not** known, cf. [Yoshida]. However, we can draw some pictures.=20

Grothendieck [Esquisse] says $\overline{\mathcal{M}}_{0,5}(\mathbb{C})$ is 'un petit joyaux'. Its real points $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ map to $\overline{\mathcal{M}}_{0,4}(\mathbb{R}) \times \overline{\mathcal{M}}_{0,4}(\mathbb{R}) = 3DT^2$ by selecting two distinct subsets of four points. To get the full space, we need to blow up (ie, add crosscaps) at the three configurations defined by triple coincidences, resulting in $T^2\#3\mathbb{R}P^2$.

Here is a more symmetric picture, defined by blowing up four points on $\mathbb{R}P^2$. Both pictures are **tesselated** by pentagons, though this is easier to see in the second picture.=20 This is a regular polytope with twelve pentagonal faces: it is the dodecahedron's 'evil twin'.=20

In general, there is a surjective map

$$\Sigma_k \times_{D_k} K_{k-3} \to \overline{\mathcal{M}}_{0,k}(\mathbb{R})$$

(where D_k is the dihedral group of order $2k$) which is 2^n to=20 1 on codimension n faces: in general, these moduli spaces are=20 tesselated by Stasheff **associahedra**.

There is a commutative diagram

$$\begin{array}{ccc} \text{Config}^*(\mathbb{R}) & \longrightarrow & \text{Config}^*(\mathbb{C}) \\ \downarrow & & \downarrow \\ = \overline{\mathcal{M}}_{0,*+1}(\mathbb{R}) & \longrightarrow & \overline{\mathcal{M}}_{0,*+1}(\mathbb{C}) \end{array}$$

The space in the upper right corner is homotopy-equivalent to the little disks operad, and the space in the upper left corner is the classical A_∞ operad $\{\Sigma_* \times K_{*-1}\}$ (made **permutative**, ie endowed with an action of the symmetric group. The left vertical map defines the tesselation; thus the mosaic operad is a kind of (quasicommutative) quotient of the A_∞ operad. Fukaya shows that the A_∞=20 operad acts on Floer cohomology, but I believe that action passes through this quotient. The diagram above is a fiber product of spaces,=20 but it is not quite a fiber product of operads.=20

Theorem of Davis, Januszkiewicz, and Scott: the tesselation defines a piecewise negatively curved metric on $\overline{\mathcal{M}}_{0,*}(\mathbb{R})$; these spaces are therefore $K(\pi, 1)$'s!

Remark: The quotient of the Fulton-MacPherson compactification of Config$^*(\mathbb{R}P_1)$ by the circle group $\mathbb{T}$ is also aspherical!

§3 Operads in groups (and groupoids)

Observation: π_1 of an operad is an operad in groups . . . provided you're careful about base-points.=20

Example $\{1, \ldots, n \in \mathbb{C}\}$ (with that order) defines a basepoint $* \in$ Config$^n(\mathbb{C})$; but the natural action of Σ_n moves it around (by changing the order).=20

Recall that a space has a fundamental **groupoid**, with respect to a system of basepoints: it is a category, with the points as objects, and homotopy classes of paths between them as morphisms.=20

Note also that a surjective homomorphism $\phi : G \to H$ of groups defines a groupoid $[H/G]$ with H as set of objects, and

$$\text{mor}(h_0, h_1) = 3D\{g \in G \mid \phi(g)h_0 = 3Dh_1\}$$

as morphisms. With this notation,

$$\pi(\mathsf{Config}^n(\mathbb{C}) \text{ rel } \Sigma_n(*)) \cong [\Sigma_n/\mathsf{Br}_n]$$

where $\mathsf{Br}_n \to \Sigma_n$ is the standard homomorphism. Thus the=20 fundamental groupoid of the little disks operad is the braid operad.=20

Def'n the braid **category** $\mathbf{B}$ has integers n as objects, with Br_n as its endomorphisms. [There are no morphisms between distinct integers.] This is a (universal) braided monoidal category, with tensor product $\mathbf{B} \times \mathbf{B} \to \mathbf{B}$=20 defined by juxtaposition $(n, m) \mapsto n + m$.=20

A standard construction [cf. Kassels] defines a functor

$$[\Sigma_n/\mathsf{Br}_n] \to \mathsf{Func}(\mathbf{B}^n, \mathbf{B})$$

which makes the category $\mathbf{B}$ an **algebra** over the braid operad. = More generally, any braided monoidal category is an algebra over the braid operad.=20

The operad $\overline{\mathcal{M}}_{0,*+1}(\mathbb{R})$ defines a similar category: there is an exact sequence

$$\pi_1(\overline{\mathcal{M}}_{0,*+1}(\mathbb{R})) \rightarrowtail= \pi_1(\overline{\mathcal{M}}_{0,*+1}(\mathbb{R})_{h\Sigma_*}) = 20 \twoheadrightarrow \Sigma_*$$

in which the fundamental group of the homotopy quotient plays the role of the braid group. There is a similar tensor category **D**, which is a kind of universal example of an algebra over the associated operad in groupoids.=20

Here are some questions and speculations:

i) do these fundamental groups act in some natural way on Fukaya's cohomology?=20

ii) does **D** have an interpretation in terms of cyclic=20 operads with a trace?

iii) are these fundamental groups in some sense Galois groups for solutions of Calogero-Moser systems [of points moving on the line] analogous to the role played by the braid groups in the Knizhnik-Zamolodchikov equations?

iv) Does the rank of $H_1(\overline{\mathcal{M}}_{0,k+1}(\mathbb{R}))$ equal $\binom{k}{3}$?