From modified log-Sobolev inequalities to Beckner inequalities and moment estimates

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Notation

- $(\mathcal{X}, \mathcal{B}, \mu)$ - probability space
- $\mathcal{E} : \mathrm{Dom}(\mathcal{E}) \times \mathrm{Dom}(\mathcal{E}) \longrightarrow \mathbb{R}$; $\mathrm{Dom}(\mathcal{E}) \subseteq L^0(\mathcal{X};\mu)$

$\mathcal{E}(f,g) = \mathcal{E}(g,f)$

$\mathcal{E}(f,f) \geqslant 0$

$\mathcal{E}(f_1+f_2,g) = \mathcal{E}(f_1,g) + \mathcal{E}(f_2,g)$

$\mathcal{A} \subseteq \mathrm{Dom}(\mathcal{E})$

$\mathcal{E}(f,g) = \mu(\Gamma(f,g))$ for some $\Gamma : \mathcal{A} \times \mathcal{A} \longrightarrow L_0(\mathcal{X},\mu)$

$\Gamma(f,f) = \Gamma(f)$

+ some technical assumptions on $\mathcal{E}, \Gamma$ and domains

Examples

- $\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf)$; L - generator of a reversible Markov semigroup

  $\longrightarrow \quad \mathcal{E}(f,g) = -\mu(fLg)$

- $\Gamma(f,g) = \nabla f \cdot \nabla g$

- $\Gamma(f,g) = \int (f(y) - f(x))(g(y) - g(x)) \cdot Q_x(dy)$

  $\mu(dx) Q_x(dy) = \mu(dy) Q_y(dx)$

- η - Poisson process with intensity λ

$$\begin{cases} D_x^+ F(\eta) = F(\eta + \delta_x) - F(\eta); \\ D_x^- F(\eta) = F(\eta) - F(\eta - \delta_x) \quad \text{if} \quad x \in \operatorname{supp} \eta \end{cases}$$

$$\mathcal{E}(F,G) = \int_{\mathcal{X}} \mathbb{E}\left[(D_x^+ F)(D_x^+ G)\right] \lambda(dx)$$

Mecke's formula $\Rightarrow$ $\mathcal{E}(F,G) = \mathbb{E}\,\Gamma(F,G) = \mathbb{E}\Big[\frac{1}{2}\int_{\mathcal{X}} D_x^- F(\eta) D_x^- G(\eta)\, \eta(dx)$

$$+ \frac{1}{2}\int D_x^+ F(\eta) D_x^+ G(\eta)\, \lambda(dx)\Big]$$

Functional inequalities and moment estimates

$\mathcal{E}(f,f) = \mu(\Gamma(f))$

- Poincaré : $\lambda \mathrm{Var}_\mu(f) \leq \mathcal{E}(f,f)$

$$\overset{\text{Milman}}{\Longrightarrow} \quad \| f - \mu(f) \|_r \leq r K \left\| \sqrt{\frac{\Gamma(f)}{\lambda}} \right\|_r \qquad \forall r \geq 2$$

- log-Sobolev (LS) : $g_1 \mathrm{Ent}_\mu(f^2) \leq \mathcal{E}(f,f)$

$$\overset{\text{Aida-Stroock}}{\Longrightarrow} \quad \| f - \mu(f) \|_r \leq \sqrt{r} K \left\| \sqrt{\frac{\Gamma(f)}{g_1}} \right\|_r \qquad \forall r \geq 2$$

- modified log-Sobolev (mLS) : $g_0 \mathrm{Ent}_\mu(f) \leq \mathcal{E}(f, \log f)$

$$\Longrightarrow \quad \| f - \mu(f) \|_r \leq \ ? \qquad \forall r \geq 2$$

Functional inequalities and moment estimates

$\mathcal{E}(f,f) = \mu(\Gamma(f))$

- Poincaré: $\lambda \operatorname{Var}_\mu(f) \le \mathcal{E}(f,f)$

$$\xrightarrow{\text{Milman}} \quad \| f - \mu(f) \|_r \le r K \left\| \sqrt{\frac{\Gamma(f)}{\lambda}} \right\|_r \qquad \forall r \ge 2$$

- log-Sobolev (LS): $\rho_1 \operatorname{Ent}_\mu(f^2) \le \mathcal{E}(f,f)$

$$\xrightarrow{\text{Aida-Stroock}} \quad \| f - \mu(f) \|_r \le \sqrt{r} K \left\| \sqrt{\frac{\Gamma(f)}{\rho_1}} \right\|_r \qquad \forall r \ge 2$$

- modified log-Sobolev (mLS): $\rho_0 \operatorname{Ent}_\mu(f) \le \mathcal{E}(f, \log f)$

$$\xrightarrow{\text{APS}} \quad \| f - \mu(f) \|_r \le \sqrt{r} \cdot K \left\| \sqrt{\frac{\Gamma(f)}{\rho_0}} \right\|_r \qquad \forall r \ge 2$$

Remark

Bobkov-Tetali: $4 \rho_1 \le \rho_0$;

$\mu \sim \operatorname{Poiss}(\lambda)$; $\mathcal{E}(f,g) = \sum_{n \ge 0} (f(n+1) - f(n))^2 \mu(n)$

Bobkov-Ledoux: $\rho_1 = 0$, Dai Pra et al.: $\rho_0 = 1/\lambda$

Toolbox: BBLM estimates + Beckner inequalities

- Thm. (BBLM): $\| f - \mu(f) \|_r \leq \sqrt{r} \cdot K \, \| \sqrt{\hat{\Gamma}(f)} \|_r$, where

$$\hat{\Gamma}(f,g)(x) = \int_{\mathcal{X}} (f(x) - f(y))(g(x) - g(y)) \, \mu(dy)$$

Main idea:

$$\forall p \in (1,2] \quad \mu(f^p) - \mu(f)^p \leq \mathrm{Cov}(f, f^{p-1}) = \mu(\hat{\Gamma}(f, f^{p-1}))$$

+ iterative argument

- Beckner ineq. (Bec): $\alpha_p \left[\mu(f^p) - \mu(f)^p \right] \leq \mathcal{E}(f, f^{p-1})$; $\quad p \in (1,2]$

$$\overset{\text{APS}}{\Longrightarrow} \sqrt{\left(\inf_{p \in (1,2]} \alpha_p \right)} \cdot \| f - \mu(f) \|_r \leq \sqrt{r} \cdot K \, \| \sqrt{\Gamma(f)} \|$$

Remark: (Bec) $\Longrightarrow$ (mLS) with $\varrho_0 \geq \lim_{p \to 1^+} \alpha_p$ (easy)

Thm. (APS): (mLS) $\Longrightarrow$ (Bec) with $\inf_{p \in (1,2]} \alpha_p \geq \varrho_0 / 6$

Moreover: $\lim_{p \to 1^+} \alpha_p \geq \varrho_0$.

Applications: Glauber dynamics

$\mathcal{X} = E^n$, $X = (X_1, \dots, X_n) \sim \mu$
↳ Polish space

X_i' s.t. $\mathbb{P}(X_i' \in \cdot \mid X = x) = \mathbb{P}(X_i \in \cdot \mid X_{\{i\}^c} = x_{\{i\}^c})$

$X^i = (X_1, \dots, X_{i-1}, X_i', X_{i+1}, \dots, X_n)$

$$\begin{cases} Lf(x) = \sum_{i=1}^n \mathbb{E}[f(X^i) - f(X) \mid X = x] \\ \mathcal{E}(f,g) = \sum_{i=1}^n \mathbb{E}\,(f(X^i) - f(X))(g(X^i) - g(X)) \\ \Gamma(f) = \sum_{i=1}^n \mathbb{E}[(f(X^i) - f(X))^2 \mid X] \end{cases}$$

Cor (APS): If GD satisfies (mLS) with $\rho_0 > 0$ then $\forall f : \mathcal{X} \to \mathbb{R}, \forall r \geq 2$

$$\| f(X) - \mathbb{E} f(X) \|_r \leq K \sqrt{r} \left\| \sqrt{\frac{\Gamma(f)}{\rho_0}} \right\|_r \overset{\text{Jensen}}{\leq} K \frac{\sqrt{r}}{\sqrt{\rho_0}} \left\| \left(\sum_i (f(X^i) - f(X))^2 \right)^{1/2} \right\|_r$$

⟶ e.g. Ising model, hardcore model, exponential random graphs

Applications: Poisson space $\mathcal{N}$

recall: η - Poisson process with intensity λ

$$\begin{cases} D_x^+ F(\eta) = F(\eta + \delta_x) - F(\eta) ; \\ D_x^- F(\eta) = F(\eta) - F(\eta - \delta_x) \quad \text{if} \quad x \in \operatorname{supp} \eta \end{cases}$$

$$\mathcal{E}(F, G) = \int_{\mathcal{X}} \mathbb{E}\left[(D_x^+ F)(D_x^+ G)\right] \lambda(dx)$$

$$\overset{\text{Mecke}}{=} \mathbb{E}\Big[\underbrace{\tfrac{1}{2}\int_{\mathcal{X}} D_x^- F(\eta) D_x^- G(\eta)\, \eta(dx) + \tfrac{1}{2}\int D_x^+ F(\eta) D_x^+ G(\eta)\, \lambda(dx)}_{=:\ \Gamma(F,G)}\Big]$$

Wu: η satisfies (mLS) with $s_0 = 1$ $\overset{\text{APS}}{\Longrightarrow}$ $\forall\, F: \mathcal{N} \to \mathbb{R}$ $\forall\, r \geq 2$

$$\| F - \mathbb{E}F \|_r \leq K \sqrt{r}\, \| \sqrt{\Gamma(F)} \|_r$$

↪ Complements Bachmann-Peccati for unbounded $\sqrt{\Gamma(F)}$

⟶ suprema of stochastic integrals, U-statistics

Other applications:

- Cauchy distribution in $\mathbb{R}^n$
- symmetric group: suprema of Hoeffding statistics
- measures satisfying the stochastic covering property
- zero-range processes
- abstract moment estimates for polynomials