Higher-order model checking with GFP-models

GFP model

$$\mathcal{D} = \langle \{D\}_{A \in Types}, \{\omega_A\}_{A \in Types}, \{c_\mathcal{D}\}_{c \in \Sigma} \rangle$$

- D_o is a finite lattice,
- $D_{A \to B}$ is the set of monotone functions from D_A to D_B,
- ω_A is the greatest element of D_A,
- for $c \in \Sigma$ of type A, $c_\mathcal{D} \in D_A$.

Variable assignment $\vartheta : \mathit{Vars} \to \bigcup \{D_A : A \in \mathit{Types}\}$

if $\vartheta(x^A)$ defined then $\vartheta(x^A) \in D_A$.

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Semantics

$$\begin{aligned}
[\![c]\!]^{\vartheta}_{\mathcal{D}} &= c_{\mathcal{D}} \\
[\![x]\!]^{\vartheta}_{\mathcal{D}} &= \vartheta(x) \\
[\![MN]\!]^{\vartheta}_{\mathcal{D}} &= [\![M]\!]^{\vartheta}_{\mathcal{D}}([\![N]\!]^{\vartheta}_{\mathcal{D}}) \\
[\![\lambda x^A.M]\!]^{\vartheta}_{\mathcal{D}} &= \text{the function mapping } d \in D_A \text{ to } [\![M]\!]^{\vartheta[x \mapsto d]}_{\mathcal{D}} \\
[\![Y F^A.M]\!]^{\vartheta}_{\mathcal{D}} &= \textsf{GFP} \text{ of } [\![\lambda F^A.M]\!]^{\vartheta}_{\mathcal{D}}
\end{aligned}$$

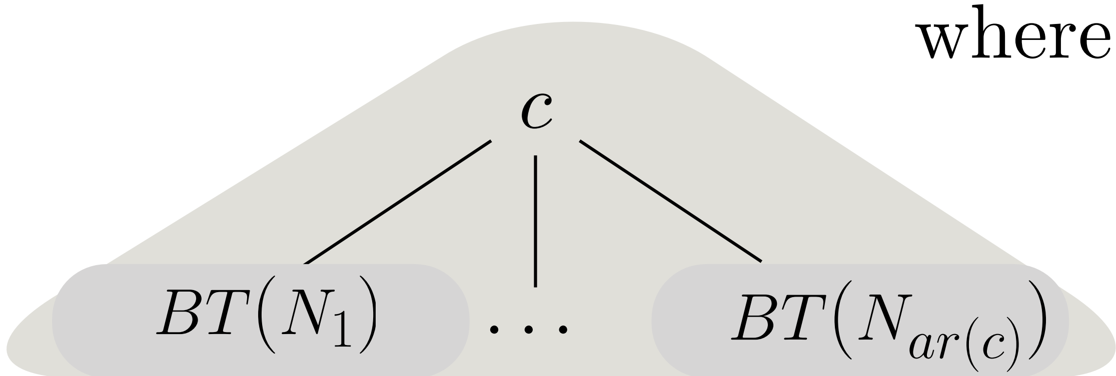
$[\![\lambda F^A.M]\!]$ is monotone.

Stable under β-reduction: $[\![\lambda x.M]\!] = [\![M[N/x]]\!]$.

Stable under δ-reduction: $[\![Y F.M]\!] = [\![M[Y F.M/F]]\!]$.

The unique result is (in some cases) a ranked tree

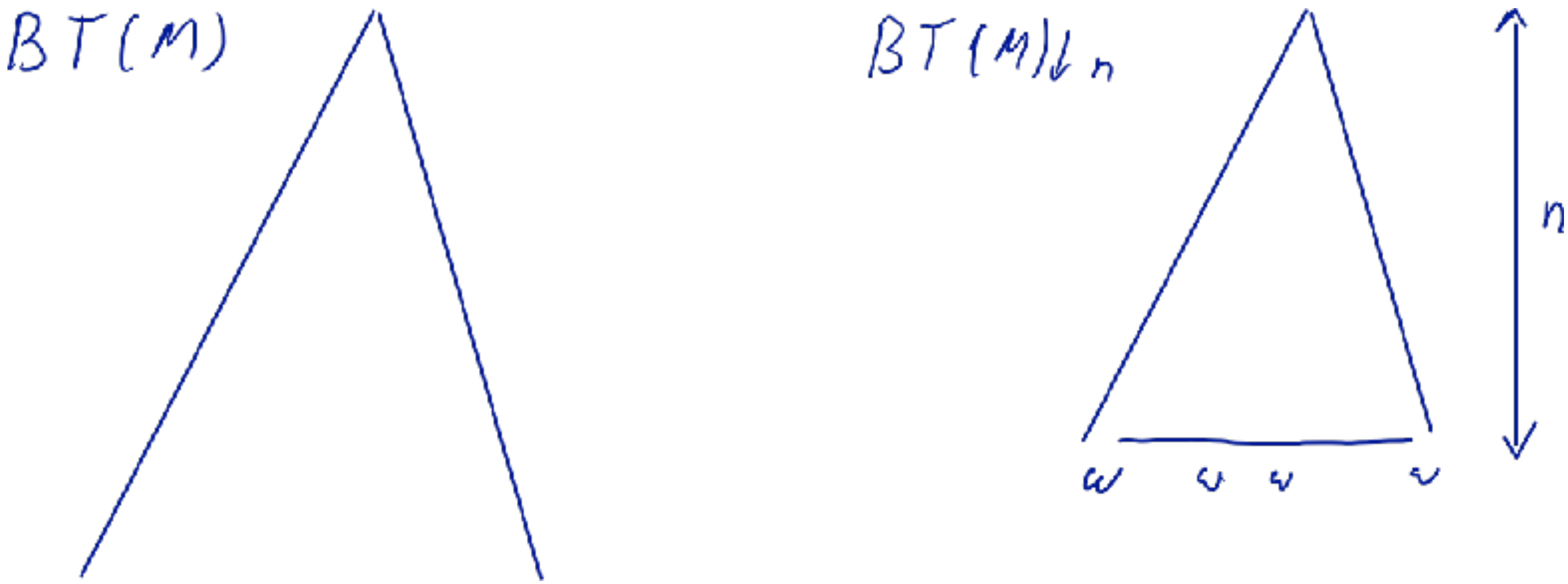
Tree signature:
All constants of have type of the form $o \to \cdots \to o \to o$, or just o.



where $M \to_\beta^* cN_1 \ldots N_{ar(c)}$

Cor: A normal form of a term $M : 0$ over a tree signature is a finite ranked tree.

$BT(M)\downharpoonleft_n$ is $BT(M)$ where nodes at level n are replaced by ω of appropriate types.



$$[\![BT(M)]\!]^{\vartheta}_{\mathcal{D}} = \bigcap \{ [\![BT(M)\downharpoonleft_n]\!]^{\vartheta}_{\mathcal{D}} : n \in \mathbb{N} \}$$

Proposition [Extended soundness]: $[\![M]\!]^{\vartheta}_{\mathcal{D}} = [\![BT(M)]\!]^{\vartheta}_{\mathcal{D}}$

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The language recognized by $F \subseteq D_o$ is the set of closed λY-terms M of type o with $[\![M]\!]_{\mathcal{D}} \in F$.

TAC automaton (trivial accepting conditions)

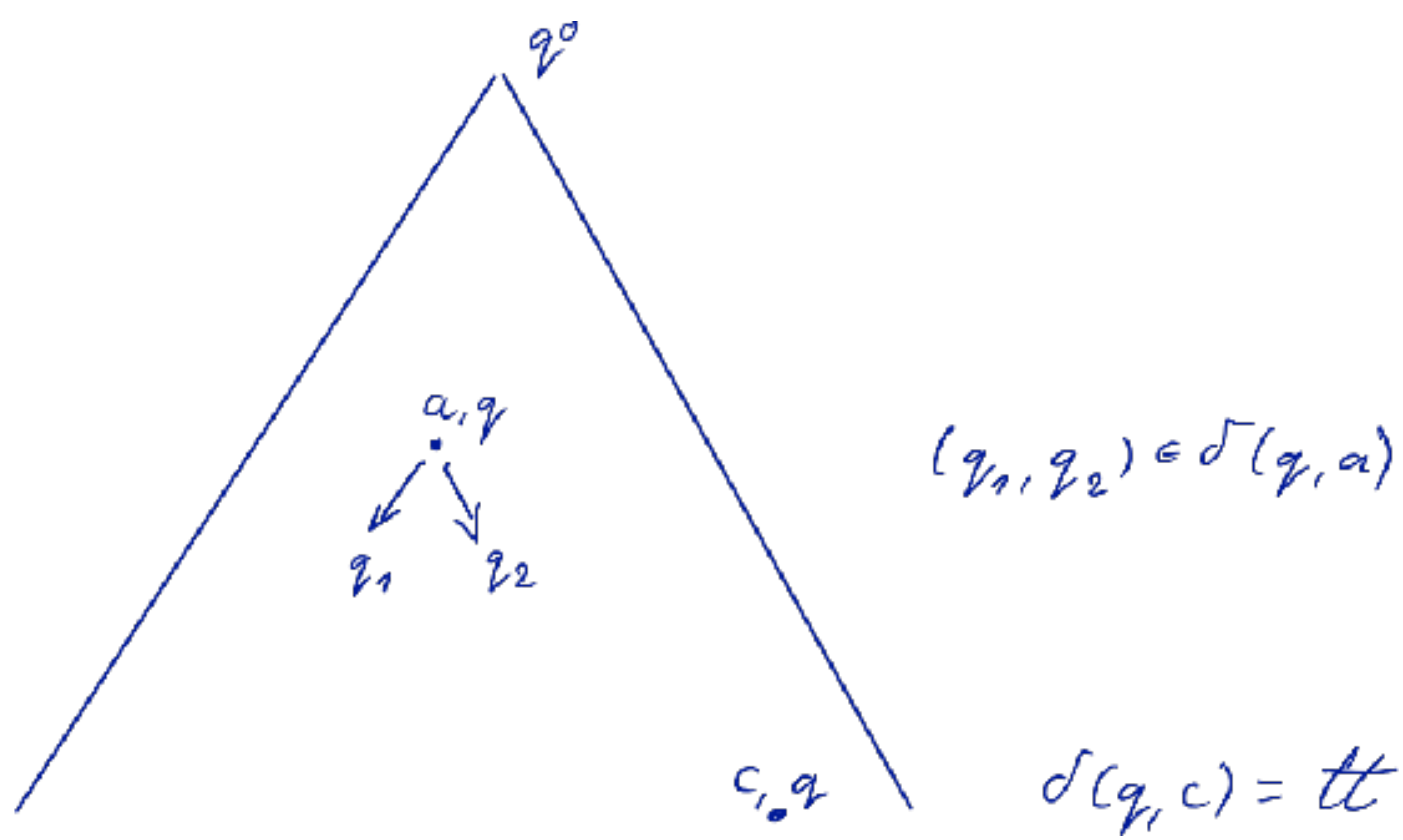
Signature $\Sigma = \Sigma_0 \cup \Sigma_2$.

$$\mathcal{A} = \langle Q, \Sigma, q^0, \delta_0, \delta_2 \rangle$$

$$\delta_0 : Q \times (\Sigma_0 \cup \{\omega\}) \to \{\mathit{ff}, \mathit{tt}\}$$

$$\delta_2 : Q \times \Sigma_2 \to \mathcal{P}(Q \times Q)$$

ω**-blind condition:** $\delta_0(q, \omega) = tt$ for all $q \in Q$.



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Def. $L(\mathcal{A}) = \{M : M$ closed, of type o, $BT(M)$ accepted by $\mathcal{A}\}$.

Prop. $L(\mathcal{A})$ is recognized by a finitary `GFP` model.

$$BT(M) \text{ accepted by } \mathcal{A} \quad \text{iff} \quad [\![M]\!]_{\mathcal{D}_{\mathcal{A}}} \in F_{\mathcal{A}}.$$

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$$D_o = \mathcal{P}(Q)$$

For $c \in \Sigma_0$: $\quad c_D = \{q : \delta(q, c) = tt\}$

For $a \in \Sigma_2$: $\quad a_D(S_1, S_2) = \{q : \delta_2(q, a) \cap (S_1 \times S_2) \neq \emptyset\}$

$$F = \{S : q^0 \in S\}$$

$$BT(M) \text{ accepted by } \mathcal{A} \quad \text{iff} \quad [\![M]\!]_{\mathcal{D}} \in F.$$

From models to automata

Fix a model $\mathcal{D} = \langle \{D_A\}, \{w_A\}, \{c_\mathcal{D}\}_{c\in\Sigma}\rangle$

Take $Q = D_o$.

Consider automaton $\mathcal{A}(q) = \langle Q, \Sigma, q, \delta_0, \delta_2\rangle$

$$\begin{aligned} \delta_0(q,c) &= tt \qquad \text{if } q \leq c_\mathcal{D} \\ \delta_2(q,a) &= \{(q_1,q_2) : q \leq a_\mathcal{D}(q_1,q_2)\} \end{aligned}$$

Lemma: For every closed term M of type o:

$$BT(M) \in \mathcal{A}_q \text{ iff } q \leq [\![M]\!]_\mathcal{D}.$$

Thm [Salvati, W.]

Let L be a set of closed λY-terms of type o.

> L is recognized by a finitary GFP model iff L is a Boolean combination of languages of ω-blind TAC automata.

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Cor:

For M, N closed λY-terms of type o over tree signature:

$$BT(M) = BT(N) \quad \text{iff} \quad [\![M]\!]_{\mathcal{D}} = [\![N]\!]_{\mathcal{D}} \text{ in all finitary GFP models.}$$

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Thm [Statman]

For every closed λ-term M there is a finitary model $\mathcal{D}$ s.t.
for every term N:

$$BT(M) = BT(N) \qquad \text{iff} \qquad [\![M]\!]_{\mathcal{D}} = [\![N]\!]_{D}.$$

Thm [Loader]

Let $\mathcal{D}$ be a nontrivial standard model.

The following problem is undecidable:

$$\text{given } d \in \mathcal{D} \text{ is there a } \lambda\text{-term } M \text{ with } [\![M]\!]_{\mathcal{D}} = d.$$