

Single use register automata for data words

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Abstract

We introduce a new automaton model for data words, called *single use register automata*. These are like register automata for data words (introduced by Kaminski and Francez), with the restriction that every read access to a register destroys the register contents. We prove that the following models are equivalent: (a) one-way deterministic single use register automata; (b) two-way deterministic single use register automata; (c) languages recognised by orbit-finite semigroups.

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1 Introduction

The literature on automata for infinite alphabets is full of depressing diagrams like [11, Figure 1] or [3, Figure 1.1], which describe countless models that satisfy only the trivial relationships (deterministic \subseteq nondeterministic, one-way \subseteq two-way, etc.).

One narrative explaining this sad state of affairs is proposed in [5] and developed in the lecture notes [3]. The narrative is that automata for infinite alphabets are “finite automata” when the word “finite” is interpreted to mean “orbit-finite”, which roughly means “finite up to permutations of letters”. Orbit-finite sets have some closure properties of the finite sets, e.g. products, subsets, unions, and intersections. These closure properties are enough for some constructions to hold (see the equivalence of pushdown automata and context-free grammars [3, Section 5.3]). However other closure properties fail – orbit-finite sets are neither closed under taking powersets, nor under taking function spaces ($X \rightarrow Y$). This means that some natural constructions in automata theory fail: determinisation (uses powerset), converting a deterministic automaton into a semigroup (uses function spaces), or converting a two-way automaton into a one-way automaton (uses powerset or function spaces, depending on whether one uses the construction of Rabin and Scott [13] or of Shepherdson [14]). Orbit-finite sets additionally do not allow choice functions which, for example, implies that orbit-finite Turing machines do not determinise [6].

In this paper, we study the connection between deterministic automata and semigroups in their infinite-alphabets version. We use the model of deterministic register automata, introduced by Kaminski and Francez [10], which is equivalent to orbit-finite deterministic automata [3, Theorem 6.5], but has a more concrete syntax which makes it easier to understand. We also consider the notion of orbit-finite semigroups that comes from [2]. As usual, each such semigroup yields an automaton, but the converse translation fails, because



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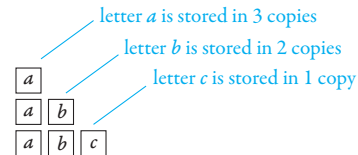
of the problems with function spaces:

► **Example 1.** Let \mathbb{A} be the set of data values. Consider the language $L \subseteq \mathbb{A}^+$ of words where the first letter appears again later in the word [2, Section 10]. This language is recognised by a deterministic register automaton, which loads the first letter into a register, and then compares this register to all subsequent positions. The language L is, however, not recognised by any homomorphism $h : \mathbb{A}^+ \rightarrow S$ with S orbit-finite. The intuitive reason for this is that $h(w)$ would need to remember all the letters that appear in w , because w is typically part of a bigger word where the first letter is unknown.

As witnessed by the above example, not every deterministic automaton can be converted into a semigroup. The purpose of this paper is to explain which deterministic register automata *can be* converted into orbit-finite semigroups. Our explanation is based on the following *single use restriction*¹: every question asked about a register has the side-effect of erasing that register (see Section 2 for a more precise definition). The automaton in Example 1 violates the single use restriction by comparing the register storing the first letter to all input letters. Our main contribution is Theorem 11, which says that single use register automata recognise exactly the same languages as orbit-finite semigroups. Furthermore one-way and two-way automata with the single use restriction have the same expressive power (the same as orbit-finite semigroups), which shows the remarkable robustness of the single use restriction.

► **Example 2.** Here is an example of a language where a single use automaton exists, but requires extra ideas. Consider the language “there are at most three distinct letters in the input word, not counting repetitions”. There is a natural (not single use) register automaton which recognises this language: use three registers to store the distinct atoms that have been seen so far, and if a fourth atom comes up, then reject.

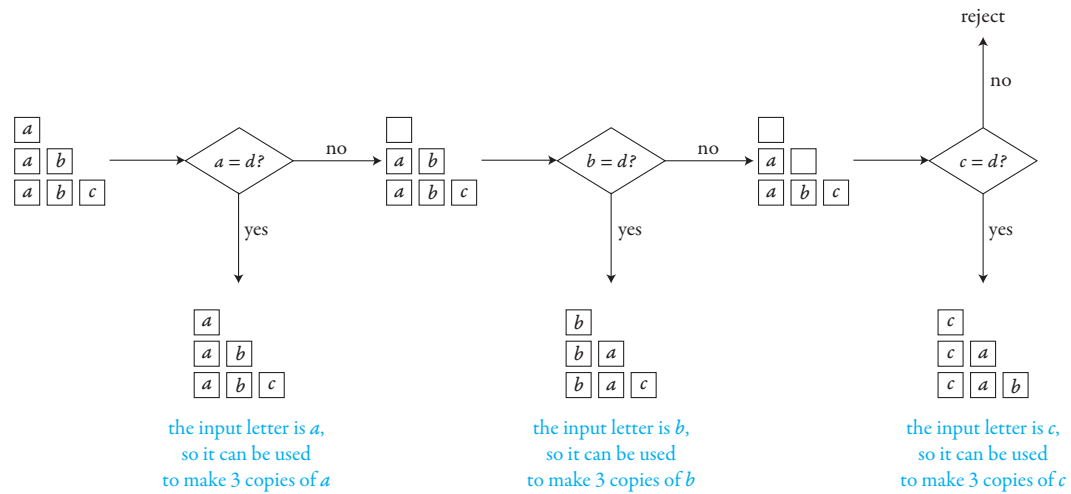
This automaton, however, violates the single use restriction, because each new input letter is compared to all the registers. Here is a solution, which conforms to the single use restriction. The general idea is that once the automaton has seen three distinct letters a, b, c , it stores them in six registers as explained in Figure 1. Assume that a new input letter d is read. The behaviour of the automaton (when it already has three atoms in the registers) is explained in the flowchart in Figure 2. A similar flowchart is used for the corner cases when the automaton has seen less than three letters so far. This construction generalises to any language of the form “at most n distinct letters”.



■ **Figure 1** Six registers.

Related work. Apart from the original paper on orbit-finite semigroups [2], the work most closely related to this paper is the result by Colcombet, Ley and Puppis [7], which says that orbit-finite semigroups have the same expressive power as *rigidly guarded* MSO^\sim . Rigidly guarded MSO^\sim is a subset of monadic second-order logic on data words, with a certain syntactic restriction on the equality tests, which ensures that its expressive power matches that of orbit-finite semigroups. Combining the main result of [7] with the main result from this paper, we see that all of the following formalisms are equivalent: orbit-finite

¹ This terminology is based on the transducer literature, see e.g. [1].



■ **Figure 2** Updating the six registers.

semigroups, rigidly guarded MSO^{\sim} , one-way single use automata, and two-way single use automata. Our proof uses techniques from [7], but at the first glance, the results seem to be incomparable. We do not see a direct translation (in either of the directions) between rigidly guarded MSO^{\sim} and single use register automata, other than passing through an orbit-finite semigroup. The classical approach of translating an MSO formula to an automaton fails in the projection step $(\exists_X.\varphi)$, because it uses the powerset construction which preserves neither orbit-finiteness nor the single use restriction. To preserve orbit-finiteness, the authors of [7] introduce a *projectability* condition. Our attempts at adapting the projectability condition to single use automata were unsuccessful, because the projection operation would lead to register automata that are orbit-finite but not single use. We also did not see any direct constructions for the opposite direction, translating single use register automata to rigidly guarded MSO^{\sim} , because the guards cannot use all the free variables from their context. In the future, we would like to investigate in more detail the exact relationship between single use automata and rigidly guarded MSO^{\sim} .

2 Register automata and the single use restriction

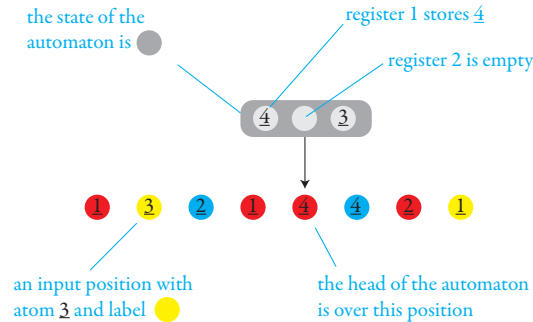
Register automata, as an automaton model for recognising languages of data words, were introduced by Kaminski and Francez in [10], under the name of *finite memory automata*. In this paper, we use the name *register automata*, following terminology from [11]. We are interested in register automata that are deterministic, but possibly two-way.

Register automata without the single use restriction. We begin by recalling the usual notion of (deterministic) register automata for data words. Later in this section we describe the single use restriction, which leads to the model that is introduced in this paper.

For the rest of this paper fix a countably infinite set \mathbb{A} , whose elements are called *atoms*. The idea is that atoms can only be compared with respect to equality. The input to a register automaton is a *data word*, which is a finite word where every position stores a pair: (label from a finite alphabet, atom). The automaton has a head which moves around the input word and a finite set of registers, which are used to store and compare atoms from the input word.

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Here is a picture of a configuration of the automaton:



A more formal definition follows below.

► **Definition 3.** A (deterministic) two-way register automaton consists of:

- a finite set Σ of labels;
- a finite set Q of states with an initial state $q_0 \in Q$;
- a finite set R of register names;
- a finite set C of atoms, called the constants;
- a transition function which maps each state $q \in Q$ to an element of:

$$\underbrace{\text{questions}}_{\text{question that is asked}} \times \underbrace{(Q \times \text{actions})}_{\text{what to do if the question has a yes answer}} \times \underbrace{(Q \times \text{actions})}_{\text{what to do if the question has a no answer}}$$

where the allowed questions and actions are defined as follows:

- Questions. Each question has a yes/no answer:
 1. Does the input position under the head have label $a \in \Sigma$?
 2. Is the head on the first/last input position?
 3. Does the input position under the head store the same atom as register $r \in R$?
 4. Do registers $r, s \in R$ store the same atom?
- Actions.
 1. Move the head to the previous/next position.
 2. Accept/reject.
 3. Move the atom stored in the position under the head into register $r \in R$.
 4. Erase the contents of register $r \in R$.
 5. Move constant $a \in C$ into register $r \in R$.

In most cases we are interested in automata without constants. Such automata will be called *equivariant*. The language recognised by a two-way register automaton is defined as follows. For an input data word with labels from Σ , a *configuration* of the automaton is a triple (position of the input word, state, register valuation), where a register valuation is a partial function from registers to atoms. The automaton begins in the *initial configuration*, where the head is over the first input position, the state is q_0 , and all registers are undefined. In each configuration, the transition function is applied in the natural way, yielding a new successor configuration. If the automaton executes an *accept* action, then the input is accepted. Otherwise, the input is rejected. The input can be rejected for the following reasons: (a) the head leaves the input word, by moving left on the first position or moving right on the last position; or (b) the computation enters an infinite loop; or (c) a reject action is performed. An automaton is called one-way if it never uses the action “move the head to the previous position”. Every one-way automaton can be easily modified so that (a) and (b) never happen, and a two-way automaton can be modified so that (a) never happens.

Two-way automata are more expressive than one-way automata [11, Figure 1]. For example a two-way automaton can recognise the language “all letters are distinct”, which is not recognised by any one-way automaton.

Single use register automata. We now define *single use register automata*, which are the main topic of this paper. To the best of our best knowledge, this is a new model. The syntax of a (both one-way and two-way) single use register automaton is the same as the syntax for a register automaton as in Definition 3, only the semantics are changed so that each question that asks about the value of a register has the side effect of erasing the contents of that register. More precisely, the side effects of questions

3. Does the input position under the head store the same atom as register $r \in R$?

4. Do registers $r, s \in R$ store the same atom?

are: erase the content of the register r and erase the contents of the registers r and s , respectively. One could also view the single use restriction as a syntactic fragment of the register automata from Definition 3, where every question about the register contents is necessarily followed by an erasing action. In this paper, we only consider deterministic automata².

3 Orbit-finite semigroups

In this section, we introduce another model for recognising properties of data words, namely orbit-finite semigroups. These were introduced in [2] and later studied in [7]. The papers [2, 7] use monoids instead of semigroups, but the difference plays no role in the theory.

3.1 Atoms and orbit-finiteness

An orbit-finite semigroup is a semigroup which happens to be an orbit-finite set with atoms. This is why we start this section by defining orbit-finite sets with atoms. Since the usual definition [3, Section 3] can take a lot of space, we present below a more concise alternative, namely the sets of tuples of atoms modulo definable partial equivalence relations.

A set $X \subseteq \mathbb{A}^k$ is called *definable* if there is a formula of first-order logic with k free variables which is true in exactly the tuples from X . The formula is allowed to use equality and constants from \mathbb{A} . For example, the formula $x_1 = x_2 \wedge x_1 \neq \underline{2}$ defines the diagonal of \mathbb{A}^2 , minus the pair $(\underline{2}, \underline{2})$. Since \mathbb{A} has quantifier elimination, we can assume that all first-order formulas are quantifier-free.

A *partial equivalence relation* is very similar to an equivalence relation, except that not every element needs to belong to an equivalence class – it is a binary relation that is transitive and symmetric, but not necessarily reflexive. A partial equivalence relation \sim on \mathbb{A}^k is called *definable* if it can be defined using a formula with $2k$ variables. The set of equivalence classes is denoted by \mathbb{A}^k / \sim .

► **Definition 4** (Orbit-finite sets). *An orbit-finite set is any finite disjoint union*

$$\mathbb{A}^{k_1} / \sim_1 + \dots + \mathbb{A}^{k_n} / \sim_n$$

² One could also think about nondeterministic register automata with the single use restriction. This model strictly extends deterministic single use automata (it can recognise the language in Example 1), and is incomparable in expressive power with deterministic (not single use) automata (it can recognise the language “some two letters are equal”, but not the complement of the language in Example 1). Finally, the single use restriction does not seem to offer any particular advantage for nondeterministic automata: emptiness remains decidable, and universality remains undecidable.

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for $k_1, \dots, k_n \in \{0, 1, 2, \dots\}$ and definable partial equivalence relations \sim_1, \dots, \sim_n .

In fact, the disjoint union is not needed, i.e. one can take $n = 1$, because the disjoint union can be simulated using higher dimension tuples. Nevertheless, we use the disjoint union because it makes certain constructions cleaner. In particular, it will be useful in the definition of straight sets introduced later in this section. As shown in [3, Theorem 3.7] Definition 4 is equivalent to the usual definition of orbit-finite sets (up to finitely supported isomorphisms). Orbit-finite sets are easily seen to be closed under sums, intersections, and Cartesian products.

Supports. Let π be a permutation of the atoms, i.e. a bijection $\mathbb{A} \rightarrow \mathbb{A}$. A permutation can be applied to tuples of atoms, elements of orbit-finite sets, and subsets of orbit-finite sets in the natural way. Every orbit-finite set is contained in a finite union of orbits under the action of atom permutations, which explains the name. One of the key notions for sets built out of atoms is the notion of support. Let x be an object to which one can apply permutations of atoms (e.g. x can be an atom, an element of an orbit-finite set, or a subset of an orbit-finite set). We say that x is supported by a tuple of atoms \bar{a} if

$$\pi(\bar{a}) = \bar{a} \quad \text{implies} \quad \pi(x) = x \quad \text{holds for every permutation of the atoms } \pi.$$

We say that x is *equivariant* if it is supported by the empty tuple of atoms, and that x is *finitely supported* if it is supported by some finite tuple of atoms. An example of an equivariant set is \mathbb{A}^2 , and an example of a finitely supported but not equivariant orbit-finite set is the set $\mathbb{A} - \{\underline{1}\}$, where $\underline{1}$ is a fixed atom. A key result about supports is that every finitely supported object admits a *least support*, which is a finite tuple of atoms that is contained in every other finite support [9, Proposition 3.4], see also [3, Theorem 6.1]. Orbit-finite sets are closed under taking finitely supported subsets [3, Lemma 3.8].

Failure of choice. One of the difficulties with orbit-finite sets is that choice axioms can fail, as illustrated by the following example.

► **Example 5.** The surjective function $(a, b) \mapsto \{a, b\}$ which maps an ordered pair of atoms to the corresponding unordered pair has no finitely supported one-sided inverse ([3, Example 9]). This means there is no finitely supported way of choosing an ordering for each unordered pair of atoms.

Straight sets and choice. The difficulties with choice illustrated in Example 5 stem from symmetries like $\{a, b\} = \{b, a\}$ which are created by quotienting tuples with respect to equivalence relations. To avoid such difficulties, we consider straight sets: an orbit-finite set is called *straight* if all of the equivalence relations \sim_i are partial identities (i.e. each equivalence class has one element), in which case each summand is simply a definable subset of \mathbb{A}^{k_i} . We adopt the convention that straight sets are written in red, like \mathbf{X} . The following lemma says that orbit-finite sets can be represented via straight sets, and that every finitely-supported function can be lifted to work with such representations.

► **Lemma 6.** *To every orbit-finite set X one can associate a straight set \mathbf{X} and a finitely supported function $r_X : \mathbf{X} \rightarrow X$, called its straight representation, so that the following hold.*

1. *for every $x \in \mathbf{X}$, x and the represented element $r_X(x)$ have equal least supports;*
2. *every finitely supported function $f : X \rightarrow Y$ between orbit-finite sets can be lifted to a*

finitely supported function $f : X \rightarrow Y$ which makes the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ r_X \downarrow & & \downarrow r_Y \\ X & \xrightarrow{f} & Y \end{array}$$

Proof. The first item is a corollary of the representation result in [3, Theorem 6.3], which says that for every orbit-finite set X there is a finitely supported function $r_X : X \rightarrow X$ from some straight set, which does not change least supports. The second item about lifting functions is then a corollary of [3, Claim 6.9]. ◀

One of the benefits of straight sets is that some problems with choice can be avoided:

► **Lemma 7 (Uniformisation).** *Let $R \subseteq X \times Y$ be a finitely supported binary relation on straight sets, such that for every $x \in X$ the set $xR = \{y : R(x, y)\}$ is nonempty. Then R can be uniformised, i.e. there is a finitely supported function $f : X \rightarrow Y$ such that all $x \in X$ satisfy $f(x) \in xR$.*

Proof. Let \bar{a} be the least support of R . Choose³ a tuple \bar{c} of atoms which contains \bar{a} plus $2n$ fresh atoms, where n is the maximal size of a least support in Y .

▷ **Claim 8.** For every x there is some $y \in xR$ such that every atom from y appears in x or \bar{c} .

Proof. The function $x \mapsto xR$ is supported by \bar{a} , it follows that the set xR is supported by the tuple $\bar{a}x$. Choose some $y \in xR$. Because xR is supported by $\bar{a}x$, any atom from y that does not appear in $\bar{a}x$ can be replaced by any other atom not appearing in $\bar{a}x$, at least as long as the equality type is preserved. Since \bar{c} was chosen to be big enough, we can assume without loss of generality that every atom in y appears either in x or in \bar{c} . ◀

For $x \in X$, consider the following order on xR : tuples are ordered lexicographically, with each coordinate ordered by its first appearance in the tuple $\bar{a}x$. The function which maps x to the least element from xR according to the above ordering is easily seen to be supported by \bar{a} , and it is the uniformisation required by the claim. ◀

3.2 Semigroups

Semigroups. Recall that a semigroup is a set S equipped with an associative product operation. An *orbit-finite semigroup* is a semigroup where the underlying set is an orbit-finite set with atoms, and where the product operation is finitely supported. If Σ is an orbit-finite set with atoms, then a language $L \subseteq \Sigma^+$ is *recognised by an orbit-finite semigroup* if there is a finitely supported semigroup homomorphism into an orbit-finite semigroup $h : \Sigma^+ \rightarrow S$, and a finitely supported accepting set $F \subseteq S$ such that $L = h^{-1}(F)$.

► **Example 9.** Consider the language “at most three distinct letters” that was studied in Example 2. To recognise this language, we use the following orbit-finite semigroup. The underlying set consists of a distinguished error element 0, together with all nonempty sets of atoms that have size at most three. The underlying set has four orbits: the error element,

³ The choice of \bar{c} makes the construction in the lemma non-equivariant in the following sense: the function f might not be equivariant even if R is. This non-equivariance could be avoided by assuming that each set xR is finite, but we will use the lemma in situations where xR is infinite.

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sets of size one, sets of size two, and sets of size three. The semigroup operation is set union, with 0 being produced if at least one of the arguments is 0 or if the resulting set contains more than three elements. The homomorphism h is defined by

$$w \in \mathbb{A}^+ \mapsto \begin{cases} \text{set of letters in } w & \text{if } w \text{ contains at most three distinct letters} \\ 0 & \text{otherwise} \end{cases}$$

This homomorphism recognises the language in question – the accepting set is the entire semigroup except 0.

Without atoms, every deterministic finite automaton can be converted into a semigroup, by considering the semigroup of state transformations $Q \rightarrow Q$. This construction does not work in the orbit-finite setting, because orbit-finite sets are not closed under taking function spaces. For this reason, orbit-finite semigroups have strictly smaller expressive power than register automata:

► **Example 10.** Assume that the input alphabet is \mathbb{A} , and consider the language “the first letter appears in some other position of the word”. This language is clearly recognised by a (multiple-use) register automaton, which stores the first letter in a register and then compares it to all other letters in the word. This language, however, is not recognised by any orbit-finite semigroup: if there existed a homomorphism h recognising this language, $h(w)$ would have to store all the atoms that appear in w , and this cannot be done in an orbit-finite way. (A more formal argument uses the notion of syntactic semigroup [2, Section 3.2]).

4 Equivalence

The main result of this paper is that orbit-finite semigroups are equivalent to (deterministic) single use automata.

► **Theorem 11.** *For every language of data words L , the following conditions are equivalent:*

1. L is recognised by a one-way deterministic single use automaton;
2. L is recognised by a two-way deterministic single use automaton;
3. L is recognised by an orbit-finite semigroup.

The implication $1 \Rightarrow 2$ is immediate. The remaining implications are proved in the next two sections.

Spurious constants. In our definition of single use automata, we allow the use of atom constants, which are not subject to the single use discipline. This feature is used in two ways: (a) the automata can recognise languages that are not necessarily equivariant; (b) in the translation from semigroups to automata, we will introduce constants by applying Lemma 7. As a consequence of item (b), when converting a semigroup into a one-way automaton, the resulting automaton will have atom constants even if the semigroup was equivariant. The following lemma shows that these spurious constants can be eliminated.

► **Lemma 12.** *If a language L is supported by \bar{a} and recognised by a single use automaton, then it is also recognised by a single use automaton which only uses constants from \bar{a} .*

Proof. First, let us prove the theorem for any equivariant L . Take any \mathcal{A} that recognises L . For any atom permutation π , define $\pi(\mathcal{A})$ to be equal to \mathcal{A} , with π applied to its constant set. First, note that if an automaton \mathcal{A} recognises the language L , then $\pi(\mathcal{A})$ recognises $\pi(L)$, which is equal to L , because L is equivariant. For every word w , we can find such

π_w , that $\pi_w(A)$ doesn't contain any atom from w (this is because w only has finitely many atoms). This means that for every w , the automaton $\pi_w(\mathcal{A})$ will answer “no” when checking the equality of a constant with some atom other than itself. Because of that we can construct a constant free automaton \mathcal{A}' , that for every w simulates $\pi_w(\mathcal{A})$: Instead of atoms from C \mathcal{A}' uses atom free (and therefore finite-state compatible) set of $|C|$ dummy constants, that always return “no” when compared with real atoms.

If the language L is finitely supported (say by \bar{a}), but not equivariant, we can use the same technique to eliminate all the non \bar{a} constants from any automaton that recognises L . ◀

A corollary of the above lemma and Theorem 11 is that for equivariant languages, orbit-finite semigroups have the same expressive power as single use automata (one-way or two-way) with no constants.

5 From two-way automata to semigroups

In this section we show that every language recognised by a two-way single use automaton is also recognised by an orbit-finite semigroup. To illustrate the central issue, consider first the situation where we want to convert a (not necessarily single use) *one-way* automaton into a semigroup. The natural construction – corresponding to what is done for finite alphabets – is to use the function space $X \rightarrow X$, where X is the set of pairs (state, register valuation). The problem is that even the subset of the function space that corresponds to actual input words might not be orbit-finite. The following lemma explains when the resulting function space is orbit-finite.

► **Lemma 13.** *Let X, Y be orbit-finite sets, and let $F \subseteq X \rightarrow Y$ be finitely supported, and such that every $f \in F$ is finitely supported. Then F is orbit-finite if and only if there is a $k \in \mathbb{N}$ such that all $f \in F$ have least support of size at most k .*

Proof. For the left-to-right implication, we observe that if a function $f \in F$ has support of size k , then the same is true for all functions in its orbit. Therefore, if F is orbit-finite and every $f \in F$ has finite least support, then the size of least supports in F is bounded.

Consider now the right-to-left implication. Choose some tuple $\bar{a} \in \mathbb{A}^k$. There are finitely many functions $f : X \rightarrow Y$ that are supported by \bar{a} , because the graph of each such function is a union of \bar{a} -orbits of $X \times Y$, and there are finitely many such orbits. Every function $X \rightarrow Y$ with support of size at most k can be obtained by taking a function $f : X \rightarrow Y$ supported by \bar{a} , and applying an atom automorphism. The result follows. ◀

Our proof strategy is to show, that thanks to the single use restriction, we can transform any two-way single use automaton into a semigroup, using a standard construction inspired by [14] without losing orbit-finiteness. Fix a two-way single use automaton \mathcal{A} . Let Q be the set of states and let n be the number of registers. We consider a data word w , as an infix of some unknown larger input word. Define the *Shepherdson function* δ_w :

$$\underbrace{Q \times (\mathbb{A} + \perp)^n}_{\text{state and register valuation at the start of the run}} \times \underbrace{\{\leftarrow, \rightarrow\}}_{\text{does the run enter from the left or right}} \times \underbrace{2^{\{\leftarrow, \rightarrow\}}}_{\text{does } w \text{ contain the first/last positions of the input word}} \rightarrow \{\checkmark, \mathbf{X}\} \cup \underbrace{(Q \times (\mathbb{A} + \perp)^n)}_{\text{state and register valuation at the end of the run}} \times \underbrace{\{\leftarrow, \rightarrow\}}_{\text{does the run exit from the left or right}}$$

to be the function that describes runs of \mathcal{A} in the usual way (see [14, Proof of Theorem 2]). The run is taken until \mathcal{A} either exists w , accepts/rejects, or loops forever. Whenever \mathcal{A} asks if a position is first/last in the word, it gets answered according to the third argument.

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The value \mathbf{X} represents runs which either use a “reject” action, or enter an infinite loop. By the same reasoning as in [14], the Shepherdson functions are compositional: one can equip the set $S = \{\delta_w : w\}$ with a semigroup structure so that $w \mapsto \delta_w$ becomes a semigroup homomorphism, i.e. $(\delta_w) \cdot (\delta_v) = \delta_{wv}$. It is easy to see that the acceptance of a word w depends only on (a finitely supported property of) δ_w , and therefore the language recognised by the automaton is also recognised by the homomorphism $w \mapsto \delta_w$. It remains to show that the semigroup S used by that homomorphism is orbit-finite: The semigroup S consists of functions from one orbit-finite set to another, and each of these functions is finitely supported (δ_w is supported by the atoms that appear in w and in the automaton). Orbit-finiteness of S (and the proof of $2 \Leftarrow 3$ of Theorem 11) will follow, therefore, from combining the following Lemma 13 with the following:

► **Lemma 14.** *There is some $k \in \mathbb{N}$ such that every Shepherdson function δ_w is supported by at most k atoms.*

Proof. For the purpose of this proof, we view a run as a sequence of transitions, where each transition indicates: source state, question, action, and the target state. Note, that the set of transitions is finite. Fix a state $q \in Q$ and $d \in \{\leftarrow, \rightarrow\}$. For a valuation \bar{a} of the registers, define $\rho_{\bar{a}}$ to be the sequence of transitions – possibly infinite – that are executed by the automaton, assuming that it enters the word from direction d , in state q and with register valuation \bar{a} . A transition used in a run $\rho_{\bar{a}}$ is called *important* if it asks a question about a register which had not been overwritten since the beginning of the run. For $i \in \{0, 1, \dots\}$ define $\rho_{\bar{a}}^i$ to be the prefix of $\rho_{\bar{a}}$ that ends right before the $(i + 1)$ -st important transition; if $\rho_{\bar{a}}$ has $\leq i$ important transitions, then $\rho_{\bar{a}}^i$ is the entire run $\rho_{\bar{a}}$.

▷ **Claim 15.** For every $i \in \{0, 1, \dots\}$ the function $\bar{a} \mapsto \rho_{\bar{a}}^i$ is supported by at most 2^i atoms.

Proof. We first observe that $\rho_{\bar{a}}^0$ does not check any of the registers, and therefore it does not depend on the tuple \bar{a} . In other words, the function $\bar{a} \mapsto \rho_{\bar{a}}^0$ is constant, and therefore has empty support. Consider now $\rho_{\bar{a}}^1$. This run depends on the answer to the question in the first important transition, which might compare some register to some atom c in the input word (note that c does not depend on \bar{a} , since $\rho_{\bar{a}}^0$ does not depend on \bar{a}). Therefore, after the first important transition, the run splits into at most two possible values of $\rho_{\bar{a}}^1$. Continuing this analysis, we see that there are at most four possible values for $\rho_{\bar{a}}^2$, and so on... ◀

The number of important transitions in a run is at most the number of registers n , since each question overwrites the contents of a register. It follows therefore from the above claim that the function $\bar{a} \mapsto \rho_{\bar{a}}$ has support of size $< 2^{n+1}$. The lemma follows, since there are finitely many choices for the state q and the direction d and each of the the runs $\rho_{\bar{a}}$ can extract at most n atoms from the word w to the final state. ◀

Note that Lemma 14 is false without the single use restriction. For example, if we take an automaton which accepts if the input words containing the atom in register 1, then the support of δ_w will contain all the atoms that appear in w .

6 From semigroups to one-way automata

In this section we prove the implication $3 \Rightarrow 1$ in Theorem 11, i.e. that every orbit-finite semigroup can be converted into a one-way single use automaton.

6.1 Transducers

In the interest of modularity, instead of constructing a big automaton, we define a chain of transducers (automata which output words). Each transducer inputs the output of the previous transducer in the chain, and adds more information to it, so that their composition computes the value of the semigroup homomorphism.

Transducers. We begin by defining the transducer model we will use in the proof. One-way single use transducers are essentially the same as one-way single use automata, with two differences: (1) the input alphabet is an arbitrary straight set, i.e. a single input letter might contain several atoms; and (2) there is an output alphabet, also a straight set, and there are output commands used to generate letters from the output alphabet. Recall that a straight set is a disjoint union $X = X_1 + \dots + X_k$ where each X_i is a finitely supported subset of \mathbb{A}^{n_i} for some $n_i \in \{0, 1, 2, \dots\}$. We write an element of a straight set as $i(a_1, \dots, a_{n_i})$, where $i \in \{1, \dots, k\}$ indicates which component of the disjoint union is used. The identifier i is called the *label*.

► **Definition 16.** *A one-way single use register transducer has the same syntax as a one-way single use register automaton, except that:*

- *it has two straight orbit-finite sets Σ and Γ , called the input and output alphabets;*
- *its transition function can use the following questions and actions:*
 - Questions.
 1. *Is the label of the input letter equal to l ?*
 2. *Is the head in the last position?*
 3. *Is the i -th atom in the current letter equal to the atom in the register $r \in R$?*
 4. *Do registers $r, s \in R$ store the same value?*
 - Actions.
 1. *Move the head to the next position.*
 2. *Accept/reject.*
 3. *Set register $r \in R$ to the i -th atom of the current letter.*
 4. *Erase the contents of register $r \in R$.*
 5. *Move constant $a \in C$ into register $r \in R$.*
 6. *For registers $r_1, \dots, r_n \in R$ and label i , output $i(r_1, r_2, \dots, r_n)$*

Since one-way single use register transducers are the only transducers that are used in this paper, we will simply call them “transducers” from now on.

The single use restriction is implemented as follows: whenever questions 3 or 4 are asked, the corresponding registers are erased, namely r and $\{r, s\}$, respectively. Likewise for the action 6, in which case the corresponding registers are $\{r_1, \dots, r_n\}$. The semantics of a transducer is a partial function $f : \Sigma^* \rightarrow \Gamma^*$ defined in the natural way. The domain of the function consists of words where the accept action is performed. The following lemma summarises the basic properties of transducers that we use in this paper.

► **Lemma 17.** *Basic properties of transducers:*

1. *The domain of a transducer is recognised by a one-way single use automaton (provided that the input alphabet has the form of $L \times \mathbb{A}$, for some finite set of labels L).*
2. *Partial functions recognised by transducers are closed under composition;*
3. *If $f : \Sigma^* \rightarrow \Gamma^*$ is recognised by a transducer and $|$ is not in Σ , then the function*

$$w_1 | \dots | w_n \mapsto f(w_1) | \dots | f(w_n) \quad \text{where } w_1, \dots, w_n \text{ do not contain } |.$$
is also recognised by a transducer.
4. *If $h : \Sigma^k \rightarrow \Gamma$ is finitely supported, then the following is recognised by a transducer:*

$$a_1 \dots a_n \in \Sigma^* \mapsto h(a_1, \dots, a_k) h(a_2, \dots, a_{k+1}) \dots h(a_{n-k+1}, a_n) \in \Gamma^*$$

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Proof. (Item 1 of Lemma 17) Since the input alphabet is of the form: (finite label, atom) the only action or question that cannot be simulated by an automaton are the “output” commands. Since they do not affect the domain of the transducer, the simulating automaton can just ignore them. ◀

Before proceeding to the proofs of other items in Lemma 17 we prove the following claims:

▷ **Claim 18.** For every transducer \mathcal{A} , there is a limit $m \in \mathbb{N}$, such that if \mathcal{A} doesn’t move forward for more than m steps, it will loop forever, never accepting the input word.

Proof. Let \mathcal{A} have q states, r registers and c constants, and let t be equal to the biggest support in \mathcal{A} ’s input alphabet. Observe, then that as long as \mathcal{A} stays in one place it only has

$$m = q \cdot (t + c + r)^r$$

possible configurations. This is because every of its k registers can only hold:

- one of the at most t atoms that appear in the current letter, or
- one of the c constants of \mathcal{A} , or
- one of the at most r atoms that were present in \mathcal{A} ’s registers, when it first entered this position,

which gives $t + c + r$ possible valuations of each of the registers. This means that if \mathcal{A} stays in one place for more than m steps, it will visit one of the possible configurations for the second time, and fall into a loop. ◀

▷ **Claim 19.** Extending the set of actions with “Swap contents of registers $i, j \in R$ ” does not extend the class of definable transductions.

Proof. In order to simulate the “swap” actions, a transducer may additionally keep a permutation of all registers in its finite state (since there are only finitely many register permutations), that will be updated every time a “swap” action is executed. ◀

Now we are ready to prove next two items from Lemma 17.

Proof. (Item 2 of Lemma 17) We are given a transducer \mathcal{A} defining a partial function $g : \Sigma^* \rightarrow \Gamma^*$, a transducer \mathcal{B} defining a partial function $h : \Gamma^* \rightarrow \Delta^*$ and we construct a transducer \mathcal{C} defining the partial function $h \circ g$. Take m as defined in Claim 18 for \mathcal{A} . Transducer \mathcal{C} keeps m copies of \mathcal{A} , one copy \mathcal{B} and works by repeating the following two steps, until \mathcal{B} finishes its run.

1. At the beginning, \mathcal{C} runs each copy of \mathcal{A} until they try to execute the “output” command. Instead of outputting, \mathcal{C} saves the values in the registers and proceeds to the next step.
2. Having learned \mathcal{A} ’s output, \mathcal{C} simulates \mathcal{B} until it asks to go forward – whenever \mathcal{B} asks for input, \mathcal{C} takes it from one of the copies of \mathcal{A} . Thanks to Claim 18, we know that if \mathcal{C} runs out values to feed to \mathcal{B} , it may conclude that \mathcal{B} is looping and execute the “reject” command.

◀

Proof. (Item 3 of Lemma 17) We are given a transducer \mathcal{A} recognizing a partial function $f : \Sigma^* \rightarrow \Gamma^*$ and we construct a transducer \mathcal{C} defining the partial function:

$$w_1|w_2|w_3|\cdots|w_n \mapsto f(w_1)|f(w_2)|f(w_3)|\cdots|f(w_n)$$

Take m as defined in Claim 18 for \mathcal{A} . Transducer \mathcal{C} simulates \mathcal{A} on each of the words w_i : it reads every input letter in m copies and moves forward to look ahead and see if the next

letter is equal to $|$ (in case \mathcal{A} asks the “is the head in last position” question). With this information \mathcal{C} can simulate \mathcal{A} until it asks to go forward. When that happens, \mathcal{C} reads the input letter in m copies, goes forward, and continues the simulation of \mathcal{A} . ◀

Before proving bullet 4 from Lemma 17, we prove the following claim:

▷ **Claim 20.** For every orbit-finite, straight sets Σ and Γ and for every $k \in \mathbb{N}$, every finitely supported function:

$$f : \Sigma^k \rightarrow \Gamma$$

Can be represented as a $\Sigma^* \rightarrow \Gamma^*$ transducer, that outputs $f(\text{input})$ if input has exactly k letters, or otherwise is undefined.

Proof. Let \bar{c} be the least support of f . Suppose for now that $\Sigma = \mathbb{A}^m$, so Σ^k is the same as \mathbb{A}^{km} . The value of $f(\bar{x})$ depends almost entirely on the equality type of $\bar{x}\bar{c}$:

- the finite label of $f(\bar{x})$, depends only on the equality type of $\bar{x}\bar{c}$
- every atom of $f(\bar{x})$ is equal to one of the atoms in $\bar{x}\bar{c}$ and the position of the j th atom of $f(\bar{x})$ in $\bar{x}\bar{c}$ depends only on j and on the equality type of $\bar{x}\bar{c}$.

A transducer can easily compute $\bar{x}\bar{c}$ equality type, using no more than $|\bar{x}\bar{c}|$ copies of each of the atoms. Each atom from $\bar{x}\bar{c}$ will be present in the output of f no more than l times, where l is the size of the biggest least support in Γ . This means that f can be implemented by a transducer \mathcal{C} whose constants all the elements from \bar{c} . It reads the first k letters and saves them in $|\bar{c}| + m + l$ copies. After that it asserts that it is in the last position (or rejects the input word) and uses the above observations to calculate the result.

To extend this proof to a general case, we notice that if Σ is an L -indexed sum of tuples, then for a fixed sequence l of k labels, the function $f_l : \mathbb{A}^{kl} \rightarrow \Gamma$ can be computed by a transducer \mathcal{A}_l , using the observation above. There are only finitely many of l 's, so we can combine all the \mathcal{A}_l 's to construct a transducer that computes f . ◀

Proof. (item 4 of Lemma 17) We construct the required function as a composition of three transducers:

1. group the input letters into windows of k and outputs them separated with the $|$ sign (in order to do this we need to store k copies of each of the input atoms)
2. apply h to each of the block – constructed as the iterated (as defined in Lemma 17 item 3) version of the transducer defined in Claim 20
3. filter out the $|$ signs

6.2 Simulating a semigroup with a transducer

This section is devoted to proving Lemma 21 below, which says that a transducer can compute the product operation in a semigroup. The transducer inputs a sequence of semigroup elements, given by straight representations (see Lemma 6) and outputs a single letter which represents the the product.

► **Lemma 21.** *Let S be an orbit-finite semigroup, and fix a straight representation $S \rightarrow S$ by applying Lemma 6. There is a transducer which does the following:*

- **Input.** $s_1, \dots, s_n \in S$.

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■ **Output.** A representation s of the product $s_1 \cdots s_n$ ⁴.

Before proving the lemma, we use it to prove the implication from semigroups to one-way automata in Theorem 11. Let L be a language that is recognised by an orbit-finite semigroup. Consider the following composition of transducers:

1. replace each input letter by its value under the semigroup homomorphism;
2. compute the product in the semigroup;
3. if the first letter is in the accepting set, then accept, otherwise reject.

All three steps are transducers, with the second step corresponding to Lemma 21, and the other steps corresponding to Lemma 17. Also by Lemma 17, the composition of the three steps is a transducer, and therefore its domain is recognised by a one-way single use automaton. This proves that every language recognised by an orbit-finite semigroup is recognised by a one-way single use automaton.

The rest of Section 6.2 is devoted to proving Lemma 21.

Plan of the proof and Green's relations. Fix S and S as in the lemma. To make notation lighter, we assume that S and its product operation are equivariant; the proof in the general case is done in essentially the same way.

In the proof, we consider the following infix ordering on a semigroup: $s \in S$ is called an *infix* of $t \in S$ if t admits a decomposition $t = xsy$ where each of x, y is either empty or an element of S . Being an infix is a transitive and reflective relation, and the associated equivalence classes are called \mathcal{J} -classes. The infix ordering is one of the semigroup orderings known as Green's relations, see [8] for a survey. There are also Green's relations for the prefix ordering (its equivalence classes are called \mathcal{R} -classes) and the suffix ordering (its equivalence classes are called \mathcal{L} -classes); these are defined analogously to the infix ordering except that the x part is required to be empty for prefixes and the y part is required to be empty for suffixes.

Define the \mathcal{J} -height of a semigroup to be the maximal length of a strictly increasing chain with respect to the infix ordering. By [2, last line of proof of Lemma 9.3] if two elements of an orbit-finite semigroup are in the same orbit, then they are either in the same \mathcal{J} -class, or they are incomparable with respect to the infix relation. A corollary is that an orbit-finite semigroup has finite \mathcal{J} -height⁵. The proof of Lemma 21 is by induction on the \mathcal{J} -height of the semigroup S . The proof has two parts. The first one reduces the general case to the case where the product is smooth in the following sense: a product of elements $s_1, \dots, s_n \in S$ is *smooth* if all of s_1, \dots, s_n as well as their product are in the same \mathcal{J} -class. The second part of the proof shows how to compute smooth products.

Reduction to the smooth product case. In this section, we show how computing products reduces to computing smooth products. Define $T \subseteq S$ to be those elements which have no proper infixes (i.e. every infix is in the same \mathcal{J} -class). By definition, T is a union of \mathcal{J} -classes, which form an antichain with respect to the infix ordering. It is also easy to see that $S - T$ is a finitely supported subsemigroup of S (possibly empty) with strictly smaller \mathcal{J} -height, to which the induction assumption can be applied. In the reduction, we

⁴ We use here, and elsewhere in the proof, the convention that changing the font colour from red to black goes from a representation to the represented element. The convention works only one way, since there can be many representations of the same element.

⁵ This argument crucially depends on the atoms having equality only. For example, if the atoms are the ordered rational numbers, then the semigroup of atoms equipped with max has one orbit, but it is totally ordered by the infix relation, and therefore it has infinite \mathcal{J} -height.

use the following description of smooth products, which is a standard application of Green’s relations.

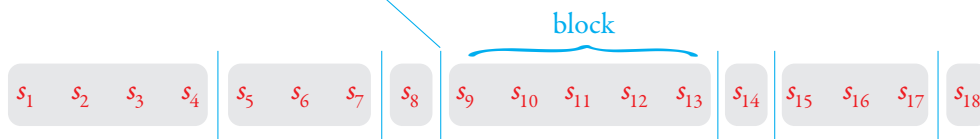
► **Lemma 22.** *Let $s_1, \dots, s_n \in T$. The following conditions are equivalent:*

1. *the product $s_1 \cdots s_n$ is smooth;*
2. *s_1, \dots, s_n and all consecutive products $s_i s_{i+1}$ are in the same \mathcal{J} -class.*

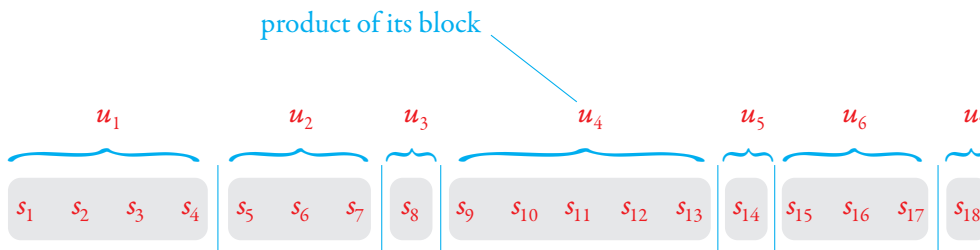
Proof. The non-trivial implication is $2 \Rightarrow 1$. By [2, Theorem 5.1], every orbit-finite semigroup is locally finite, which means that every finitely generated subsemigroup is finite. As observed in [2, Section 5], Green’s relations can be used in locally finite semigroups, and not just finite ones. In particular, one can apply the following result [8, Lemma 2] about Green’s relations to get the implication $2 \Rightarrow 1$: if b, c, d, bc, cd are in the same \mathcal{J} -class, then also bcd is in this \mathcal{J} -class. ◀

Let us continue with the reduction from general products to smooth products. Suppose that we have a transducer which computes smooth products, and we want to compute a product of $s_1, \dots, s_n \in S$, given $s_1, \dots, s_n \in S$. Define an *interval* to be a connected set of positions $I \subseteq \{1, \dots, n\}$. The product of an interval is defined to be the semigroup product of all s_i with $i \in I$. An interval is called *smooth* if the corresponding product is smooth. Define a *block* to be maximal inclusionwise interval where every two consecutive positions form a smooth interval. By Lemma 22, blocks are smooth intervals, and every smooth interval is contained in a block. Here is a picture of the partition into blocks:

the vertical line separates two elements if their product is in a different infix class



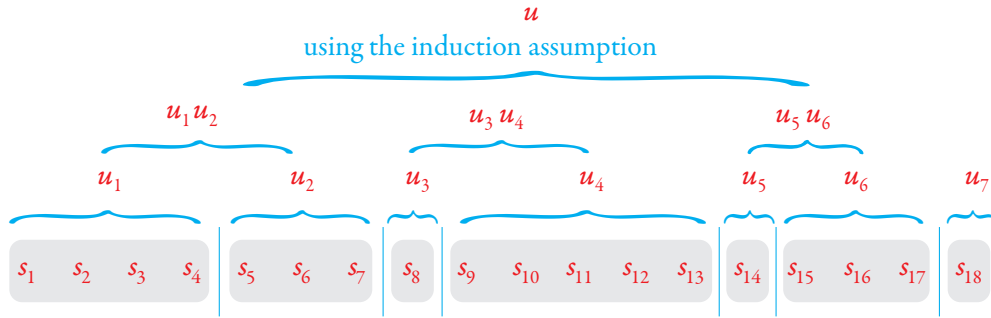
Suppose that we are given representations $s_1, \dots, s_n \in S$. A transducer can compute the partition into blocks, by placing a separator symbol between every two consecutive positions which do not form a smooth interval. There is no trouble with the single use restriction, since the placement of the separators can be done by using a buffer which stores the last two input letters. Compute the products of the blocks, using the assumption that we know how to compute smooth products, as shown in the following picture:



If we group the products of blocks in groups of two, then the resulting products will be in $S - T$. This is because contrary to elements in T , products of two consecutive blocks have proper infixes. Therefore, the product can be computed by combining the induction assumption, for the smaller semigroup $S - T$, with item 3 of Lemma 17. Here is the picture:



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If the number of blocks is odd, then the value of the last block can be added to the product at the end of the computation. This completes the reduction of computing products to computing smooth products.

Computing smooth products. We show a transducer which computes smooth products:

- **Input.** $s_1, \dots, s_n \in S$.
- **Output.** A representation of the product of $s_1 \cdots s_n$, provided that it is smooth.

This completes the proof of Lemma 21.

We compute the smooth product in four steps, as described below. Each of the steps can be viewed as a transducer, and therefore the entire construction is obtained by transducer composition. We assume below that $n > 1$, i.e. there is more than one input letter – otherwise the transducer can simply copy its input letter into the output.

1. We assume that the input to the transducer is a smooth product – let J be the \mathcal{J} -class which contains s_1, \dots, s_n and their product. In this step, we decorate each input letter with an idempotent from J , such that all those idempotents have the same least support. To choose these idempotents, we use the following claim.

- ▷ **Claim 23.** There is a finitely supported partial function $E : S \rightarrow S$ such that:
- a. For every $s \in S$, $E(s)$ represents an idempotent in the \mathcal{J} -class of s , or is undefined if there is no such idempotent;
 - b. If $s_1, s_2 \in S$ represent \mathcal{J} -equivalent semigroup elements, then $E(s_1)$ and $E(s_2)$ have the same least support.

Proof. Choose a tuple \bar{c} of atoms which contains more than twice the number of atoms needed to support any element of S (and therefore also of S , since representations have the same supports as represented elements). These atoms will be the support of the function E .

The set of \mathcal{J} -classes in S is itself an orbit-finite set – as a quotient of an orbit-finite set under a finitely supported equivalence relation – and therefore one can talk about the support of a \mathcal{J} -class. By the same argument as in the proof of Lemma 7, if a \mathcal{J} -class contains an idempotent, then it contains an idempotent e which satisfies:

- * The least support of e is contained in the union of:
 - the tuple \bar{c} fixed at the beginning of this proof;
 - the least support of the \mathcal{J} -class of e .

Call an idempotent *special* if it satisfies property (*) above, and the set of atoms from \bar{c} in its least support is smallest with respect to some fixed linear ordering of subsets of \bar{c} . We define E to be the uniformisation (using Lemma 7) of the following relation:

$$\{(s, e) : e \text{ is a special idempotent in the } \mathcal{J}\text{-class of } s\}$$

Now, the only thing left to show is that this E satisfies the property 1b of the Claim: The function which maps a semigroup element to its \mathcal{J} -class is equivariant, by the assumption that the semigroup is equivariant, and equivariant functions can only make supports smaller. It follows that the least support of every special idempotent e contains the entire least support of the \mathcal{J} -class of e . By definition, all special idempotents in the same \mathcal{J} -class use the same constants from \bar{c} in their least support. Summing up, all special idempotents in the same \mathcal{J} -class have the same least support, namely the least support of the \mathcal{J} -class, plus some fixed constants from \bar{c} that depend only on the \mathcal{J} -class. This observation extends to representations, since the representations have the same least supports as the represented elements. ◀

Using item 4 of Lemma 17, apply the function from the above claim to each input letter, yielding a word of the form $(s_1, e_1), \dots, (s_n, e_n)$. The product is smooth and $n > 1$, which means that $(J \cdot J) \cap J$ is nonempty, so J must contain an idempotent [12, Corollary 2.25], and therefore e_1, \dots, e_n are defined. The product is smooth, so by the second item in Claim 23 all of e_1, \dots, e_n have the same least support.

2. Let e_1, \dots, e_n be as in the previous step. In this step, we replace all of the idempotents by the same one, say e_1 , as stated in the following claim.

▷ Claim 24. There is a transducer which does this:

- **Input.** $(s_1, e_1), (s_2, e_2), \dots, (s_n, e_n) \in S^2$, where e_1, \dots, e_n have equal least supports.
- **Output.** $(s_1, e_1), (s_2, e_1), \dots, (s_n, e_1) \in S^2$

Proof. As the following claim illustrates, the single use condition can be lifted for the atoms that appear in each letter:

▷ Claim 25. For every transducer \mathcal{A} there exists m such that, if \mathcal{A} holds at least m copies of a value $v \in \mathbb{A}$, and the value v is present in the currently seen letter, then \mathcal{A} can produce any number of extra copies of v .

Proof. Let m be equal to the size of the biggest least support in \mathcal{A} 's input alphabet. This is enough copies of v for \mathcal{A} to locate it in the current letter and to load it into as many registers as required. ◀

We can use this claim to define the following transducer performing the requested transductions:

- a. read e_1 in $m + 1$ copies;
- b. for every letter (s_i, e_i) :
 - i. output (s_i, e_1) , using one of the copies of e_1 ;
 - ii. use the remaining copies to restore the missing copy of e_1 using atoms from e_i .

3. Let the list of pairs produced in the previous step be $(s_1, e), \dots, (s_n, e)$. In this step, each semigroup element s_i is decomposed into a prefix that ends with e and a suffix that begins with e , as stated in the following claim.

▷ Claim 26. There is a finitely supported partial function $F : S^2 \rightarrow S^2$ which does this:

- **Input.** (s, e) , where e is an idempotent from the \mathcal{J} -class of s .

- **Output.** A pair (x, y) such that $s = xy$, $x = xe$ and $y = ey$.

Proof. Suppose that e is an idempotent in the \mathcal{J} -class of s . If e is an idempotent in the \mathcal{J} -class of s , then s contains $e = ee$ as an infix, and therefore there is at least one factorisation (x, y) as required in the statement of the claim. To produce F , apply Lemma 7. ◀

Using item 4 of Lemma 17, apply the function from the above claim to each pair produced in the previous step, yielding a word of the form $(x_1, y_1), \dots, (x_n, y_n)$ where each x_i represents a semigroup element ending with e and each y_i represents a semigroup element beginning with e .

4. For $i \in \{1, \dots, n-1\}$ define g_i to be the product $y_i x_{i+1}$. Using item 4 of Lemma 17, transform the output from the previous step into $x_1, g_1, \dots, g_{n-1}, y_n$. The product represented by the above elements is the same as the product of s_1, \dots, s_n . Since the first letter x_1 and the last letter y_n can be incorporated in a final step, it suffices to show that the product of the g_i can be computed by a transducer. The crucial observation is in the following claim.

▷ Claim 27. All of g_1, \dots, g_n have the same least support.

Proof. Define \mathcal{H} -equivalence to be the meet of the \mathcal{L} - and \mathcal{R} -equivalence relations: an \mathcal{H} -equivalence class is defined to be any nonempty intersection of an \mathcal{L} -equivalence class and an \mathcal{R} -equivalence class. Each g_i begins with e , ends with e , and is in the \mathcal{J} -class of e . It follows that each g_i is in the same \mathcal{H} -class as e ; this is because \mathcal{R} -classes and \mathcal{L} -classes in a given \mathcal{J} -class form an antichain [2, Lemma 7.1]. Like for any idempotent, the \mathcal{H} -class of e is a group [8, Lemma 11]. We finish the proof by stating that in an orbit-finite group with an equivariant product operation, all elements have the same least support (see [3, Solution to Exercise 84], or [7, Lemma 2.14]). ◀

Because all g_1, \dots, g_{n-1} have the same least support, their product can be computed by a transducer using the same idea as in Claim 24.

7 Conclusions

Our main direction for future work is to consider two-way single use transducers. We conjecture that these are equivalent to both (a) a single use variant of streaming string transducers [1]; and (b) the list functions defined in [4] extended by a datatype \mathbb{A} with an equality test $\mathbb{A}^2 \rightarrow \{=, \neq\}$. If true, this conjecture would underscore the robustness of the single use model.

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