

MSO+ ∇ is undecidable

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Abstract—This paper is about an extension of monadic second-order logic over the full binary tree, which has a quantifier saying “almost surely a branch $\pi \in \{0, 1\}^\omega$ satisfies a formula $\varphi(\pi)$ ”. This logic was introduced by Michalewski and Mio; we call it MSO+ ∇ following notation of Shelah and Lehmann. The logic MSO+ ∇ subsumes many qualitative probabilistic formalisms, including qualitative probabilistic CTL, probabilistic LTL, or parity tree automata with probabilistic acceptance conditions. We show that it is undecidable to check if a given sentence of MSO+ ∇ is true in the full binary tree¹.

I. INTRODUCTION

Probability has been present in the theory of verification since the very beginning. An early example [22], [23] is the following question: given an LTL formula and a Markov chain, decide if almost all (in the sense of measure) runs of the system satisfy the formula. Another early example [14] is: given a formula of probabilistic CTL, decide if there is some Markov chain where the formula is true (the complexity of the problem is settled in [10]). The same question for the more general logic CTL* is answered in [15, Theorem 1 and 2, and Section 15]. Other variants of these logics have been considered in [13], [2]. More recent work tries to synthesize controllers for probabilistic systems, see e.g. [4, Theorem 15].

Is there a master theorem, which unifies all decidability results about probabilistic logics? An inspiration for such a master theorem would be Rabin’s famous result [19] about decidability of monadic second-order logic over infinite trees. Rabin’s theorem immediately gives most decidability results (if not the optimal complexities) about temporal logics, including satisfiability questions for (non-probabilistic) logics like LTL, CTL* and the modal μ -calculus. Maybe there is a probabilistic extension of Rabin’s theorem, which does the same for probabilistic logics?

Quite surprisingly, the question about a probabilistic version of Rabin’s theorem has only been asked recently, by Michalewski and Mio [17]. It is rather easy to see that any decidable version of MSO must be qualitative rather than quantitative (*i.e.* probabilities can be compared to 0 and 1, but not to other numbers), since otherwise one could express problems like “does a given probabilistic automaton accept some word with probability at least 0.5”, which are known to be undecidable [18], see also [12]. Even when probabilities are qualitative, one has to be careful to avoid undecidability. For example, the following problem is undecidable [1, Theorem 7.2]: given a Büchi automaton, decide if there is some ω -word that is accepted with a non-zero probability (assuming that runs of the automaton are chosen at random, flipping a coin for each transition). This immediately implies [17, Theorem 1] undecidability for a natural probabilistic extension of MSO, which has a quantifier of the form “there is a non-zero probability of picking a set X of positions that satisfies $\varphi(X)$ ”, both for infinite words and infinite trees.

Michalewski and Mio propose a different probabilistic extension of MSO, which does not admit any straightforward reductions from known undecidable problems, like the ones for probabilistic Büchi automata mentioned above. Their idea—which only makes sense for trees and not words—is to extend MSO over the infinite binary tree by a quantifier which says that a property $\varphi(\pi)$ of branches is true almost surely, assuming the coin-flipping measure on infinite branches in the complete binary tree. The logic proposed by Michalewski and Mio is obtained from Rabin’s MSO by adding the probabilistic quantifier for branches. We write MSO+ ∇ for this logic². As explained in [17], MSO+ ∇ directly expresses qualitative problems like: model checking Markov chains for LTL objectives, their generalisations such as $2\frac{1}{2}$ player games with ω -regular objectives, or emptiness for

¹Independently and in parallel another proof of this result was given employing different techniques in [3].

²In [17] the quantifier is denoted by \forall_π^1 , but in this paper we denote it by ∇ , following the notation used by Shelah and Lehmann in [15].

various automata models with probability including the qualitative tree languages from [11]. These results naturally lead to the question [17, Problem 1]: is the logic $\text{MSO}+\nabla$ decidable?

A positive result about $\text{MSO}+\nabla$ was proved in [5], [7]: the weak fragment of $\text{MSO}+\nabla$ is decidable. In the weak fragment, the set quantifiers $\forall X$ and $\exists X$ of MSO range only over finite sets³. The decidability proof uses automata: for every formula of the weak fragment there is an equivalent automaton of a suitable kind [5, Theorem 8], and emptiness for these automata is decidable [7, Theorem 3]. Combining these results, one obtains decidable satisfiability⁴ for the weak fragment of $\text{MSO}+\nabla$. The weak fragment of $\text{MSO}+\nabla$ is still powerful enough to subsume problems like satisfiability for qualitative probabilistic CTL^* . Nevertheless, the decidability of the full logic $\text{MSO}+\nabla$ remained open.

This paper proves that the full logic $\text{MSO}+\nabla$ is undecidable, *i.e.* it is undecidable if a sentence of the logic is true in the full binary tree, thus answering [17, Problem 1]. Independently and in parallel another proof of this result is given in [3], by proving that the emptiness problem of qualitative universal parity tree automata is undecidable.

II. THE LOGIC

In this section we describe the logic $\text{MSO}+\nabla$.

Our logic is an extension of Rabin’s MSO over the full binary tree, so we begin by describing that. We write $\mathbf{2}$ for the set $\{0, 1\}$. The full binary tree is the tree where nodes are identified with $\mathbf{2}^*$, finite words over the alphabet $\{0, 1\}$. The ancestor order \leq is the prefix relation. We write $|x| \in \mathbb{N}$ for the length of a bit sequence $x \in \mathbf{2}^*$. To express properties of the full binary tree, we use monadic second-order logic (MSO). This logic which has two types of variables

$$\underbrace{X, Y, Z, \dots}_{\text{sets of nodes}} \quad \underbrace{x, y, z, \dots}_{\text{nodes}}$$

³Actually, the papers prove decidability for a stronger logic, where set quantifiers range over “thin” sets, which are a common generalisation of finite sets and infinite branches.

⁴For weak logics the satisfiability problem “is a given formula true in some infinite labelled binary tree” is in general more difficult than the model checking problem “is a given formula true in the unlabelled binary tree”. For general MSO , this difference disappears, as set quantification can be used to guess labellings.

which can be quantified existentially and universally. To compare nodes and sets of nodes we use predicates

$$x \in X \quad \underbrace{x \leq y}_{\text{ancestor}} \quad \underbrace{x = y0}_{\text{left child}} \quad \underbrace{x = y1}_{\text{right child}} .$$

By Rabin’s Theorem, there is an algorithm which inputs a sentence of MSO , and says if the sentence is true in the full binary tree, see [21] for a survey of the topic.

The idea behind $\text{MSO}+\nabla$ is to extend MSO with probabilistic quantification over branches⁵. A *branch* is defined to be an element of $\mathbf{2}^\omega$. Probability for sets of branches is measured using the *coin-tossing* measure on $\mathbf{2}^\omega$, which is the unique complete probabilistic measure \mathbb{P} that satisfies

$$\mathbb{P}[x \cdot \mathbf{2}^\omega] = 2^{-|x|},$$

for all $x \in \mathbf{2}^*$. The logic $\text{MSO}+\nabla$ extends MSO by adding a new type of variable

$$\underbrace{\pi, \sigma, \tau, \dots}_{\text{branches}}$$

along with a membership test $x \in \pi$ (for membership tests, a branch is identified with the set of nodes that are its finite prefixes). To bind branches, the logic $\text{MSO}+\nabla$ has a probabilistic quantifier

$$\nabla \pi. \phi(\pi),$$

which says that there exists a set $R \subseteq \mathbf{2}^\omega$, such that R has defined measure equal to 1, and every branch in R satisfies ϕ . Intuitively, it means that $\phi(\pi)$ holds for a *randomly chosen* branch. This completes the definition of $\text{MSO}+\nabla$.

We now give some examples that illustrate the expressive power of $\text{MSO}+\nabla$.

Example II.1. This example is from [5, Section 3]. Consider the formula

$$\exists X \left\{ \begin{array}{l} \underbrace{\forall x. \exists y. (y \geq x \wedge y \in X)}_{\text{every node has a descendant in } X} \\ \underbrace{\neg \nabla \pi. (\exists x. x \in \pi \wedge x \in X)}_{\text{with positive probability, } \pi \text{ avoids } X} \end{array} \right.$$

⁵There is an alternative way of adding probability to MSO , namely by having a quantifier which says that $\varphi(X)$ is true almost surely, assuming that the set of nodes X is chosen uniformly at random. This logic is already known to be undecidable [17, Theorem 1], even for ω -words, thanks to a straightforward reduction from emptiness for probabilistic Büchi automata with an almost sure acceptance condition [1].

This sentence is true. To see why, consider

$$X = \bigcup_{n \geq 2} X_n \quad \text{where } X_n = \{x0^n : x \in \mathbf{2}^n\}.$$

Every node x in the full binary tree has a descendant in X , namely $x0^{|x|}$. The probability of a branch visiting X_n is $1/2^n$, and therefore the probability of visiting X is at most

$$\frac{1}{2} = \sum_{n \geq 2} \frac{1}{2^n}.$$

(In fact, the probability of visiting X is smaller, because the events of visiting X_n are not independent.) It follows that the probability of avoiding X is positive, and therefore X makes the formula true. One can show that there is no set X which makes the formula true and which is regular when seen as a language $X \subseteq \mathbf{2}^*$. This implies that the family of sets X which make the formula true cannot be defined in MSO.

The above example shows that for formulas with free set variables – which can be seen as describing languages of labelled trees – the logic $\text{MSO} + \nabla$ is strictly more expressive than MSO.

Example II.2. Following [11], consider a nondeterministic parity automaton on infinite trees, where a run is considered accepting if the parity condition is satisfied almost surely. The existence of an accepting run can be easily expressed in the $\text{MSO} + \nabla$, by guessing a labelling of the tree with states and then checking the acceptance condition using the quantifier ∇ . The same idea works for more general acceptance conditions, e.g. a conjunction of two acceptance conditions: an almost surely parity condition, and a usual (all paths) parity condition. Such automata are considered in [5], [7], [4].

Example II.3. Consider the following variant of qualitative probabilistic CTL. This logic is used to define properties of labelled trees $t : \mathbf{2}^* \rightarrow \Sigma$. The atomic formulas check the label of the root, and Boolean combinations are allowed. There is a probabilistic version of the until operator: if φ_1, φ_2 are already defined formulas then also

$$\nabla(\varphi_1 \cup \varphi_2),$$

is a formula, which is true in a tree if almost surely a branch π has the property that for some $y \in \pi$, the subtree of y satisfies φ_2 , and for all $x < y$, the subtree of x satisfies φ_1 . For every formula φ of this

logic, one can easily write a sentence of $\text{MSO} + \nabla$ that is true if and only if φ is true in some labelled tree. The same kind of translation would work for many generalisations of the logic, e.g. one could add an operator that checks if all (not almost all) paths satisfy a given property, or parity counting, etc.

The formulas in Examples II.1, II.2 and II.3 are all of the form

$$\exists X_1 \dots \exists X_n. \varphi(X_1, \dots, X_n) \quad (1)$$

where φ uses only ∇ , quantification over finite sets of nodes, and (non-probabilistic) quantification over branches. By [5], [7], the truth of such sentences is decidable⁶. The purpose of this paper is to prove that, if we allow formulas that are more complicated than (1), then the logic becomes undecidable.

III. UNDECIDABILITY

The main result of this paper is undecidability of the logic $\text{MSO} + \nabla$, as stated in the following theorem.

Theorem III.1. *There is no algorithm which decides whether or not a given sentence of $\text{MSO} + \nabla$ is true in the full binary tree.*

The main ingredient in the undecidability proof is showing that $\text{MSO} + \nabla$ can express a certain asymptotic counting property. Once the counting property has been defined, a routine encoding of Minsky machines can be used to establish undecidability. We now describe this asymptotic counting property.

Define an *interval* to be a finite path in the complete binary tree, *i.e.* a set of the form

$$\{z : x \leq z \leq y\} \quad \text{for some } x, y \in \mathbf{2}^*, x < y.$$

The nodes x and y are called the *source* and *target* of the interval, respectively. The *interior* of the interval $[x, y]$ is the set

$$\text{Int}([x, y]) \stackrel{\text{def}}{=} \{z : x < z < y\}.$$

The *length* of an interval is the cardinality of its interior.

Let \mathcal{I} be a family of intervals. If all intervals in \mathcal{I} are pairwise disjoint, then the family is uniquely determined by the sets

$$\text{source}(\mathcal{I}), \text{target}(\mathcal{I}) \subseteq \mathbf{2}^*$$

⁶In the paper [3], which gives an alternative proof of the main result in this paper, it is shown that universality is undecidable for the tree automata described in Example II.2. It follows that the theory of $\text{MSO} + \nabla$ is undecidable even after prepending universal set quantifiers in (1). Our undecidability proof uses formulas with a more complex quantifier structure.

of its sources and targets. We only consider families of intervals that are pairwise disjoint, and therefore from now on, when we say *family of intervals*, we mean a family of pairwise disjoint intervals. We write $\mathcal{I}, \mathcal{K}, \mathcal{J}$ for such families.

For a family of intervals \mathcal{I} and a node x that is the source of some interval \mathcal{I} , we write $\mathcal{I}(x)$ for the length of the corresponding interval (which is unique by assumption that all intervals are pairwise disjoint). If π is a branch, then we write $\mathcal{I}(\pi)$ for the sequence

$$\mathcal{I}(x_1), \mathcal{I}(x_2), \dots$$

where x_1, x_2, \dots are all of the sources of \mathcal{I} that appear in π , ordered by increasing depth. See Figure 1.

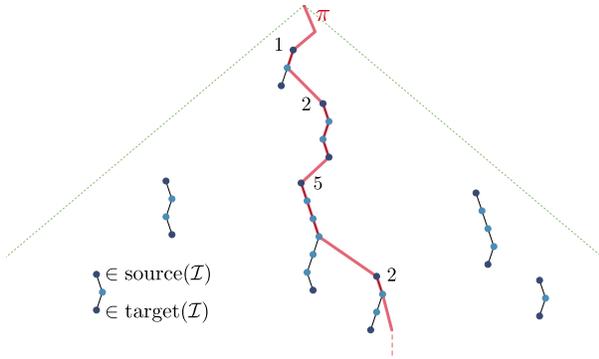


Figure 1. Here we have the sequence $\mathcal{I}(\pi) = 1, 2, 5, 2, \dots$

The sequence $\mathcal{I}(\pi)$ is a sequence of natural numbers, whose length may be finite or infinite. We say that a sequence of natural numbers is *eventually constant* if it has infinite length, and it has the same number on all but finitely many positions. Here is an example:

$$1, 2, 5, 2, 1, \overbrace{2, 2, 2, 2, 2, \dots}^{\text{only 2}}$$

If \mathcal{I} is a family of pairwise disjoint intervals, then we write

$$\mathbb{P}[\mathcal{I} \text{ is eventually constant}]$$

for the probability of choosing a branch π such that $\mathcal{I}(\pi)$ is eventually constant. The main technical result of this paper is that $\text{MSO}+\nabla$ can express that this probability is 1. The family \mathcal{I} is represented by its sources and targets.

Theorem III.2. *There is a formula $\varphi(X, Y)$ of $\text{MSO}+\nabla$ which is true if and only if*

$$\mathbb{P}[\mathcal{I} \text{ is eventually constant}] = 1$$

for some⁷ family of intervals \mathcal{I} where

$$X = \text{source}(\mathcal{I}) \quad Y = \text{target}(\mathcal{I}).$$

Once we have proved the above lemma, undecidability of the logic follows by a routine reduction from the halting problem for Minsky machines. The general idea is to write the computation of the Minsky machine, repeated infinitely often, on each branch of the tree, and to use eventually constant sequences to check if the counter values in consecutive configurations are consistent. This reduction is discussed in Section VI. The remaining part of the paper is devoted to proving Theorem III.2.

Note how the property

$$\mathbb{P}[\mathcal{I} \text{ is eventually constant}] = 1$$

is asymptotic in two ways: (a) it allows sequences that are not eventually constant on a set of branches with zero probability, and (b) on each branch there can be a finite delay before the constant tail starts.

IV. BOUNDEDNESS PROPERTIES

The proof of Theorem III.2 builds on ideas developed in the undecidability proofs from [9], [6] for the logic $\text{MSO}+\text{U}$, which is quantitative extension of MSO that talks about boundedness. In this section, we establish a connection with $\text{MSO}+\text{U}$, by showing that $\text{MSO}+\nabla$ can express various boundedness properties for families of intervals. In the next section, we build on this connection, and known results about $\text{MSO}+\text{U}$, to express the language of eventually constant sequences in Theorem III.2.

For a family \mathcal{I} of pairwise disjoint intervals, let

$$\mathbb{P}[\liminf \mathcal{I} < \infty] \quad (2)$$

be the probability of choosing a branch π such that

$$\liminf \mathcal{I}(\pi) < \infty.$$

The measured event is that the \liminf is both defined (i.e. \mathcal{I} is visited infinitely often) and finite. In other words (2) is the probability of choosing a branch such that $\mathcal{I}(\pi)$ contains some natural number infinitely often. The following lemma shows that $\text{MSO}+\nabla$ can express positive probability of (2). When we say that a formula of $\text{MSO}+\nabla$ expresses a property of a family of intervals \mathcal{I} , we assume that \mathcal{I} is given by two sets, representing its sources and targets, as in Theorem III.2.

⁷The family of intervals \mathcal{I} is unique, if it exists.

Roughly, the main observation is as follows. If the intervals get progressively longer, then the probability of a branch visiting targets infinitely often drops to zero. Otherwise, if the intervals have bounded length, then almost every branch that visits sources infinitely often must also visit targets infinitely often. This phenomenon can be observed in Example II.1. In this example every node has a descendant in X , but in order to avoid these descendants with positive probability, they need to be progressively more and more distant.

Lemma IV.1. *MSO+ ∇ can express*

$$\mathbb{P}[\liminf \mathcal{I} < \infty] > 0.$$

Proof. We show that the property in the statement of the lemma is equivalent to the following property, which is definable in MSO+ ∇ (see Appendix B).

(*) there exists $\mathcal{I}' \subseteq \mathcal{I}$ such that

$$\mathbb{P}\left[\underbrace{\mathcal{I}' \text{ io}}_{\substack{\text{a branch visits} \\ \text{sources of } \mathcal{I}' \\ \text{infinitely often}}} \right] > 0$$

and all $\mathcal{K} \subseteq \mathcal{I}'$ satisfy

$$\mathbb{P}[\mathcal{K} \text{ io} \Rightarrow \text{target}(\mathcal{K}) \text{ io}] = 1.$$

(\Rightarrow) We first show that the property in the statement of the lemma implies (*). For $n \in \mathbb{N}$, define \mathcal{I}_n to be the intervals in \mathcal{I} that have length exactly n . The event in the statement of the lemma says that with positive probability, there is some n such that a branch passes through \mathcal{I}_n infinitely often. By countable additivity of measures, for some n there is positive probability of seeing sources from \mathcal{I}_n infinitely often. Define $\mathcal{I}' = \mathcal{I}_n$. To establish (*), we prove the following claim.

Claim IV.2. If all intervals in \mathcal{K} have length n then

$$\mathbb{P}[\mathcal{K} \text{ io} \Rightarrow \text{target}(\mathcal{K}) \text{ io}] = 1.$$

Proof. Consider the complement of the event in the claim, that is:

$$\left[\mathcal{K} \text{ io} \wedge \underbrace{\text{target}(\mathcal{K}) \text{ fo}}_{\substack{\text{a branch visits} \\ \text{targets of } \mathcal{K} \\ \text{only finitely often}}} \right].$$

It is equal to:

$$\bigcup_{x \in \mathbf{2}^*} [\mathcal{K} \text{ io} \wedge A(x)],$$

where by $A(x)$ we denote the event of a branch passing through x and not visiting *any* target of \mathcal{K} after x . If x_0 is a source of \mathcal{K} , conditional on visiting x_0 , the probability of the event $A(x_0)$ is at most $1 - 1/2^{n+1}$; to avoid every target below x_0 , we have to avoid first the target corresponding to x_0 which is at distance $n+1$ since every interval in \mathcal{K} has length n . In other words, when going down the tree from x_0 whenever we visit a source, the relative probability of further avoiding targets is at most $1 - 1/2^{n+1}$, which means that

$$\mathbb{P}[\mathcal{K} \text{ io} \wedge A(x_0)] \leq \lim_{k \rightarrow \infty} \left(1 - \frac{1}{2^{n+1}}\right)^k = 0.$$

This proves that the complement of the event in the claim has probability zero⁸. \square

(\Leftarrow) We now show that (*) implies the property in the statement of the lemma. Let then \mathcal{I}' be as (*). Since the property in the statement of the lemma is closed under adding intervals to a family, it is enough to show

$$\mathbb{P}[\liminf \mathcal{I}' < \infty] > 0.$$

We will show a stronger property, namely

$$\mathbb{P}[\limsup \mathcal{I}' < \infty] > 0. \quad (3)$$

An interval in $I \in \mathcal{I}'$ is called a *record breaker* if it is strictly longer than all intervals in \mathcal{I}' with sources that are ancestors of the source of I .

Claim IV.3. Almost surely, the sources of record breakers are visited finitely often.

Proof. Define A_n to be the branches which see the target of some record breaker after having already seen at least n sources of record breakers. By definition, on each branch, the n -th record breaker has length at least n , and therefore the probability of seeing its target is at most $1/2^{n+1}$. It follows that the probability of A_n is at most

$$\frac{1}{2^n} = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots$$

Branches that visit infinitely many targets of record breakers belong to all sets A_n , and therefore they have probability zero.

We have thus established that almost surely targets of record breakers are seen finitely often. If we set \mathcal{K} to be the record breakers, then we know by (*)

⁸A more direct (but abstract) proof of this claim can be given using Lévy's zero-one law.

that almost surely sources of record breakers are seen finitely often, thus establishing the claim. \square

A branch π sees record breakers infinitely often if and only if the sequence $\mathcal{I}'(\pi)$ has infinite lim sup. Therefore, it follows from the claim that almost surely the sequence $\mathcal{I}'(\pi)$ has finite length, or it is infinite but has finite lim sup. Since there is positive probability of visiting \mathcal{I}' infinitely often, we get (3). \square

Building on the above lemma, we now show how $\text{MSO}+\nabla$ can characterise branches π where $\mathcal{I}(\pi)$ is unbounded.

Definition IV.4. A set of nodes X is called a *characteristic* for a family of intervals \mathcal{I} if

$$\mathbb{P}[X \text{ io} \iff (\limsup \mathcal{I} = \infty)] = 1. \quad (4)$$

Recall the notion of record breakers that was used in the proof of Lemma IV.1. It is not hard to see that the record breakers are a characteristic, and therefore every family of intervals admits at least one characteristic. The following lemma shows that being a characteristic can be described in $\text{MSO}+\nabla$ (we assume, as usual, that a family of intervals is given by its sources and targets).

Lemma IV.5. *There is a formula of $\text{MSO}+\nabla$ which says that X is a characteristic of \mathcal{I} .*

Proof. We say that Y is a *semi-characteristic* of \mathcal{I} if

$$\mathbb{P}[Y \text{ io} \Rightarrow (\limsup \mathcal{I} = \infty)] = 1. \quad (5)$$

It is not hard to see that X is a characteristic of \mathcal{I} if and only if every semi-characteristic satisfies

$$\mathbb{P}[Y \text{ io} \Rightarrow X \text{ io}] = 1.$$

Therefore, to prove the lemma, it is enough to define semi-characteristics in $\text{MSO}+\nabla$.

We claim that (5) is equivalent to

(*) There exists $\mathcal{K} \subseteq \mathcal{I}$ which is unbounded and

$$\mathbb{P}[Y \text{ io} \Rightarrow \mathcal{K} \text{ io}] = 1.$$

We say that a family \mathcal{K} is *unbounded* if

$$\mathbb{P}[\mathcal{K} \text{ io} \Rightarrow (\limsup \mathcal{K} = \infty)] = 1.$$

Being unbounded is equivalent to saying:

there exists $\mathcal{K}' \subseteq \mathcal{K}$ such that

$$\begin{aligned} \mathbb{P}[\mathcal{K} \text{ io} \iff \mathcal{K}' \text{ io}] &= 1 \text{ and} \\ \mathbb{P}[\mathcal{K}' \text{ io} \Rightarrow (\liminf \mathcal{K}' = \infty)] &= 1. \end{aligned} \quad (6)$$

To see this take \mathcal{K}' to be the record breakers for the forward implication; the converse is immediate. Further, the condition (6) (and therefore also (*)) is definable in $\text{MSO}+\nabla$, since the second conjunct is the complement of the property from Lemma IV.1.

The implication (*) \Rightarrow (5) is trivial. For the converse implication, we take \mathcal{K} to be the record breakers. Then the family \mathcal{K} is unbounded and we have (*) because for every branch π , $\limsup \mathcal{I}(\pi) = \infty$ if and only if \mathcal{K} appears infinitely often in π . \square

V. EVENTUALLY CONSTANT INTERVALS

We call elements of \mathbb{N}^ω *number sequences*. They are denoted by f, g, h . In the previous section, we have essentially encoded number sequences on branches using intervals, and demonstrated that the probabilistic quantifier can be used to say that the encoded number sequences are bounded (they have finite lim sup) almost surely. How is boundedness useful for expressing the eventually constant language in Theorem III.2? To answer this question, we first need to define asymptotic mixes. The ideas are borrowed from the proof of undecidability of $\text{MSO}+\cup$ in [9].

If $X = \{x_0 < x_1 < \dots\} \subseteq \mathbb{N}$ then by $f|_X$ we denote the subsequence of f taking only positions from X , i.e. $f|_X = (f(x_0), f(x_1), \dots) \in \mathbb{N}^* \cup \mathbb{N}^\omega$.

Definition V.1 (Asymptotic equivalence). Given $f, g \in \mathbb{N}^\omega$, we say that f is *asymptotically equivalent* to g , denoted $f \sim g$, if f and g are bounded on the same sets of positions, i.e. for all $X \subseteq \mathbb{N}$, either both $f|_X$ and $g|_X$ are bounded or both are unbounded. If f is not asymptotically equivalent to g we write $f \not\sim g$.

A *vector sequence* is an element of $(\mathbb{N}^+)^{\omega}$, e.g.:

$$(4, 7, 6) (2, 3) (10) (1, 1, 1) \dots$$

We denote vector sequences by $\mathbf{f}, \mathbf{g}, \mathbf{h}$. We say that a number sequence $f \in \mathbb{N}^\omega$ is an *extraction* of \mathbf{f} (denoted $f \in \mathbf{f}$) if for each $n \in \mathbb{N}$ the number $f(n)$ is a component of $\mathbf{f}(n)$ (written simply $f(n) \in \mathbf{f}(n)$).

Definition V.2 (Asymptotic mix). Given two vector sequences \mathbf{f}, \mathbf{g} we say that \mathbf{f} is an *asymptotic mix* of \mathbf{g} if for all $f \in \mathbf{f}$ there exists $g \in \mathbf{g}$ such that $f \sim g$.

A vector sequence \mathbf{f} has *dimension* d if every vector in it has dimension d . Notice that each vector of a vector sequence must be non-empty and therefore, $d \geq 1$ always. The following lemma (that we state without a proof) makes a crucial connection between the dimension and asymptotic mixes, the latter being

a property of boundedness of the components of vector sequences.

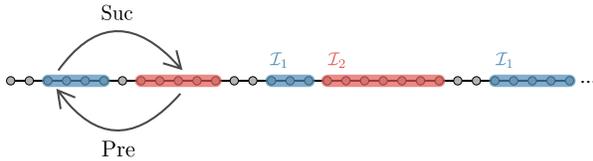
Lemma V.3 ([9] Lemma 2.1). *Let $d \in \mathbb{N}$, $d > 0$. There exists a vector sequence of dimension d which is not an asymptotic mix of any vector sequence of dimension $d - 1$ (nor any smaller dimension).*

We will encode vector sequences with two families of intervals \mathcal{K} and \mathcal{I} , by *wrapping* the former over the latter. The lengths of \mathcal{K} will encode the dimensions, and those of \mathcal{I} will encode the components. We want to express that \mathcal{K} is eventually constant. The rough idea is as follows. If \mathcal{K} is not eventually constant then it must alternate between two lengths (we can ask for it to be bounded), say 5 and 3. We then check whether this is the case by employing Lemma V.3.

But it is not yet clear how we are to express asymptotic equivalence and mixes in $\text{MSO} + \nabla$, so we do this first in the next two technical subsections.

A. Asymptotic equivalence

Consider MSO on infinite words for a moment. Suppose that we encode two number sequences with families of intervals $\mathcal{I}_1, \mathcal{I}_2$. *A priori* it is not possible to express $\mathcal{I}_1 \sim \mathcal{I}_2$ in the logic⁹, unless we impose some restriction, such that there is some MSO definable function that given the n th interval of \mathcal{I}_1 outputs the position of the n th interval of \mathcal{I}_2 . The simplest way of having this is to require that the intervals in \mathcal{I}_1 and \mathcal{I}_2 are alternating:



If $\mathcal{I}_1, \mathcal{I}_2$ are arranged in such a way, the functions Pre and Suc are MSO definable (the first neighbour to the left, or right respectively) and hence we are able to quantify over subsequences which enables us to express asymptotic equivalence in the logic.

For trees we have the following definitions.

We call two families of intervals $\mathcal{I}_1, \mathcal{I}_2$ *isolated* if $\bigcup \mathcal{I}_1 \cap \bigcup \mathcal{I}_2 = \emptyset$, *i.e.* there is no node that belongs both to an interval in \mathcal{I}_1 and an interval in \mathcal{I}_2

Definition V.4 (Precedes). Let $\mathcal{I}_1, \mathcal{I}_2$ be isolated families of intervals. We say that \mathcal{I}_1 *precedes* \mathcal{I}_2 if for all $x' \in \text{source}(\mathcal{I}_2)$ there exists $x \in \text{source}(\mathcal{I}_1)$ such

⁹Even if we are allowed to speak about boundedness.

that $x < x'$ and there is no node strictly between x and x' that is a source of \mathcal{I}_1 or \mathcal{I}_2 .

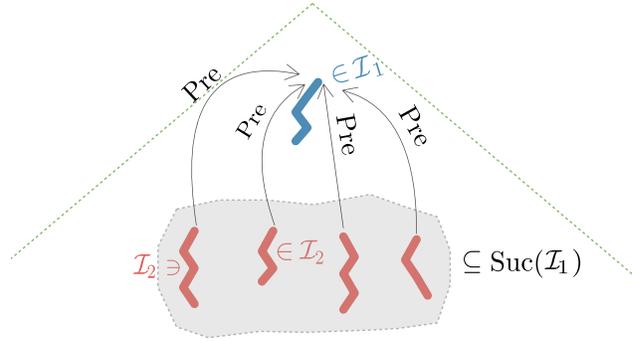
The fact that \mathcal{I}_1 precedes \mathcal{I}_2 induces a function $\text{Pre}: \text{source}(\mathcal{I}_2) \rightarrow \text{source}(\mathcal{I}_1)$ that maps $x' \mapsto x$ as in the definition above. Additionally, for a family $\mathcal{I} \subseteq \mathcal{I}_1$, we define:

$$\text{Suc}(\mathcal{I}) \stackrel{\text{def}}{=} \{[x', y'] \in \mathcal{I}_2 : \text{Pre}(x') \in \text{source}(\mathcal{I})\} \subseteq \mathcal{I}_2,$$

and dually, for $\mathcal{I} \subseteq \mathcal{I}_2$ we put

$$\text{Pre}(\mathcal{I}) \stackrel{\text{def}}{=} \{[x, y] \in \mathcal{I}_1 : \exists x' \in \text{source}(\mathcal{I}). \text{Pre}(x') = x\}.$$

For the sake of readability we will use the functions Pre and Suc without additional parameters, assuming that the families \mathcal{I}_1 and \mathcal{I}_2 are known from the context. The picture on trees looks as follows:



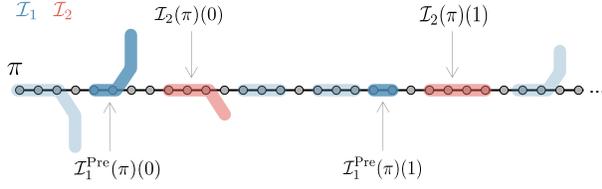
In a branch π , it might be the case that between consecutive intervals in \mathcal{I}_2 , there are many sources of intervals from \mathcal{I}_1 , so the encoding of the two sequences is not alternating, hence the following definition.

Definition V.5 (Preceding subsequence). Let $\mathcal{I}_1, \mathcal{I}_2$ be isolated families of intervals such that \mathcal{I}_1 precedes \mathcal{I}_2 . Assume that π is a branch where \mathcal{I}_2 appears infinitely often. By $\mathcal{I}_1^{\text{Pre}}(\pi)$ we denote the subsequence of $\mathcal{I}_1(\pi)$ that we get by applying \mathcal{I}_1 only to the nodes x for which there exists $x' \in \pi \cap \text{source}(\mathcal{I}_2)$ such that $\text{Pre}(x') = x$.

Notice that in the above definition we require x' to belong to π , *a priori* we might have $\text{Pre}(x') = x$ for some $x' \in \text{source}(\mathcal{I}_2)$ outside π but for no such node in π (in that case $\mathcal{I}_1(x)$ is not taken into $\mathcal{I}_1^{\text{Pre}}(\pi)$). Observe additionally that if \mathcal{I}_1 precedes \mathcal{I}_2 and \mathcal{I}_2 appears infinitely often in a branch π then $\mathcal{I}_1^{\text{Pre}}(\pi)$ is a number sequence (*i.e.* it is infinite). However, we are not claiming that $\mathcal{I}_1^{\text{Pre}}$ is a family of intervals.

Typically, on a branch π where \mathcal{I}_2 appears infinitely often we have: a few intervals of \mathcal{I}_1 then one interval in \mathcal{I}_2 and so on. The sequence $\mathcal{I}_1^{\text{Pre}}(\pi)$ is taking into

account only the intervals that immediately precede those of \mathcal{I}_2 . It looks as follows:



Remark V.6. Consider $\mathcal{I}_1, \mathcal{I}_2$ two isolated families of intervals such that \mathcal{I}_1 precedes \mathcal{I}_2 . Let π be a branch on which \mathcal{I}_2 appears infinitely often. In that case the two sequences $\mathcal{I}_1^{\text{Pre}}(\pi)$ and $\mathcal{I}_2(\pi)$ are both defined. Let $x'_k \in \pi \cap \text{source}(\mathcal{I}_2)$ be the k th source of an interval in \mathcal{I}_2 on π (it has $k - 1$ strict ancestors in $\text{source}(\mathcal{I}_2)$). Then, by the definitions of the respective sequences:

$$\begin{aligned} \mathcal{I}_2(\pi)(k) &= \mathcal{I}_2(x'_k), \\ \mathcal{I}_1^{\text{Pre}}(\pi)(k) &= \text{Pre}(\mathcal{I}_2)(\text{Pre}(x'_k)). \end{aligned}$$

This means that the two number sequences are in a sense *synchronised* and the function Pre maps between the corresponding sources.

In other words, number sequence encodings \mathcal{I}_2 and $\mathcal{I}_1^{\text{Pre}}$ are alternating as in the case of infinite words, which facilitates quantifying over their subsequences.

As a consequence it is easier to express asymptotic equivalence between $\mathcal{I}_2(\pi)$ and $\mathcal{I}_1^{\text{Pre}}(\pi)$.

Lemma V.7. *Let $\mathcal{I}_1, \mathcal{I}_2$ be isolated families of intervals, such that \mathcal{I}_1 precedes \mathcal{I}_2 . Then we can express in MSO+ ∇ that:*

$$\mathbb{P}[\mathcal{I}_2 \text{ io} \wedge \mathcal{I}_1^{\text{Pre}} \not\sim \mathcal{I}_2] > 0.$$

The formula used to express the property in the lemma above utilizes the fact that Pre and Suc are MSO-definable and quantifies over subsets of $\text{Pre}(\mathcal{I}_2)$ and \mathcal{I}_2 . The proof is in Appendix A1.

B. A characterization of asymptotic mixes

Having built tools to express asymptotic equivalence, we now move on to asymptotic mixes. In this section we give the definition of *separation* which is equivalent to asymptotic mixes.

Remark V.8. The reason why we give this equivalent definition of asymptotic mixes is that it will allow us in the sequel to partition certain sets of branches into countably many subsets (one for each bound

b), for the purpose of then using the \aleph_0 -additivity of the measure. Thereby allowing us to *pull out* one existential quantifier.

For a vector sequence \mathbf{f} denote by $\min(\mathbf{f}) \in \mathbf{f}$ (respectively $\max(\mathbf{f}) \in \mathbf{f}$) the number sequences that pick the minimal (respectively maximal) component of every vector. For a number sequence $f \in \mathbb{N}^\omega$ and $b \in \mathbb{N}$ we write $f \leq b$ if for all $n \in \mathbb{N}$ we have $f(n) \leq b$.

Definition V.9 (Separation). Let \mathbf{f}, \mathbf{g} be two vector sequences and $b \in \mathbb{N}$. We say that b *separates* \mathbf{f} from \mathbf{g} if one of the following holds:

- $\exists X. \min(\mathbf{f}|_X) \leq b$ and $\min(\mathbf{g}|_X)$ is unbounded,
- $\exists X. \max(\mathbf{g}|_X) \leq b$ and $\max(\mathbf{f}|_X)$ is unbounded.

Lemma V.10. *Let \mathbf{f}, \mathbf{g} be two vector sequences. Then \mathbf{f} is not an asymptotic mix of \mathbf{g} if and only if there exists $b \in \mathbb{N}$ that separates \mathbf{f} from \mathbf{g} .*

Proof. We start with the forward implication. Given a number sequence f we define the *best response* $g_f \in \mathbf{g}$ for $n \in \mathbb{N}$ as

$$g_f(n) = \arg \min_{x \in \mathbf{g}(n)} |f(n) - x|.$$

So g_f is the choice of components in \mathbf{g} that minimize the distance to f .

Since \mathbf{f} is not an asymptotic mix of \mathbf{g} , there exists $f \in \mathbf{f}$ such that for all $g \in \mathbf{g}$, $f \not\sim g$; in particular we have $f \not\sim g_f$. This means that there exists $X \subseteq \mathbb{N}$ such that one of the following holds:

- $f|_X$ is bounded and $g_f|_X$ is unbounded,
- $g_f|_X$ is bounded and $f|_X$ is unbounded.

By the definition of g_f , in the first case $\min(\mathbf{g}|_X)$ is unbounded while $\min(\mathbf{f}|_X)$ is clearly bounded (by some $b \in \mathbb{N}$). In the second case we have $\max(\mathbf{g}|_X) \leq b$ for some b while $\max(\mathbf{f}|_X)$ is unbounded. Therefore, there exists $b \in \mathbb{N}$ that separates \mathbf{f} from \mathbf{g} .

For the backward implication, assume that b separates \mathbf{f} from \mathbf{g} . In the first case of Definition V.9 it suffices to construct $f \in \mathbf{f}$ by picking a component smaller than b if it exists, and an arbitrary component otherwise. In the second case, we pick the maximal component. \square

C. Wrappings

Let us now explain in more detail how vector sequences are encoded using families of intervals. Recall the definition of Int from page 3.

Definition V.11 (Wrappings). Let \mathcal{I}, \mathcal{K} be families of intervals. We say that \mathcal{K} *wraps* \mathcal{I} if $\text{Int}(\mathcal{K}) =$

source(\mathcal{I}) and for each interval $[x, y] \in \mathcal{K}$ we have $\text{Len}([x, y]) \geq 1$.

Let \mathcal{I}, \mathcal{K} be families of intervals such that \mathcal{K} wraps \mathcal{I} and take $[x, y] \in \mathcal{K}$. Then $\text{Int}([x, y]) = \{x_1, x_2, \dots, x_{\mathcal{K}(x)}\}$ such that $x < x_1 < \dots < x_{\mathcal{K}(x)} < y$ and $\mathcal{K}(x) \geq 1$. All the x_i s are sources of some intervals in \mathcal{I} . Define:

$$\vec{\mathcal{K}}(\mathcal{I}, x) = (\mathcal{I}(x_1), \mathcal{I}(x_2), \dots, \mathcal{I}(x_{\mathcal{K}(x)})).$$

Extend this definition to branches π in such a way that if \mathcal{K} appears infinitely often in π then $\vec{\mathcal{K}}(\mathcal{I}, \pi)$ is a vector sequence: if $\pi \cap \text{source}(\mathcal{K}) = \{x_0 < x_1 < \dots\}$ then $\vec{\mathcal{K}}(\mathcal{I}, \pi)(k)$ equals $\vec{\mathcal{K}}(\mathcal{I}, x_k)$.

In this way we can encode vector sequences using two families of intervals \mathcal{I}, \mathcal{K} . The lengths of intervals in the outer layer \mathcal{K} are the dimensions of the vectors, while the lengths of the intervals in \mathcal{I} are the components. We illustrate this in Figure 2.

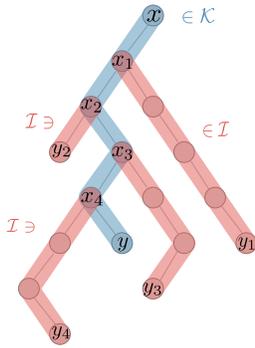


Figure 2. In this partial tree the set $[x, y]$ is an interval in \mathcal{K} , and $[x_i, y_i]$ are intervals in \mathcal{I} , $1 \leq i \leq 4$. We have $\mathcal{K}(x) = 4$, $\mathcal{I}(x_1) = 3$, $\mathcal{I}(x_2) = 0$, $\mathcal{I}(x_3) = 2$, and $\mathcal{I}(x_4) = 2$. The vector that is encoded in x is $\vec{\mathcal{K}}(\mathcal{I}, x) = (3, 0, 2, 2)$.

Just as in the case of asymptotic equivalence above, it is not possible to speak of whether a vector sequence that is encoded using families of intervals as above is an asymptotic mix of another, without imposing some structure. For this purpose, for number sequences we had the notion of a family preceding another in Definition V.4, for vector sequences we use a stronger condition.

Definition V.12 (Tail-precedes). Let $\mathcal{K}_1, \mathcal{K}_2$ be isolated families of intervals. We say that \mathcal{K}_1 *tail-precedes* \mathcal{K}_2 if for all $x' \in \text{source}(\mathcal{K}_2)$ there exists $y \in \text{target}(\mathcal{K}_1)$ such that $y < x'$ and there is no node strictly between y and x' that is a source of \mathcal{K}_1 or \mathcal{K}_2 .

Note that tail-preceding is a stronger property than preceding given in Definition V.4, therefore if \mathcal{K}_1 tail-precedes \mathcal{K}_2 , and \mathcal{K}_2 appears infinitely often in some branch π then the sequences $\mathcal{K}_1^{\text{Pre}}(\pi)$ and $\vec{\mathcal{K}}_1^{\text{Pre}}(\pi)$ are well-defined. This enables us to talk about asymptotic mixes and to apply Lemma V.7.

However, what guarantees that the relevant families of intervals are structured in such a way, *i.e.* one tail-preceding the other. This is the subject of the next lemma, which essentially says that if \mathcal{K} is bounded but not eventually constant then it is possible to find two subfamilies each of certain lengths such that one tail-precedes the other.

Lemma V.13. *Let \mathcal{K} be a family of intervals such that*

$$\mathbb{P} \left[\wedge \begin{cases} \mathcal{K} \text{ io} \\ \limsup \mathcal{K} < \infty \\ \mathcal{K} \text{ is not eventually constant} \end{cases} \right] > 0.$$

Then there exist two numbers $\ell_1 > \ell_2 \in \mathbb{N}$ and isolated $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{K}$ such that:

- every interval in \mathcal{K}_1 has length ℓ_1 ,
- every interval in \mathcal{K}_2 has length ℓ_2 ,
- \mathcal{K}_1 tail-precedes \mathcal{K}_2 , and
- $\mathbb{P}[\mathcal{K}_2 \text{ io}] > 0$.

Proof. We have assumed that there is a non-zero probability of picking a branch π such that $\mathcal{K}(\pi)$ is a sequence that is infinite, bounded, and not eventually constant. This means that with a positive probability there are two numbers that both appear infinitely often in the sequence $\mathcal{K}(\pi)$, *i.e.* the set

$$\left\{ \pi : \begin{cases} \exists \ell_1 > \ell_2 \in \mathbb{N}. \\ \mathcal{K}(\pi) \text{ contains infinitely often } \ell_1 \text{ and } \ell_2 \end{cases} \right\},$$

has non-zero probability. Consequently, as there are countably many choices of $\ell_1 > \ell_2 \in \mathbb{N}$, there exist two numbers $\ell_1 > \ell_2 \in \mathbb{N}$ such that:

$$\mathbb{P}[\mathcal{K} \text{ contains infinitely often } \ell_1 \text{ and } \ell_2] > 0.$$

Let $\mathcal{I}_1 \subseteq \mathcal{K}$ (respectively $\mathcal{I}_2 \subseteq \mathcal{K}$) be the intervals in \mathcal{K} whose length is ℓ_1 (respectively ℓ_2). The probability that both \mathcal{I}_1 and \mathcal{I}_2 appear infinitely often is non-zero. This means that:

$$\mathbb{P}[\text{source}(\mathcal{I}_1) \text{ io} \wedge \text{source}(\mathcal{I}_2) \text{ io}] > 0.$$

From Claim IV.2 we have:

$$\mathbb{P}[\text{target}(\mathcal{I}_1) \text{ io} \wedge \text{source}(\mathcal{I}_2) \text{ io}] > 0.$$

Now we prove that we can find subsets of $\mathcal{I}_1, \mathcal{I}_2$ for which the last two bullet points in the statement of the lemma hold.

Claim V.14. Let $X, Y \subseteq \mathbf{2}^*$ be such that $\mathbb{P}[X \text{ io} \wedge Y \text{ io}] > 0$. Then there exist $X' \subseteq X$ and $Y' \subseteq Y$ such that between any two nodes $x < y$ in Y' there exists a node $u \in \text{Int}([x, y])$ that belongs to X' and moreover $\mathbb{P}[Y' \text{ io} \wedge X' \text{ io}] > 0$.

Proof. We construct for all $n > 0$, sets $X_n \subseteq X$, $Y_n \subseteq Y$ and put $X' = \bigcup_{n>0} X_n$, $Y' = \bigcup_{n>0} Y_n$. For any node y we say that $x \in X$ is an X -successor of y if $x > y$ and there is no node strictly between x and y that is in X . Similarly we define Y -successors.

Let $Y_0 = \{\epsilon\}$ where ϵ is the root node and define for all $n > 0$:

$$X_n \stackrel{\text{def}}{=} \bigcup_{y \in Y_{n-1}} \{x \in X : x \text{ is an } X\text{-successor of } y\},$$

$$Y_n \stackrel{\text{def}}{=} \bigcup_{x \in X_n} \{y \in Y : y \text{ is a } Y\text{-successor of } x\}.$$

We can easily observe that for X', Y' constructed this way we have that between every two nodes in Y' there is always a node in X' (in fact, also symmetrically, the nodes in X' are separated by nodes in Y'). Let π be a branch where both X and Y appear infinitely often. Then the first non-root node in this branch that belongs to X belongs to X_1 , after which the first node that belongs to Y belongs to Y_1 , and so on. Consequently both X' and Y' also appear infinitely often in π . Therefore, $\mathbb{P}[Y' \text{ io} \wedge X' \text{ io}] > 0$. \square

Set $X = \text{target}(\mathcal{I}_1)$, $Y = \text{source}(\mathcal{I}_2)$ and apply Claim V.14 resulting in $X' \subseteq X$ and $Y' \subseteq Y$. We set \mathcal{K}_1 (respectively \mathcal{K}_2) to be the intervals whose targets are in X' (respectively sources in Y'). The statement of the lemma now can be deduced from the properties of X' and Y' . \square

Given families of intervals \mathcal{I}, \mathcal{K} such that the latter wraps the former, we are now able to sketch how to express in $\text{MSO} + \nabla$ that \mathcal{K} is eventually constant.

Proposition V.15. Let \mathcal{I}, \mathcal{K} be two families of intervals such that \mathcal{K} wraps \mathcal{I} and we have:

$$\mathbb{P} \left[\begin{cases} \mathcal{K} \text{ io} & \Rightarrow \limsup \mathcal{K} < \infty, \text{ and} \\ \mathcal{I} \text{ io} & \Rightarrow \liminf \mathcal{I} = \infty \end{cases} \right] = 1.$$

Then the following property is definable in $\text{MSO} + \nabla$:

$$\mathbb{P}[\mathcal{K} \text{ io} \wedge \mathcal{K} \text{ is not eventually constant}] > 0.$$

Let \mathcal{I}, \mathcal{K} be such that \mathcal{K} wraps \mathcal{I} . We say that $\mathcal{I}' \subseteq \mathcal{I}$ is an *extraction* of $(\mathcal{K}, \mathcal{I})$ if for all $[x, y] \in \mathcal{K}$ there is exactly one element of $\text{source}(\mathcal{I}')$ that belongs to $\text{Int}([x, y])$.

We write $\mathcal{I}_1 \leq \mathcal{I}_2$ if the sources of the two families of intervals coincide and the targets of \mathcal{I}_1 are ancestors of the targets of \mathcal{I}_2 , *i.e.* for every interval $[x, y] \in \mathcal{I}_1$ there is an interval $[x, y'] \in \mathcal{I}_2$ such that $[x, y] \subseteq [x, y']$ (equivalently $y \leq y'$).

We claim that the statement in the proposition is equivalent to the following:

(*) there exist isolated $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{K}$, where \mathcal{K}_1 tail-precedes \mathcal{K}_2 , $\mathbb{P}[\mathcal{K}_2 \text{ io}] > 0$, and if $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{I}$ are such that \mathcal{K}_i wraps \mathcal{I}_i , $i \in \{1, 2\}$ then:

$$\begin{aligned} \exists \mathcal{I}'_1 \leq \mathcal{I}_1. \quad \forall \mathcal{I}'_2 \leq \mathcal{I}_2. \\ \exists \mathcal{J}_1 \subseteq \mathcal{I}'_1 \text{ extraction of } (\mathcal{K}_1, \mathcal{I}'_1). \\ \forall \mathcal{J}_2 \subseteq \mathcal{I}'_2 \text{ extraction of } (\mathcal{K}_2, \mathcal{I}'_2). \\ \mathbb{P}[\mathcal{J}_2 \text{ io} \wedge \mathcal{J}_1^{\text{Pre}} \not\sim \mathcal{J}_2] > 0. \end{aligned} \quad (7)$$

It is not hard to see that (*) is $\text{MSO} + \nabla$ -definable, see Appendix B. For (7) use Lemma V.7.

Roughly, the intuition behind this proposition is as follows. The statement of the proposition can be equivalently written as: there exist two numbers $\ell_1 > \ell_2$ such that with nonzero probability \mathcal{K} alternates between them. But this property is hard to express in our logic; it requires counting to make sure that $\ell_1 > \ell_2$. To remedy this difficulty we make use of Lemma V.3. This lemma provides us with an important equivalence between a property that is hard to express (a) $\ell_1 > \ell_2$ and a property that we can express in our logic more easily: (b) there exists a vector sequence of dimension ℓ_1 that is not an asymptotic mix of any vector sequence of dimension ℓ_2 .

We start with an explanation of (*) and then proceed to give a sketch of the proof. The complete proof can be found in Appendix A2.

The families of intervals $\mathcal{K}_1 \subseteq \mathcal{K}$ and $\mathcal{K}_2 \subseteq \mathcal{K}$ are meant to represent two families of eventually constant intervals of two distinct lengths $\ell_1 > \ell_2$, as in Lemma V.13. Once \mathcal{K}_1 and \mathcal{K}_2 are fixed, the families \mathcal{I}_1 and \mathcal{I}_2 are defined uniquely as the families of those intervals in \mathcal{I} that are wrapped by some intervals in \mathcal{K}_1 and \mathcal{K}_2 respectively. With $\mathcal{I}'_1 \leq \mathcal{I}_1$ we will imitate the vector sequence \mathbf{f} of dimension ℓ_1 that is not an asymptotic mix of any vector sequence \mathbf{g} of dimension ℓ_2 (it exists because of Lemma V.3). The rest of (*) expresses that \mathbf{f} is not an asymptotic mix of \mathbf{g} . Thus, \mathcal{J}_1 represents a choice of $f \in \mathbf{f}$, while

\mathcal{J}_2 represents a choice of $g \in \mathbf{g}$. Finally, the last line of (*) (see (7)) says that $f \not\sim g$. Note here that, the fact that \mathcal{K}_1 tail-precedes \mathcal{K}_2 implies that \mathcal{J}_1 precedes \mathcal{J}_2 , so $\mathcal{J}_1^{\text{Pre}}$ is well-defined.

(\Rightarrow) The idea for the forward implication follows the explanation given above. We construct $\mathcal{K}_1, \mathcal{K}_2$ of respective lengths ℓ_1 and ℓ_2 using Lemma V.13. From Lemma V.3, we set \mathbf{f} to be a vector sequence of dimension ℓ_1 that is not an asymptotic mix of any vector sequence of dimension ℓ_2 . The assumption that $\mathbb{P}[\mathcal{I} \text{ io} \Rightarrow (\liminf \mathcal{I} = \infty)] = 1$ guarantees that the intervals in \mathcal{I}_1 and \mathcal{I}_2 are *long*, so with $\mathcal{I}'_1 \leq \mathcal{I}_1$ we are able imitate the vector sequence \mathbf{f} while the choice of \mathcal{I}'_2 represents a vector sequence \mathbf{g} .

At this point, to facilitate (see Remark V.8) the construction of \mathcal{J}_1 we use the equivalence between separation and asymptotic mixes described in Lemma V.10. The proof is finalized by doing a case analysis of the two cases in the definition of separation: Definition V.9. Depending on the case, we fix the extraction \mathcal{J}_1 either by picking intervals of length as small (in the first case) or as big (in the latter case) as possible from \mathcal{I}'_1 .

(\Leftarrow) The converse implication is easier, it relies on *copying*. We assume that almost surely whenever \mathcal{K} appears infinitely often then it is eventually constant (the negation of the first statement) and use this to refute the second statement. This is done by copying in the following sense. When $\mathcal{I}'_1 \leq \mathcal{I}_1$ is fixed, we find a family $\mathcal{I}'_2 \leq \mathcal{I}_2$ that copies the choice made in \mathcal{I}'_1 ; and the same for restrictions \mathcal{J}_2 based on \mathcal{J}_1 . In the end, in almost every branch we will have number sequences that are asymptotically equivalent, refuting the last line in (7). This terminates the (sketch of the) proof of Proposition V.15.

It is not hard to remove the assumption in Proposition V.15 so as to get Theorem III.2. It suffices to quantify existentially over the wrapped interval \mathcal{I} and make sure that \mathcal{K} is *sufficiently spaced*. The details can be found in Appendix A3.

VI. REDUCING TWO-COUNTER MACHINES WITH ZERO TESTS

A two-counter machine has a finite set of control states and two counters, which can be increased, decreased, and tested for zero. The question of whether such a machine has a halting run, is undecidable. In this section we will demonstrate that given a two-counter machine \mathcal{M} , we can effectively construct a formula $\Phi(\mathcal{M})$ such that \mathcal{M} has a halting run if and only if $\Phi(\mathcal{M})$ is true.

The reduction is relatively standard once equipped with Theorem III.2. The reason being that Theorem III.2 already allows us to do arithmetic in an asymptotic sense: Suppose that $\mathcal{I}_1, \mathcal{I}_2$ are two families of intervals that are isolated (that is $\bigcup \mathcal{I}_1 \cap \bigcup \mathcal{I}_2 = \emptyset$) and eventually constant on almost every branch. This means that for almost every branch π , $\mathcal{I}_i(\pi)$ defines some natural number $L_i(\pi)$. With the help of Theorem III.2 we can express, for instance, that for almost every branch π , $L_1(\pi) = L_2(\pi) + 1$, or that $L_i(\pi) = 0$ as follows.

Lemma VI.1. *Let $\mathcal{I}_1, \mathcal{I}_2$ be isolated families of intervals that appear infinitely often and are eventually constant almost surely. For almost every branch π , $\mathcal{I}_i(\pi)$ is eventually constant, equal to some number, say $L_i(\pi)$. Then, one can express in $\text{MSO} + \nabla$ the following:*

$$\mathbb{P}[L_1 = 0] = 1 \quad (8)$$

$$\mathbb{P}[L_1 = L_2 + 1] = 1. \quad (9)$$

Proof. Condition (8) is directly formalisable in $\text{MSO} + \nabla$. The formula says that for almost every branch π , after some threshold, every node in $\text{source}(\mathcal{I}_1) \cap \pi$ has a child that is in $\text{target}(\mathcal{I}_1)$.

As for Condition (9), first we can easily express that almost surely $L_1(\pi) > 0$ (this is a necessary condition for (9)). If this is the case, then we define another family \mathcal{I}_3 , such that $\text{source}(\mathcal{I}_3) = \text{source}(\mathcal{I}_1)$ and the targets $\text{target}(\mathcal{I}_3)$ are exactly the parents of the nodes in $\text{target}(\mathcal{I}_1)$, *i.e.* we move the targets of \mathcal{I}_1 to their parents thereby decreasing the lengths of intervals by 1. The set $\text{target}(\mathcal{I}_3)$ (and therefore \mathcal{I}_3) is MSO -definable and moreover for $L_3(\pi)$ defined analogously, we have $L_3(\pi) = L_1(\pi) - 1$ for almost every branch π .

Thus, to verify that $L_3(\pi) = L_2(\pi)$, *i.e.* Condition (9), it is enough to check that $\mathcal{I}_3 \cup \mathcal{I}_2$ is eventually constant, almost surely by applying Theorem III.2. This is possible because \mathcal{I}_3 and \mathcal{I}_2 are disjoint and therefore $\mathcal{I}_3 \cup \mathcal{I}_2$ is a valid family of intervals. \square

We now illustrate how a run of \mathcal{M} is encoded. It is of the form

$$(q_1, c_1^1, c_1^2), (q_2, c_2^1, c_2^2), \dots, (q_\ell, c_\ell^1, c_\ell^2),$$

where q_k are control states and c_k^1, c_k^2 is the value of the first and second counter on the k th step respectively. We will encode such a run using three families of intervals: $\mathcal{K}, \mathcal{I}_1, \mathcal{I}_2$ and a labelling by states. \mathcal{K} will be eventually constant and equal to ℓ (the length of the run), and the nodes in $\text{Int}(\mathcal{K})$ will be

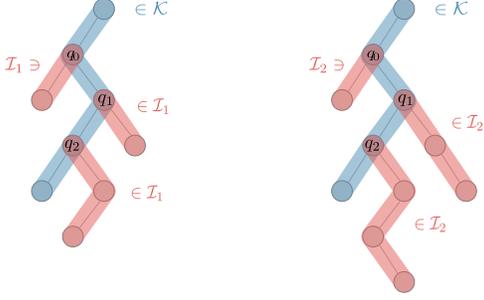


Figure 3. $\text{Int}(\mathcal{K})$ is labeled by the states q_i . Note the lengths of the intervals in \mathcal{I}_i . On the left we have the respective lengths 0,0 and 1 corresponding to the history of counter 1 in the run. On the right: 0,1 and 2.

labeled by the control states of \mathcal{M} . In other words, intervals $[x, y] \in \mathcal{K}$ will be such that $\text{Int}([x, y]) = \{x_1 < \dots < x_\ell\}$ and x_k is labeled by q_k . Further, \mathcal{K} will wrap both \mathcal{I}_1 and \mathcal{I}_2 , and x_k will be the source of an interval in \mathcal{I}_1 of length c_k^1 (i.e. $\mathcal{I}_1(x_k) = c_k^1$), as well as the source of an interval in \mathcal{I}_2 of length c_k^2 (i.e. $\mathcal{I}_2(x_k) = c_k^2$).

Example VI.2. Consider the run:

$$(q_0, 0, 0), (q_1, 0, 1), (q_2, 1, 2).$$

Its encoding with intervals is depicted in Figure 3.

In order to ascertain that a run is valid, we need to check whether the counters are being increased and decreased correctly using Lemma VI.1. For this purpose it is necessary to be able to speak about, for instance, the value of counter 1 at step k by choosing the correct subset of \mathcal{I}_1 . This can be done as follows.

If $\mathcal{I}'_1 \subseteq \mathcal{I}_1$ is a family of intervals that is an extraction of $(\mathcal{K}, \mathcal{I}_1)$, we say that $\mathcal{K}' \leq \mathcal{K}$ is *induced* by \mathcal{I}'_1 if $\text{target}(\mathcal{K}') = \text{source}(\mathcal{I}'_1)$. We say that \mathcal{I}'_1 is a *component selector* of \mathcal{I}_1 if \mathcal{K}' induced by \mathcal{I}'_1 is eventually constant with probability 1. In such a case, the lengths of the intervals in \mathcal{I}'_1 (from some moment on, along almost every branch of the tree) correspond to the values of the counter 1 at a certain step. In other words, \mathcal{I}'_1 is a component selector if on almost every branch π , there exists a number $k \in \mathbb{N}$ such that \mathcal{I}'_1 is eventually choosing exactly the k th component.

The following proposition follows directly from the ability to express Conditions (8) and (9).

Proposition VI.3. *For every two-counter machine with zero tests \mathcal{M} , we can effectively compute a formula $\phi(\mathcal{M})$ of $\text{MSO}+\nabla$, such that $\phi(\mathcal{M})$ is true if and only if \mathcal{M} halts.*

Proof. The first part of the formula $\phi(\mathcal{M})$ says: there exist families of intervals \mathcal{K} , \mathcal{I}_1 , and \mathcal{I}_2 and a labelling ρ of $\text{Int}(\mathcal{K})$ by states of \mathcal{M} such that:

- \mathcal{K} appears infinitely often and is eventually constant almost surely,
- \mathcal{K} wraps both \mathcal{I}_1 and \mathcal{I}_2 , and
- every component selector \mathcal{I}' of either \mathcal{I}_1 or \mathcal{I}_2 on almost every branch is eventually constant and the labels of ρ in the nodes $\text{source}(\mathcal{I}')$ stabilise almost surely.

This implies that for $i = 1, 2$ and almost every branch π , $\vec{\mathcal{K}}(\mathcal{I}_i, \pi)$ is a vector sequence that is eventually constant equal to some vector $(c_1^i, c_2^i, \dots, c_\ell^i)(\pi)$. Moreover, on almost every branch π the labels of the nodes in $\text{Int}(\mathcal{K})$ must also stabilise to some sequence $(q_1, \dots, q_\ell)(\pi)$.

The second part of the formula uses component selectors as well as the conditions from Lemma VI.1 to test the relationship between the values $(c_n^1, c_n^2, q_n, c_{n+1}^1, c_{n+1}^2, q_{n+1})(\pi)$ to verify that on almost every branch $(c_1^1, \dots, c_\ell^1)(\pi)$, $(c_1^2, \dots, c_\ell^2)(\pi)$, and $(q_1, \dots, q_\ell)(\pi)$ is a valid run of \mathcal{M} . This is done by requiring that the values of counters and the labeling in any two consecutive component selectors respect the transition relation of \mathcal{M} .

If the formula is true then the witnessing families \mathcal{K} , \mathcal{I}_1 , \mathcal{I}_2 , and a labelling ρ must almost surely encode (the unique) accepting run of \mathcal{M} . Conversely, if \mathcal{M} has an accepting run then one can easily choose families as above such that each interval $[x, y] \in \mathcal{K}$ encodes in fact this single run. This implies that the above $\text{MSO}+\nabla$ formula must be true in that case. \square

Theorem III.1 is a corollary of Proposition VI.3.

VII. CONCLUSIONS

The undecidability result from this paper, together with the undecidability results about $\text{MSO}+\text{U}$ from [9], [6], lead to the following fundamental question: is there *any* quantifier that can be added to MSO on infinite words (or trees), while retaining decidability? Of course a negative answer would require formalising what “quantifier” means. A natural direction is to use the abstract approach from [16], which precludes positive answers that involve adding unary predicates as discussed in [20].

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APPENDIX

A. Proofs in Section V

1) Proof of Lemma V.7:

Lemma V.7. *Let $\mathcal{I}_1, \mathcal{I}_2$ be isolated families of intervals, such that \mathcal{I}_1 precedes \mathcal{I}_2 . Then we can express in $\text{MSO}+\nabla$ that:*

$$\mathbb{P}[\mathcal{I}_2 \text{ io} \wedge \mathcal{I}_1^{\text{Pre}} \not\sim \mathcal{I}_2] > 0.$$

Proof. We claim that the property in the statement of the lemma is equivalent to

(*) either

(†₁) there exists $\mathcal{I} \subseteq \text{Pre}(\mathcal{I}_2)$ such that

$$\mathbb{P} \left[\wedge \begin{cases} \text{Suc}(\mathcal{I}) \text{ io} \\ \limsup \mathcal{I} < \infty \\ \limsup \text{Suc}(\mathcal{I}) = \infty \end{cases} \right] > 0, \text{ or}$$

(†₂) there exists $\mathcal{I} \subseteq \mathcal{I}_2$ such that

$$\mathbb{P} \left[\wedge \begin{cases} \mathcal{I} \text{ io} \\ \limsup \mathcal{I} < \infty \\ \limsup \text{Pre}(\mathcal{I}) = \infty \end{cases} \right] > 0.$$

The property (*) is $\text{MSO}+\nabla$ -definable since Pre and Suc are MSO -definable and for checking the boundedness we can use Lemma IV.5. See Appendix B.

(\Rightarrow) Assume that there exists a set of branches $R \subseteq \mathbf{2}^\omega$ that has a non-zero probability, such that for each branch $\pi \in R$, \mathcal{I}_2 appears infinitely often in π and there exists a set of positions $X_\pi \subseteq \mathbb{N}$ on which the sequence $\mathcal{I}_1^{\text{Pre}}(\pi)$ is bounded but $\mathcal{I}_2(\pi)$ is not (the dual case is analogues, see below). By \aleph_0 -additivity of the measure, this implies that there exists $b \in \mathbb{N}$ such that:

$$\mathbb{P} \left[\begin{cases} \mathcal{I}_2 \text{ io, and} \\ \exists X \subseteq \mathbb{N}. \begin{cases} \mathcal{I}_1^{\text{Pre}} \upharpoonright_X \equiv b, \text{ and} \\ \limsup \mathcal{I}_2 \upharpoonright_X = \infty \end{cases} \end{cases} \right] > 0. \quad (10)$$

Take $\mathcal{I} \subseteq \text{Pre}(\mathcal{I}_2)$ as the family of intervals that have length equal to b . Take any branch π in the set from (10) and let $X_\pi \subseteq \mathbb{N}$ be a witness. Clearly, X_π must be infinite and therefore \mathcal{I} appears infinitely often in π and $\limsup \mathcal{I}(\pi) = b < \infty$. On the other hand, $\limsup \text{Suc}(\mathcal{I})(\pi) = \infty$ because $\text{Suc}(\mathcal{I})(\pi)$ contains as a subsequence the lengths of intervals in \mathcal{I}_2 that are measured in $\mathcal{I}_2 \upharpoonright_{X_\pi}$, see Remark V.6. It means in particular that $\text{Suc}(\mathcal{I})$ appears infinitely often in π . Therefore, (†₁) holds for \mathcal{I} and such π , which means that the probability there is positive.

In the dual case, when for each $\pi \in R$ there is X_π such that sequence $\mathcal{I}_1^{\text{Pre}}(\pi) \upharpoonright_{X_\pi}$ is unbounded but $\mathcal{I}_2(\pi) \upharpoonright_{X_\pi}$ is bounded, we know that there exists $b \in \mathbb{N}$ such that:

$$\mathbb{P} \left[\begin{cases} \mathcal{I}_2 \text{ io, and} \\ \exists X \subseteq \mathbb{N}. \begin{cases} \limsup \mathcal{I}_1^{\text{Pre}} \upharpoonright_X = \infty, \text{ and} \\ \mathcal{I}_2 \upharpoonright_X \equiv b \end{cases} \end{cases} \right] > 0. \quad (11)$$

In that case we take $\mathcal{I} \subseteq \mathcal{I}_2$ as the family of intervals of length equal to b . For each branch π in the set from (11) and its witness X_π we have: $\mathcal{I} \text{ io}$ in π ; $\limsup \mathcal{I}(\pi) = b < \infty$; and $\limsup \text{Pre}(\mathcal{I})(\pi) = \infty$ — notice that the sequence $\text{Pre}(\mathcal{I})(\pi)$ contains the sequence $\mathcal{I}_1^{\text{Pre}}(\pi) \upharpoonright_{X_\pi}$ as, possibly strict, subsequence. However, as the latter is unbounded, also the former must be unbounded. Therefore, (†₂) holds.

(\Leftarrow) Assume that (†₁) is true and fix $\mathcal{I} \subseteq \text{Pre}(\mathcal{I}_2)$. Take any branch π in the set measured in (†₁). Since $\text{Suc}(\mathcal{I})$ appears infinitely often in π , by the definition of Suc we have \mathcal{I}_2 also appears infinitely often in π . We will show that $\mathcal{I}_1^{\text{Pre}}(\pi) \not\sim \mathcal{I}_2(\pi)$.

For $k \in \mathbb{N}$, denote by $\text{source}_k(\mathcal{I}_2)$ the set of sources of \mathcal{I}_2 that have exactly $k-1$ strict ancestors that are also sources of \mathcal{I}_2 . Let X_π be the set of numbers k such that $\pi \cap \text{source}_k(\mathcal{I}_2) \cap \text{source}(\text{Suc}(\mathcal{I})) \neq \emptyset$. Then $\mathcal{I}_2(\pi) \upharpoonright_{X_\pi} = \text{Suc}(\mathcal{I})(\pi)$ is unbounded by the assumption. On the other hand, $\mathcal{I} \subseteq \text{Pre}(\mathcal{I}_2)$ and by the definition of X_π we know that $\mathcal{I}_1^{\text{Pre}}(\pi) \upharpoonright_{X_\pi}$ is a subsequence of $\mathcal{I}(\pi)$ and is therefore bounded. This concludes the proof that $\mathcal{I}_1^{\text{Pre}}(\pi) \not\sim \mathcal{I}_2(\pi)$.

Finally, consider the last case that (†₂) holds and fix $\mathcal{I} \subseteq \mathcal{I}_2$ witnessing that. Take a branch π from the set measured in (†₂). The fact that \mathcal{I} appears infinitely often in π implies directly that \mathcal{I}_2 also appears infinitely often in π . Take X_π as the set of numbers k such that $\pi \cap \text{source}_k(\mathcal{I}_2) \cap \text{source}(\mathcal{I}) \neq \emptyset$. Then $\mathcal{I}_2(\pi) \upharpoonright_{X_\pi} = \mathcal{I}(\pi)$ is bounded. However, $\mathcal{I}_1^{\text{Pre}}(\pi) \upharpoonright_{X_\pi}$ contains $\text{Pre}(\mathcal{I})(\pi)$ as a subsequence and therefore is unbounded. Thus, $\mathcal{I}_1^{\text{Pre}}(\pi) \not\sim \mathcal{I}_2(\pi)$. \square

2) Proof of Proposition V.15: This section of the appendix is devoted to the proof of Proposition V.15.

Proposition V.15. *Let \mathcal{I}, \mathcal{K} be two families of intervals such that \mathcal{K} wraps \mathcal{I} and we have:*

$$\mathbb{P} \left[\begin{cases} \mathcal{K} \text{ io} & \Rightarrow \limsup \mathcal{K} < \infty, \text{ and} \\ \mathcal{I} \text{ io} & \Rightarrow \liminf \mathcal{I} = \infty \end{cases} \right] = 1.$$

Then the following property is definable in $\text{MSO}+\nabla$:

$$\mathbb{P}[\mathcal{K} \text{ io} \wedge \mathcal{K} \text{ is not eventually constant}] > 0.$$

The claim is that the property in the statement of the proposition is equivalent to:

- (*) there exist isolated $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{K}$, where \mathcal{K}_1 tail-precedes \mathcal{K}_2 , $\mathbb{P}[\mathcal{K}_2 \mathbf{io}] > 0$, and if $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{I}$ are such that \mathcal{K}_i wraps \mathcal{I}_i , $i \in \{1, 2\}$ then:

$$\begin{aligned} & \exists \mathcal{I}'_1 \leq \mathcal{I}_1. \quad \forall \mathcal{I}'_2 \leq \mathcal{I}_2. \\ & \quad \exists \mathcal{J}_1 \subseteq \mathcal{I}'_1 \text{ extraction of } (\mathcal{K}_1, \mathcal{I}'_1). \\ & \quad \forall \mathcal{J}_2 \subseteq \mathcal{I}'_2 \text{ extraction of } (\mathcal{K}_2, \mathcal{I}'_2). \\ & \quad \mathbb{P}[\mathcal{J}_2 \mathbf{io} \wedge \mathcal{J}_1^{\text{Pre}} \not\sim \mathcal{J}_2] > 0. \end{aligned} \quad (7)$$

Proof of the forward implication:

Let $\mathcal{K}_1, \mathcal{K}_2$ be as in Lemma V.13, so that every interval in \mathcal{K}_1 (respectively \mathcal{K}_2) has length ℓ_1 (respectively ℓ_2), $\ell_1 > \ell_2$, \mathcal{K}_1 tail-precedes \mathcal{K}_2 , and $\mathbb{P}[\mathcal{K}_2 \mathbf{io}] > 0$. Let $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{I}$ be such that \mathcal{K}_i wraps \mathcal{I}_i for $i = 1, 2$ —notice that such $\mathcal{I}_1, \mathcal{I}_2$ are defined uniquely by these conditions.

Let \mathbf{f} be a vector sequence of dimension ℓ_1 that is not an asymptotic mix of any vector sequence of dimension ℓ_2 . It exists thanks to Lemma V.3.

We construct $\mathcal{I}'_1 \leq \mathcal{I}_1$ as follows. If $k \in \mathbb{N}$ and $x_k \in \text{source}(\mathcal{K}_1)$ has exactly k strict ancestors in $\text{source}(\mathcal{K}_2)$ then:

$$\begin{aligned} \vec{\mathcal{K}}_1(\mathcal{I}_1, x_k) &= (v_1, v_2, \dots, v_{\ell_1}), \\ \mathbf{f}(k) &= (w_1, w_2, \dots, w_{\ell_1}), \\ \vec{\mathcal{K}}_1(\mathcal{I}'_1, x_k) &= (v'_1, v'_2, \dots, v'_{\ell_1}), \end{aligned} \quad (12)$$

where $v'_i = \min(v_i, w_i)$ for $i = 1, 2, \dots, \ell_1$.

Lemma A.1. *Assume that π is a branch such that \mathcal{K}_2 appears infinitely often in π and $\liminf \mathcal{I}_1(\pi) = \infty$. Then for every $f \in \mathbf{f}$ there exists $f' \in \vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1, \pi)$ such that $f' \sim f$. In particular, $\vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1, \pi)$ is not an asymptotic mix of any vector sequence of dimension strictly smaller than ℓ_1 .*

Proof. Fix some $f \in \mathbf{f}$. Notice that for $k \in \mathbb{N}$ the vector $\vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1, \pi)(k)$ is given by the formula (12). Thus, we can construct $f' \in \vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1, \pi)$ by copying f . More formally, for all $k \in \mathbb{N}$, if $f(k)$ is the i th component of \mathbf{f} then also $f'(k)$ is the i th component of $\vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1, \pi)(k)$.

We prove that $f' \sim f$. Let $X \subseteq \mathbb{N}$, and suppose that $f|_X$ is bounded. Then $f'|_X$ is bounded as well, since construction, we have that for all $n \in \mathbb{N}$, $f'(n) \leq f(n)$ (see (12)). If on the other hand $f|_X$ is unbounded, then so is $f'|_X$, as a consequence of the fact that $\lim \mathcal{I}_1(\pi) = \infty$.

Now assume that $\vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1, \pi)$ is an asymptotic mix of a vector sequence \mathbf{g} of dimension strictly smaller

than ℓ_1 . In that case \mathbf{f} must be an asymptotic mix of \mathbf{g} : for each $f \in \mathbf{f}$ there exists $f' \in \vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1, \pi)$ given by the above construction such that $f \sim f'$; moreover by assumption there exists $g \in \mathbf{g}$ such that $f' \sim g$; and thus $f \sim g$; a contradiction. \square

Fix some $\mathcal{I}'_2 \leq \mathcal{I}_2$ and take a branch π on which \mathcal{K}_2 appears infinitely often and $\liminf \mathcal{I}_1(\pi) = \infty$ (the assumptions on \mathcal{K}_2 and \mathcal{I} guarantee that with a positive probability a random branch has these properties). By the lemma above $\vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1, \pi)$ is not an asymptotic mix of $\vec{\mathcal{K}}_2(\mathcal{I}'_2, \pi)$. This means that we have:

$$\mathbb{P} \left[\left\{ \mathcal{K}_2 \mathbf{io}, \text{ and } \vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1) \text{ is not an asymp. mix of } \vec{\mathcal{K}}_2(\mathcal{I}'_2) \right\} \right] > 0.$$

Lemma V.10 implies that

$$\mathbb{P} \left[\left\{ \mathcal{K}_2 \mathbf{io}, \text{ and } \exists b \in \mathbb{N}. b \text{ separates } \vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1) \text{ from } \vec{\mathcal{K}}_2(\mathcal{I}'_2) \right\} \right] > 0.$$

And thus, by countable additivity of measures, there must exist $b \in \mathbb{N}$ such that:

$$\mathbb{P}[\mathcal{K}_2 \mathbf{io} \wedge b \text{ separates } \vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1) \text{ from } \vec{\mathcal{K}}_2(\mathcal{I}'_2)] > 0.$$

From the definition of separation we now have the following two cases:

$$\mathbb{P} \left[\left\{ \mathcal{K}_2 \mathbf{io}, \text{ and } \exists X. \left\{ \begin{array}{l} \min(\vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1)|_X) \leq b, \text{ and} \\ \min(\vec{\mathcal{K}}_2(\mathcal{I}'_2)|_X) \text{ is unbd.} \end{array} \right. \right\} \right] > 0, \quad (13)$$

and

$$\mathbb{P} \left[\left\{ \mathcal{K}_2 \mathbf{io}, \text{ and } \exists X. \left\{ \begin{array}{l} \max(\vec{\mathcal{K}}_2(\mathcal{I}'_2)|_X) \leq b, \text{ and} \\ \max(\vec{\mathcal{K}}_1^{\text{Pre}}(\mathcal{I}'_1)|_X) \text{ is unbd.} \end{array} \right. \right\} \right] > 0. \quad (14)$$

The first case:

Construct an extraction $\mathcal{J}_1 \subseteq \mathcal{I}'_1$ of $(\mathcal{K}_1, \mathcal{I}'_1)$ by picking any interval whose length is smaller than b (if there is none, we pick arbitrarily). We fix an extraction $\mathcal{J}_2 \subseteq \mathcal{I}'_2$ of $(\mathcal{K}_2, \mathcal{I}'_2)$, and prove that

$$\mathbb{P}[\mathcal{J}_2 \mathbf{io} \wedge \mathcal{J}_1^{\text{Pre}} \not\sim \mathcal{J}_2] > 0.$$

Since \mathcal{K}_1 precedes \mathcal{K}_2 (tail-preceding is a stronger property), we know that for $x \in \text{source}(\mathcal{K}_2)$, $\text{Pre}(x)$ is well-defined, it is the first ancestor of x in $\text{source}(\mathcal{K}_1)$. Let $\mathcal{K}'_2 \subseteq \mathcal{K}_2$ be the family on which we keep only those intervals $[x, y] \in \mathcal{K}_2$ such that $\vec{\mathcal{K}}_1(\mathcal{I}'_1, \text{Pre}(x))$ has a component that is smaller than b . Then (13) implies that $\mathbb{P}[\mathcal{K}'_2 \mathbf{io}] > 0$.

For $x \in \text{source}(\mathcal{K}'_2)$ define $M(x)$ to be the minimal component in the vector $\vec{\mathcal{K}}'_2(\mathcal{I}'_2, x)$. On a branch π where \mathcal{K}'_2 appears infinitely often, there are infinitely many nodes $x_0 < x_1 < \dots$ belonging to $\text{source}(\mathcal{K}'_2)$; define:

$$M(\mathcal{K}'_2)(\pi) = M(x_0), M(x_1), \dots \in \mathbb{N}^\omega.$$

Then (13) implies that:

$$\mathbb{P}[\mathcal{K}'_2 \mathbf{io} \wedge \limsup M(\mathcal{K}'_2) = \infty] > 0.$$

Finally define $\mathcal{K}''_2 \subseteq \mathcal{K}'_2$ to be the record breakers with respect to the function M , *i.e.* for all $x, x' \in \text{source}(\mathcal{K}''_2)$, if $x < x'$ then $M(x) < M(x')$. From the inequality above it follows that:

$$\mathbb{P}[\mathcal{K}''_2 \mathbf{io} \wedge (\liminf M(\mathcal{K}''_2) = \infty)] > 0, \quad (15)$$

where $M(\mathcal{K}''_2)(\pi)$ is the number sequence resulting from applying M only to the sources of the intervals in \mathcal{K}''_2 . Let $\mathcal{J}'_2 \subseteq \mathcal{J}_2$ be such that every element of $\text{source}(\mathcal{J}'_2)$ belongs to some interval in \mathcal{K}''_2 . Since the intervals in \mathcal{K}''_2 have length ℓ_2 , the sources of \mathcal{J}'_2 are always at a distance smaller than ℓ_2 than the respective source of \mathcal{K}''_2 : if $x' \in \text{source}(\mathcal{J}'_2)$ and $x' \in \text{Int}([x, y]) \in \mathcal{K}''_2$ then $|x'| - |x| \leq \ell_2$. Therefore, as a consequence of Lemma IV.2 and (15) we have

$$\mathbb{P}[\mathcal{J}'_2 \mathbf{io} \wedge (\liminf \mathcal{J}'_2 = \infty)] > 0.$$

But by construction, the intervals in \mathcal{J}'_2 are preceded by intervals in \mathcal{J}_1 whose length is smaller than b , hence we have proved that

$$\mathbb{P}[\mathcal{J}_2 \mathbf{io} \wedge \mathcal{J}_1^{\text{Pre}} \not\sim \mathcal{J}_2] > 0.$$

The second case:

Construct $\mathcal{J}_1 \subseteq \mathcal{I}'_1$ extraction of $(\mathcal{K}_1, \mathcal{I}'_1)$ by picking intervals with the maximal length. We fix an extraction $\mathcal{J}_2 \subseteq \mathcal{I}'_2$ of $(\mathcal{K}_2, \mathcal{I}'_2)$, and prove that

$$\mathbb{P}[\mathcal{J}_2 \mathbf{io} \wedge \mathcal{J}_1^{\text{Pre}} \not\sim \mathcal{J}_2] > 0.$$

Let $\mathcal{K}'_2 \subseteq \mathcal{K}_2$ be the family that keeps only those $[x, y] \in \mathcal{K}_2$ for which $\vec{\mathcal{K}}'_2(\mathcal{I}'_2, x)$ has all components smaller than b . Then (14) implies that $\mathbb{P}[\mathcal{K}'_2 \mathbf{io}] > 0$. Let $\mathcal{J}'_2 \subseteq \mathcal{J}_2$ be such that every source of an interval in \mathcal{J}'_2 belongs to an interval in \mathcal{K}'_2 . Since the intervals in \mathcal{K}'_2 all have length ℓ_2 , the distance between a node in $\text{source}(\mathcal{K}'_2)$ and its first descendant in $\text{source}(\mathcal{J}'_2)$ is at most ℓ_2 , so applying Claim IV.2 we have that $\mathbb{P}[\mathcal{J}'_2 \mathbf{io}] > 0$. While every interval in \mathcal{J}'_2 has length at most b , (14) implies that there is a non-zero probability that $\mathcal{J}_1^{\text{Pre}}$ is unbounded, *i.e.*

$$\mathbb{P}[\mathcal{J}'_2 \mathbf{io} \wedge \mathcal{J}_1^{\text{Pre}} \not\sim \mathcal{J}'_2] > 0.$$

This concludes the proof of the forward implication. *Proof of the converse implication:*

Assume that

$$\mathbb{P}[\mathcal{K} \mathbf{io} \Rightarrow \mathcal{K} \text{ is eventually constant}] = 1.$$

Let $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{K}$ be such that \mathcal{K}_1 tail-precedes \mathcal{K}_2 , and $\mathbb{P}[\mathcal{K}_2 \mathbf{io}] > 0$. Consider \mathcal{I}_1 and \mathcal{I}_2 as in the statement and fix $\mathcal{I}'_1 \leq \mathcal{I}_1$.

We let $\mathcal{I}'_2 \leq \mathcal{I}_2$ be such that for all $x' \in \text{source}(\mathcal{K}_2)$ the following holds: let $x = \text{Pre}(x')$ (it exists because \mathcal{K}_1 tail-precedes \mathcal{K}_2) then for all $k \in \mathbb{N}$ if both $\vec{\mathcal{K}}_1(\mathcal{I}'_1, x)$ and $\vec{\mathcal{K}}_2(\mathcal{I}'_2, x')$ have k th components defined: $(\vec{\mathcal{K}}_1(\mathcal{I}'_1, x))_k$ and $(\vec{\mathcal{K}}_2(\mathcal{I}'_2, x'))_k$ then:

$$(\vec{\mathcal{K}}_2(\mathcal{I}'_2, x'))_k = \min \left\{ (\vec{\mathcal{K}}_2(\mathcal{I}'_2, x'))_k, (\vec{\mathcal{K}}_1(\mathcal{I}'_1, x))_k \right\}.$$

When the respective components are not defined, take $(\vec{\mathcal{K}}_2(\mathcal{I}'_2, x'))_k = (\vec{\mathcal{K}}_2(\mathcal{I}'_2, x'))_k$.

Fix $\mathcal{J}_1 \subseteq \mathcal{I}'_1$ an extraction of $(\mathcal{K}_1, \mathcal{I}'_1)$. We say that \mathcal{J}_1 chooses k th component in x if $[x, y] \in \mathcal{K}_1$, $x' \in \text{source}(\mathcal{J}_1) \cap \text{Int}([x, y])$, and $|x'| - |x| = k + 1$. We construct an extraction $\mathcal{J}_2 \subseteq \mathcal{I}'_2$ of $(\mathcal{K}_2, \mathcal{I}'_2)$ by copying. More formally, consider $x' \in \text{source}(\mathcal{K}_2)$ and let $x = \text{Pre}(x')$. If \mathcal{J}_1 chooses the k th component in x then in x' we choose to \mathcal{J}_2 the k th component as well if it exists, otherwise we choose some arbitrary component.

Let π be a branch where \mathcal{K} appears infinitely often, is eventually constant, and \mathcal{I}_2 has infinite lim inf. If \mathcal{J}_2 appears infinitely often in π , we prove that from the construction above, $f \stackrel{\text{def}}{=} \mathcal{J}_1^{\text{Pre}}(\pi)$ is asymptotically equivalent to $g \stackrel{\text{def}}{=} \mathcal{J}_2(\pi)$. Let $X \subseteq \mathbb{N}$. Since \mathcal{K} is eventually constant in π after some point, from the construction above, the numbers in $f \upharpoonright_X$ are always smaller than the corresponding numbers in $g \upharpoonright_X$. Because \mathcal{I}_2 tends to infinity, we have that either both $f \upharpoonright_X$ and $g \upharpoonright_X$ are bounded or both of them are unbounded. As a consequence $\mathcal{J}_1^{\text{Pre}}(\pi) \sim \mathcal{J}_2(\pi)$.

From the assumptions and the argument above we conclude that:

$$\mathbb{P}[\mathcal{J}_2 \mathbf{io} \Rightarrow \mathcal{J}_1^{\text{Pre}} \sim \mathcal{J}_2] = 1,$$

and hence refute the second statement of the lemma and finish the proof of the converse implication. This concludes the proof of Proposition V.15.

3) *Implicit wrappings:* We demonstrate how to avoid speaking explicitly about the family of intervals \mathcal{I} in the formulation of Proposition V.15. We will give a proof of Theorem III.2 using Proposition V.15.

Definition A.2. We say that a family of intervals \mathcal{K} is *sufficiently spaced* if there exists a family \mathcal{I} such that \mathcal{K} wraps \mathcal{I} and

$$\mathbb{P}[\mathcal{I} \text{ io} \Rightarrow (\liminf \mathcal{I} = \infty)] = 1.$$

We claim that

$$\mathbb{P}[\mathcal{K} \text{ io} \Rightarrow \mathcal{K} \text{ is eventually constant}] = 1,$$

is equivalent to

- (*) $\mathbb{P}[\mathcal{K} \text{ io} \Rightarrow \limsup \mathcal{K} < \infty] = 1$ and either:
 $\mathbb{P}[\mathcal{K} \text{ io} \Rightarrow (\lim \mathcal{K} = 0)] = 1$ or
 $\mathbb{P}[\mathcal{K} \text{ io} \Rightarrow (\liminf \mathcal{K} > 0)] = 1$ and for all $\mathcal{K}' \subseteq \mathcal{K}$ that are sufficiently spaced we have:

$$\mathbb{P}[\mathcal{K}' \text{ io} \Rightarrow \mathcal{K}' \text{ is eventually constant}] = 1.$$

Notice that the definition of \mathcal{K} wrapping \mathcal{I} (see Definition V.11) implicitly implies that all the intervals $[x, y] \in \mathcal{K}$ have positive length. However, in the following lemma we prefer to allow the family \mathcal{K} to contain some intervals of length 0. This explains the additional condition in (*).

The forward implication is immediate. For the converse, assume (*). Clearly if $\mathbb{P}[\mathcal{K} \text{ io} \Rightarrow (\lim \mathcal{K} = 0)] = 1$ then \mathcal{K} is almost surely eventually constant whenever it appears infinitely often.

Now suppose towards a contradiction that there is non-zero probability that the following properties hold: \mathcal{K} appears infinitely often, is bounded, $[\liminf \mathcal{K} > 0]$, but \mathcal{K} is not eventually constant. In that case, without loss of generality we can assume that \mathcal{K} contains no intervals of length 0. Then, by Lemma V.13, there exist $\ell_1 > \ell_2 \in \mathbb{N}$ and isolated $\mathcal{K}_1 \subseteq \mathcal{K}$, $\mathcal{K}_2 \subseteq \mathcal{K}$ such that $\mathcal{K}_1 \subseteq \mathcal{K}$ contains intervals of length ℓ_1 , $\mathcal{K}_2 \subseteq \mathcal{K}$ contains intervals of length ℓ_2 , and there is non-zero probability that both \mathcal{K}_1 and \mathcal{K}_2 appear infinitely often. As \mathcal{K} contains no intervals of length 0, we know that $\ell_2 > 0$.

Take $i = 1, 2$ and $i' = 3 - i$ (i.e. the other number). For $k \in \mathbb{N}$ and $x \in \text{source}(\mathcal{K}_i)$ define $S_k(x)$ to be the set of all $x' \in \text{source}(\mathcal{K}_{i'})$ such that :

$x < x'$, and

$$\forall u \in \text{source}(\mathcal{K}_{i'}), x < u < x'. \text{Len}([x, u]) \leq k + \ell_1 + \ell_2$$

In other words, $S_k(x)$ contains the first descendants of x in $\text{source}(\mathcal{K}_{i'})$ that are at a distance at least k .

For all $n \in \mathbb{N}$ we define $X_n \subseteq \text{source}(\mathcal{K}_1)$, $Y_n \subseteq \text{source}(\mathcal{K}_2)$ as follows: let X_0 be the subset of nodes in $\text{source}(\mathcal{K}_1)$ that do not have any strict ancestors in $\text{source}(\mathcal{K}_1)$ and

$$Y_n = \bigcup_{x \in X_n} S_n(x) \quad X_{n+1} = \bigcup_{y \in Y_n} S_{n+1}(y).$$

Let $\mathcal{K}'_1 \subseteq \mathcal{K}_1$ (resp. $\mathcal{K}'_2 \subseteq \mathcal{K}_2$) contain all the intervals with sources in $\bigcup_{n \in \mathbb{N}} X_n$ (resp. in $\bigcup_{n \in \mathbb{N}} Y_n$). Put $\mathcal{K}' = \mathcal{K}'_1 \cup \mathcal{K}'_2$.

Claim A.3. For \mathcal{K}' defined as above we have $\mathbb{P}[\mathcal{K}' \text{ io}] > 0$.

Proof. Directly from the definition, because

$$[\mathcal{K}_1 \text{ io}] \cap [\mathcal{K}_2 \text{ io}] \subseteq [\mathcal{K}' \text{ io}].$$

□

Proposition A.4. *There exists a family of intervals \mathcal{I} such that \mathcal{K}' wraps \mathcal{I} and for each interval $[x, y] \in \mathcal{K}'$ if $x \in X_n \cup Y_n$ then the intervals in \mathcal{I} with sources in $\text{Int}([x, y])$ have length exactly n . In particular, $\mathbb{P}[\mathcal{I} \text{ io} \Rightarrow (\lim \mathcal{I} = \infty)] = 1$ and therefore \mathcal{K}' is sufficiently spaced.*

Proof. It is enough to observe that the intervals added to \mathcal{I} by a naive construction will not overlap with consecutive intervals of \mathcal{K}' . However, this is guaranteed by the choice of the sets $S_n(x)$ and the fact that \mathcal{K}' contains no trivial intervals. □

Finally, if π is a branch in which both \mathcal{K}_1 and \mathcal{K}_2 appear infinitely often, then in π , \mathcal{K}'_1 and \mathcal{K}'_2 also appear infinitely often. This implies that

$$\mathbb{P}[\mathcal{K}' \text{ io} \wedge \mathcal{K}' \text{ is not eventually constant}] > 0,$$

contradicting (*).

Due to the analysis from Appendix B, (*) as well as being sufficiently spaced are MSO+ ∇ definable. Therefore, Theorem III.2 follows.

B. Definability in MSO+ ∇

In this technical section we argue why all the properties gradually defined throughout the paper are in fact MSO+ ∇ definable. Therefore, the section consists of a pass through the successively defined concepts.

First, as explained in Section III, we will represent a family of intervals \mathcal{I} as a pair of sets $\sigma_{\mathcal{I}} = \text{source}(\mathcal{I})$ and $\tau_{\mathcal{I}} = \text{target}(\mathcal{I})$ of nodes of the tree. Consider the following MSO formulae ($\exists!$ stands for “there exists a unique”):

$$\begin{aligned} \phi_{\text{int}}(x, y, \sigma_{\mathcal{I}}, \tau_{\mathcal{I}}) &= x \in \sigma_{\mathcal{I}} \wedge y \in \tau_{\mathcal{I}} \wedge x < y \wedge \\ &\forall z. (x < z < y) \Rightarrow z \notin \sigma_{\mathcal{I}} \wedge z \notin \tau_{\mathcal{I}}, \end{aligned}$$

$$\begin{aligned} \phi_{\text{set}}(\sigma_{\mathcal{I}}, \tau_{\mathcal{I}}) &= \forall x \in \sigma_{\mathcal{I}}. x \notin \tau_{\mathcal{I}} \wedge \\ &\forall y \in \tau_{\mathcal{I}}. y \notin \sigma_{\mathcal{I}} \wedge \\ &\forall x \in \sigma_{\mathcal{I}}. \exists! y \in \tau_{\mathcal{I}}. \phi_{\text{int}}(x, y, \sigma_{\mathcal{I}}, \tau_{\mathcal{I}}) \wedge \\ &\forall y \in \tau_{\mathcal{I}}. \exists! x \in \sigma_{\mathcal{I}}. \phi_{\text{int}}(x, y, \sigma_{\mathcal{I}}, \tau_{\mathcal{I}}) \wedge. \end{aligned}$$

The formula $\phi_{\text{int}}(x, y, \sigma_{\mathcal{I}}, \tau_{\mathcal{I}})$ expresses that $[x, y]$ is an interval in \mathcal{I} , while $\phi_{\text{set}}(\sigma_{\mathcal{I}}, \tau_{\mathcal{I}})$ means that $(\sigma_{\mathcal{I}}, \tau_{\mathcal{I}})$ in fact represent a valid family of intervals. Notice that $\mathcal{I} \subseteq \mathcal{K}$ boils down to saying that $\phi_{\text{set}}(\sigma_{\mathcal{I}}, \tau_{\mathcal{I}})$, $\phi_{\text{set}}(\sigma_{\mathcal{K}}, \tau_{\mathcal{K}})$, and $\sigma_{\mathcal{I}} \subseteq \sigma_{\mathcal{K}}$ and $\tau_{\mathcal{I}} \subseteq \tau_{\mathcal{K}}$.

Remark A.5. Consider a representation $(\sigma_{\mathcal{I}}, \tau_{\mathcal{I}})$ of a family of intervals \mathcal{I} . Let X be a set of nodes and π be a branch (also represented as a set of nodes). Then the following conditions are MSO definable: X **fo** in π ; X **io** in π ; \mathcal{I} **io** in π .

Using the above remark, the $(*)$ property of Lemma IV.1 is easily MSO+ ∇ definable by the following formula

$$\begin{aligned} \phi_{\text{IV.1}}(\mathcal{I}) \stackrel{\text{def}}{=} & \exists \mathcal{I}' \subseteq \mathcal{I}. \\ & \phi_{\text{set}}(\sigma_{\mathcal{I}'}, \tau_{\mathcal{I}'}) \wedge \\ & \nabla \pi. \sigma_{\mathcal{I}'} \text{ io in } \pi \wedge \\ & \forall \mathcal{K} \subseteq \mathcal{I}'. \phi_{\text{set}}(\sigma_{\mathcal{K}}, \tau_{\mathcal{K}}) \Rightarrow \\ & \nabla \pi. (\sigma_{\mathcal{K}} \text{ io in } \pi \Leftrightarrow \tau_{\mathcal{K}} \text{ io in } \pi). \end{aligned}$$

To define in MSO+ ∇ the property (6), one uses the negation of the condition from Lemma IV.1: it is equivalent to saying that $\mathbb{P}[\mathcal{I} \text{ io} \Rightarrow (\liminf \mathcal{I} = \infty)] = 1$. This means that the following formula is equivalent to saying that \mathcal{I} is unbounded

$$\begin{aligned} \phi_{\text{ubnd}}(\mathcal{I}) \stackrel{\text{def}}{=} & \exists \mathcal{K} \subseteq \mathcal{I}. \\ & \phi_{\text{set}}(\sigma_{\mathcal{K}}, \tau_{\mathcal{K}}) \wedge \\ & \nabla \pi. (\sigma_{\mathcal{I}} \text{ io in } \pi \Leftrightarrow \sigma_{\mathcal{K}} \text{ io in } \pi) \wedge \\ & \neg \phi_{\text{IV.1}}(\mathcal{K}). \end{aligned}$$

Thus, using Lemma IV.5, a characteristic of a family of intervals is MSO+ ∇ definable by the following formula:

$$\begin{aligned} \phi_{\text{char}}(\sigma_{\mathcal{I}}, \tau_{\mathcal{I}}, X) \stackrel{\text{def}}{=} & \phi_{\text{set}}(\sigma_{\mathcal{I}}, \tau_{\mathcal{I}}) \wedge \\ & \exists \mathcal{K} \subseteq \mathcal{I}. \\ & \phi_{\text{set}}(\sigma_{\mathcal{K}}, \tau_{\mathcal{K}}) \wedge \\ & \phi_{\text{ubnd}}(\mathcal{K}) \wedge \\ & \nabla \pi. (X \text{ io in } \pi \Leftrightarrow \sigma_{\mathcal{K}} \text{ io in } \pi) \wedge \\ & \forall \mathcal{K}' \subseteq \mathcal{I}. \\ & (\phi_{\text{set}}(\sigma_{\mathcal{K}'}, \tau_{\mathcal{K}'}) \wedge \phi_{\text{ubnd}}(\mathcal{K}')) \Rightarrow \\ & \nabla \pi. (\sigma_{\mathcal{K}'} \text{ io in } \pi \Rightarrow \sigma_{\mathcal{K}} \text{ io in } \pi). \end{aligned}$$

Remark A.6. From that moment on we will represent families of intervals \mathcal{I} as triples $\sigma_{\mathcal{I}}, \tau_{\mathcal{I}}, X_{\mathcal{I}}$, where $\phi_{\text{char}}(\sigma_{\mathcal{I}}, \tau_{\mathcal{I}}, X_{\mathcal{I}})$ holds. Thanks to that representation, we have

$$\mathbb{P}[X_{\mathcal{I}} \text{ io} \Leftrightarrow \limsup \mathcal{I} = \infty] = 1.$$

Therefore, (up to a set of branches of measure 0) “ $\limsup \mathcal{I}(\pi) < \infty$ ” is MSO+ ∇ definable by the formula:

$$\begin{aligned} \phi_{\text{bnd}}(\mathcal{I}, \pi) \stackrel{\text{def}}{=} \\ \exists x \in \pi. \forall y \in \pi. (x < y) \Rightarrow y \notin X_{\mathcal{I}}. \end{aligned}$$

Notice that Definition V.4 is already stated as an MSO property. Moreover, the relation between x and x' given by the function Pre is also MSO definable directly from the definition. This leads to the conclusion that one can define in MSO that \mathcal{I}_1 precedes \mathcal{I}_2 and \mathcal{I}' is the effect of applying the function Pre (resp. Suc) to a family of intervals $\mathcal{I} \subseteq \mathcal{I}_2$ (resp. $\mathcal{I} \subseteq \mathcal{I}_1$). Clearly the fact that \mathcal{I}_1 and \mathcal{I}_2 are isolated is also MSO definable using our encoding.

Remark A.6 immediately implies that the $(*)$ property in the proof of Lemma V.7 in Appendix A1 is MSO+ ∇ definable—it is enough to replace each occurrence of $[\limsup \mathcal{I} < \infty]$ by ϕ_{bnd} .

Next, we investigate the properties from Section V-C. It is easy to see that the following formula defines that \mathcal{K} wraps \mathcal{I} :

$$\begin{aligned} \phi_{\text{wrap}} \stackrel{\text{def}}{=} & \phi_{\text{set}}(\mathcal{I}) \wedge \phi_{\text{set}}(\mathcal{K}) \wedge \phi_{\geq 1}(\mathcal{K}) \wedge \\ & \forall x'. x' \in \sigma_{\mathcal{I}} \Leftrightarrow \exists x, y. \phi_{\text{int}}(x, y, \sigma_{\mathcal{K}}, \tau_{\mathcal{K}}) \\ & \wedge x < x' < y, \end{aligned}$$

where $\phi_{\geq 1}(\mathcal{K})$ states that every interval in \mathcal{K} has length at least 1.

Further, Definition V.12 is itself expressed in MSO. The same holds for the notion of *extraction* and the order $\mathcal{I}_1 \leq \mathcal{I}_2$. These observations give us sufficient background to study the $(*)$ property of Proposition V.15. The only part of this property that is not directly MSO+ ∇ formalisable is (7). However, under the previous assumptions of the formula, \mathcal{J}_1 and \mathcal{J}_2 satisfy the conditions of Lemma V.7, and therefore (7) is in fact MSO+ ∇ definable.

Following the construction from the main body, observe that being *sufficiently spaced* (see Definition A.2) is definable in MSO+ ∇ . This is because the requirement $\mathbb{P}[\mathcal{I} \text{ io} \Rightarrow (\liminf \mathcal{I} = \infty)] = 1$ is just the negation of the first statement of Lemma IV.1, *i.e.* it is expressible by the formula $\neg \phi_{\text{IV.1}}(\mathcal{I})$. Thus, the $(*)$ property discussed in Section A3 is also MSO+ ∇ definable. Therefore, Theorem III.2 follows.