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TWO-WAY UNARY TEMPORAL LOGIC OVER TREES *

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Abstract. We consider a temporal logic $\mathsf{EF} + \mathsf{F}^{-1}$ for unranked, unordered finite trees. The logic has two operators: $\mathsf{EF}\varphi$, which says "in some proper descendant φ holds", and $\mathsf{F}^{-1}\varphi$, which says "in some proper ancestor φ holds". We present an algorithm for deciding if a regular language of unranked finite trees can be expressed in $\mathsf{EF} + \mathsf{F}^{-1}$. The algorithm uses a characterization expressed in terms of forest algebras.

1. Introduction

We say a logic has a decidable characterization if the following decision problem is decidable: "given as input a finite automaton, decide if the recognized language can be defined using a formula of the logic". Representing the input language by a finite automaton is a reasonable choice, since many known logics (over words or trees) are captured by finite automata.

This type of problem has been successfully studied for word languages. Arguably best known is the result of McNaughton, Papert and Schützenberger [11, 8], which says that the following three conditions on a regular word language L are equivalent: a) L can be defined in first-order logic; b) L can be defined using a star-free expression; and c) the syntactic semigroup of L does not contain a non-trivial group. Since condition c) can be effectively tested, the above theorem gives a decidable characterization of first-order logic. This result demonstrates two important features of work in this field: a decidable characterization not only gives a better understanding of the logic in question, but it often reveals unexpected connections with algebraic concepts. During several decades of research, decidable characterizations have been found for fragments of first-order logic with restricted quantification and a large group of temporal logics, see [9] and [15] for references.

For trees, however, much less is known. No decidable characterization has been found for what is possibly the most important subclass of regular tree languages, first-order logic with the descendant relation, despite several attempts [10, 7, 2]. Similarly open are chain logic [14] and the temporal logics CTL, CTL* and PDL. However, there has been some recent progress. In [5], decidable characterizations were presented for the temporal logics EF and $\mathsf{EX}+\mathsf{EF}$; while Benedikt and Segoufin [1] characterized tree languages definable in first-order

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logic with the successor relation (but without the descendant relation). Two new results give effective characterizations for some fragments of first-order logic with limited quantifier alternation. The expressive power of alternation-free formulas (i.e. boolean combinations of formulas with quantifier prefix $\exists^*$) is characterized in [4]. Properties that can be defined both with quantifier prefix $\exists^*\forall^*$ and also with quantifier prefix $\forall^*\exists^*$ are characterized in [3]. We will come back to the latter class later on in this introduction.

In this paper, we continue the line of research started in [5], by focusing on a temporal logic for trees. We consider a logic called $\mathsf{EF}+\mathsf{F}^{-1}$. This logic has two operators: $\mathsf{EF}\varphi$, which says "in some proper descendant φ holds", and $\mathsf{F}^{-1}\varphi$, which says "in some proper ancestor φ holds". Thanks to the backward modality, $\mathsf{EF}+\mathsf{F}^{-1}$ is more expressive than EF alone. For instance, the formula

$$\mathsf{EF}(a \wedge \neg\mathsf{F}^{-1}\neg b)$$

defines the class of trees where some node has label a, but all of its ancestors have label b. This is a property reminiscent of CTL, and cannot be expressed by only using EF, since it fails the identities that must be satisfied by EF-definable languages [6].

The main result in this paper is Theorem 6.2, which gives a decidable characterization of languages definable in $\mathsf{EF}+\mathsf{F}^{-1}$. Before we present this result, in Section 2 we try to justify the choice of the logic $\mathsf{EF}+\mathsf{F}^{-1}$. In Section 3 we present the algebraic formalism that will be used in the proofs. The rest of the paper is devoted to proving the main result.

I would like to thank Luc Segoufin. We spent a lot of time together trying to understand the expressive power of $\mathsf{EF}+\mathsf{F}^{-1}$. Without his input this paper would not have been possible. I would also like to thank the anonymous referees for their helpful comments.

2. Why two-way unary temporal logic

There are two reasons to consider $\mathsf{EF}+\mathsf{F}^{-1}$. The first reason is that, over words, this logic corresponds to an important and well-studied class of regular languages. The second reason is that, over trees, the logic is related to XML. We go over these reasons in Sections 2.1 and 2.2 respectively.

2.1. The word analogy.

There is a very robust class of regular word languages that has several equivalent descriptions (a survey of this class can be found in [12]):

(1) Word languages that can be defined in the temporal logic $\mathsf{F}+\mathsf{F}^{-1}$. Here $\mathsf{F}\varphi$ means "in some future position φ" and $\mathsf{F}^{-1}\varphi$ means "in some past position φ".
(2) Word languages that can be defined by a first-order formula with two variables and the left-to-right ordering of positions (but without the successor relation).
(3) Word languages that can be defined by a first-order formula (with many variables, the left-to-right ordering, but without the successor relation) with a $\forall^*\exists^*$ quantifier prefix, and also by one with an $\exists^*\forall^*$ quantifier prefix.
(4) Word languages whose syntactic semigroup belongs to the semigroup variety DA. One way of defining this variety is in terms of an identity: DA is the class of semigroups that satisfy the identity $(st)^\omega = (st)^\omega s(st)^\omega$.
(5) Word languages described by finite disjoint unions of unambiguous products (a form of regular expression).
(6) Word languages that can be recognized by "turtle automata", a type of deterministic two-way word automaton.

(7) Word languages that can be recognized by two-way deterministic automata where the states in a run are non-decreasing with respect to a given order.

An important corollary of property 4 is that membership of a regular language in the above class is decidable: it suffices to check if the syntactic semigroup of the language satisfies the DA identity.

Some of the above classes generalize easily to trees, some don't.

We will not talk about classes 5, 6 and 7. It is not clear what unambiguous expressions are for trees, likewise for the automata.

We will come back to the algebraic description in item 4 later on in the paper.

The three logically defined classes 1, 2 and 3 can be easily extended to trees. A natural counterpart of class 1 is the logic $\mathsf{EF}+\mathsf{F}^{-1}$ considered in this paper. The classes 2 and 3 can define tree languages if the order is interpreted as the ancestor/descendant ordering of tree nodes. (One could also consider variants where two partial orders of nodes are available instead of one: the ancestor/descendant order and also the left-to-right ordering of siblings. We keep to the simpler case, where siblings are unordered.) The logically defined classes diverge for trees:

- Two-variable logic is strictly more expressive than the temporal logic. The translation from temporal to two-variable logic is fairly obvious. For the converse, the problem is that $x \not\leq y \wedge y \not\leq x$ cannot be expressed in the temporal logic. For instance, the language: "there are two a's" can be defined by a two-variable formula, but cannot be defined in the temporal logic. This is because the temporal logic is bisimulation invariant, and cannot see the difference between one child with a and two children with a. (Note however, that the languages "two a's below some b", or "three a's" cannot be defined in two-variable logic.)
- As we will show at the end of this paper, the intersection of $\forall^*\exists^*$ and $\exists^*\forall^*$ is incomparable with both the two-variable and the temporal logic.

The second fragment has been considered in [3], the investigation therein shows that it is a well-behaved class of tree languages. We are left with the temporal logic and two-variable logic. Why do we choose temporal logic and not two-variable logic? The reason is that two-variable logic seems to be less robust for trees: why can "two a's" be defined, but not "three a's"? Of course it is nonetheless important to understand two-variable logic, and we leave this task as future work.

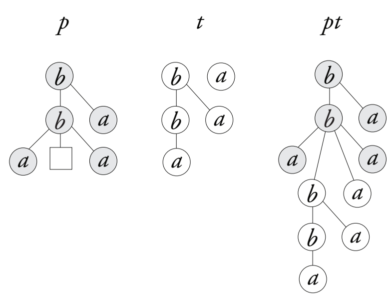
2.2. **XPath.** XPath is a formalism used to describe paths and nodes in unranked trees. There is a strong connection between XPath and two-variable logics

A set of paths is seen as a binary relation $P(x, y)$, which says when a source x can be connected with a target y. The basic idea in XPath is that one starts with atomic paths, called axes, such as "x is a descendant of y", or "x is a child of y", and then constructs longer paths using mechanisms such as concatenation. Marx and de Rijke [**?**] show that a fragment of XPath called Core XPath has exactly the same expressive power as two-variable first-order logic. (The equivalence in expressive power is for Boolean queries in XPath and sentences of two-variable logic. The equivalence also holds for unary queries in XPath and formulas of two-variable logic with one free variable; but it fails for binary queries.) Note however, that the axes considered by Marx include child and next-child, which go beyond the fragments considered in this paper. When the only axes allowed are "descendant" and "ancestor", Core XPath has exactly the same power as "our" logic $\mathsf{EF}+\mathsf{F}^{-1}$. A decidable characterization for fragments of XPath with the other axes, including the one considered by Marx, is left as future work.

3. Basic definitions

3.1. **Trees and forests.** We work with unranked finite labeled trees. We assume that an alphabet (A, B) contains two types of labels: one set of labels A that can be used in the leaves, and another set of labels B that can be used in inner nodes (i.e. not leaves). This division is convenient for the algebraic framework we use in general, and for the induction proof in this paper in particular. *Trees* are defined as follows: every leaf label $a \in A$ is a tree; if $t_1, \ldots, t_n$ are trees and $b \in B$ is an inner node label then $b(t_1 + \cdots + t_n)$ is a tree. A *forest* is a sequence of trees. As above, we concatenate forests using $+$. In particular every forest is of the form $t = t_1 + \cdots + t_n$, for some trees $t_1, \ldots, t_n$. We do not allow empty forests, so $n \geq 1$. We denote both trees and forests using letters s, t. When b is a label and t is a forest, we write bt for the tree that has label b in the root, and where the children form the forest t. In other words, we omit the parentheses and write bt instead of $b(t)$.

A *context* is a forest where exactly one leaf is labeled by a special label $\square$; this leaf is interpreted as a hole. We denote contexts by p, q. The *main path* in a context consists of the ancestors of the hole. A forest t can be substituted in place of the hole of a context p, the resulting forest is denoted $p(t)$, or sometimes pt.



There is a natural composition operation on contexts: the context pq is the unique context such that $(pq)t = p(qt)$ holds for all forests t. We allow the empty context, denoted by $\square$; this is the context where the only node in the context is the hole $\square$. The empty context satisfies $\square t = t$. Nodes of trees, forests and contexts are defined the usual way. We write x, y, z for nodes, and $x \leq y$ when x is an ancestor of y.

The reader will notice that the trees and forests we defined are sibling-ordered (i.e. $s+t$ is not the same as $t + s$). However, properties definable in our logic $\mathsf{EF} + \mathsf{F}^{-1}$ are going to be invariant under this order.

3.2. **The logic.** The logic $\mathsf{EF} + \mathsf{F}^{-1}$ is defined as follows:

- Every label – both inner node label and leaf label – is a formula; this formula holds in nodes with that label.
- Formulas are closed under boolean combinations, including negation.
- If φ is a formula, then $\mathsf{EF}\varphi$ is also a formula; it is true in a node x if there is some proper descendant $y > x$ where φ is true. Likewise for $\mathsf{F}^{-1}\varphi$, but this time y must be a proper ancestor $y < x$.

A formula φ of $\mathsf{EF}+\mathsf{F}^{-1}$ is most naturally interpreted as a unary query, i.e. in a given tree it selects a set of nodes. For instance, the formula $\mathsf{EF}\mathit{true}$ selects all inner nodes. In this paper, we are interested in tree languages, i.e. boolean queries, where a formula is either true or false in a given tree. To get a boolean query, we say a formula of $\mathsf{EF}+\mathsf{F}^{-1}$ is true in a tree if it is true in its root.

The main contribution of this paper is a characterization of the regular tree languages that can be defined by a boolean query of $\mathsf{EF}+\mathsf{F}^{-1}$. It is, however, natural to also ask for a characterization of unary queries. For instance, the first unary query below can be defined in $\mathsf{EF}+\mathsf{F}^{-1}$, but the second one cannot:

- Some ancestor of the selected node has label a, i.e. $\mathsf{F}^{-1}a$.
- Some child of the selected node has label a.

In general, a regular unary query can be given e.g. as a formula of monadic-second order logic with one free variable. Note that although the second unary query cannot be defined, the tree language "some child of the root has label a" can be defined, by the formula

$$\mathsf{EF}(a \ \wedge \mathsf{F}^{-1}\mathit{true} \wedge \neg\mathsf{F}^{-1}\mathsf{F}^{-1}\mathit{true})\ .$$

This suggests that characterizing unary queries is a nonobvious problem, which we leave as future work.

3.3. **Antichain composition principle.** A problem with $\mathsf{EF}+\mathsf{F}^{-1}$ is that it is not closed under "composition". We illustrate this problem, together with a workaround, for words; then we show the result for trees.

Consider the word languages aa and $(a+b)^*$. Both are definable in $\mathsf{F}+\mathsf{F}^{-1}$, and even only using F, but the language $(a+b)^*aa(a+b)^*$ is not. We claim however, that the concatenation of two definable languages is also definable if the place in the word where they meet can be uniquely determined in $\mathsf{F}+\mathsf{F}^{-1}$:

Lemma 3.1 (Composition for words). *Let L, K be two word languages definable in $\mathsf{F}+\mathsf{F}^{-1}$ and let φ be a $\mathsf{F}+\mathsf{F}^{-1}$ formula with the semantic property that in every word, φ holds in at most one word position. The following word language is also definable in $\mathsf{F}+\mathsf{F}^{-1}$:*

$$\{a_1\dots a_n : a_1\cdots a_i \in L,\ a_{i+1}\cdots a_n \in K,\ \textit{and } \varphi \textit{ holds in } a_1\cdots a_n \textit{ at position } i+1\}$$

Proof. We use relativization. We define ψ_1 by taking the formula defining L, and replacing each subformula ψ by $\psi \wedge \mathsf{F}\varphi$. Likewise, we define ψ_2 by taking the formula defining K, and replacing each subformula ψ by $\psi \wedge (\varphi \vee \mathsf{F}^{-1}\varphi)$. The formula for the language in the lemma is then $\psi_1 \wedge \mathsf{F}(\varphi \wedge \psi_2)$. □

For trees, the situation is more complicated. First of all, there are two notions of composition: concatenation $s+t$ for forests and composition pq for contexts. We are interested in generalizing Lemma 3.1 to composition of contexts. In our generalization though, we may need to substitute many trees simultaneously. This leads to a slightly less appealing definition, which follows.

A formula is called *antichain* if in every tree, the set of nodes where it holds forms an antichain, i.e. a set (not necessarily maximal) of nodes pairwise incomparable with respect to the descendant relation. This is a semantic property, and may not be apparent just by looking at the syntax of the formula. For instance, the first two formulas below are antichain, while the third is not:

- The node is a leaf: $\neg\mathsf{EF}\mathit{true}$.

- The node is a minimal occurrence of b: $b \wedge \neg \mathsf{F}^{-1} b$.
- The node has label b.

Using antichain formulas, we define our notion of concatenation. The ingredients are:

- An antichain formula φ.
- Disjoint tree languages $L_1, \ldots, L_n$.
- Leaf labels $a_1, \ldots, a_n$.

Let t be a tree. We define the tree

$$t[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n]$$

as follows. For each node x of t where the antichain formula φ holds, we determine the unique i such the tree language L_i contains the subtree of x. If such an i exists, we remove the subtree of x (including x), and replace x by a leaf labeled with a_i. Since φ is antichain, this can be done simultaneously for all x. Note that the formula φ may depend also on ancestors of x, while the languages L_i only talk about the subtree of x.

Lemma 3.2 (Antichain composition principle)**.** *Let φ, $L_1, \ldots, L_n$ and $a_1, \ldots, a_n$ be as above. If $L_1, \ldots, L_n$ are tree-definable, and K is a tree-definable language, then so is*

$$\{t : t[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n] \in K\} .$$

Proof. This is proved by a relativization entirely analogous to the one used in Lemma 3.1. □

The point of this lemma is that the languages L_i are taken out of their context inside the tree t. For instance L_i can say something like: "the root has label a and a child with label b",

$$L_i = \mathsf{EF}(b \ \wedge \ \mathsf{F}^{-1} a \ \wedge \ \neg \mathsf{F}^{-1} \mathsf{F}^{-1} \mathit{true}) ,$$

while in general the property "a node in the tree that has label b and a child with label b" cannot be expressed in $\mathsf{EF} + \mathsf{F}^{-1}$.

4. Forest algebra

To represent languages of trees, we will be using forest algebra. We feel that using forest algebra instead of automata simplifies the combinatorics used in our characterization. Furthermore, when using forest algebra, the key properties from Theorem 6.2 can be stated in terms of identities.

Here we only sketch out the definitions and basic properties; the reader is referred to [6] for more details. The algebras described in [6] differ slightly from those used here—mainly in that we do not allow empty forests here—but the results carry over into this setting.

A forest algebra is to a regular language of unranked trees as a semigroup is to a regular language of words. Formally, a forest algebra is an algebra with two sorts (H, V), along with some operations that satisfy a number axioms. While defining the operations and axioms, we will illustrate them on an important example, called the free forest algebra, where H is the set of all nonempty forests, and V is the set of all, possibly empty, contexts.

The operations and axioms of forest algebra are presented below. Elements of H will be denoted by h, g, f and elements of V will be denoted by v, w, u.

- A composition operation $+$ on H. This operation is required to be associative, i.e. $h + (g + f) = (h + g) + f$ holds for all $f, g, h \in H$. This makes H a semigroup, called the *horizontal semigroup*, and justifies the notation $h + g + f$. In the free forest algebra, $+$ is forest concatenation. We do not require H to contain a neutral element, e.g. there is no empty forest in the free forest algebra.

- A composition operation $\cdot$ on V. Again, this is required to be associative. We omit the $\cdot$ symbol, writing vw instead of $v \cdot w$, for $v, w \in V$. Furthermore, we require there to be a neutral element $\square \in V$, i.e. an element satisfying $v \cdot \square = \square \cdot v = v$ for all $v \in V$. In particular, V is a monoid, called the *vertical monoid*. In the free forest algebra, $\cdot$ is context composition, while $\square$ is the empty context.
- An insertion operation $V \to H \to H$. The result of this insertion is denoted by $vh \in H$. The empty context acts as the identity of this operation, i.e. $\square h = h$. The insertion operation must be a left action, i.e. it must satisfy $(vw)h = v(wh)$ for $v, w \in V$ and $h \in H$, which justifies the notation vwh. In the free forest algebra, the left action is substituting a forest into a context. There is an faithfulness requirement: distinct contexts $v, w \in V$ must induce different functions.
- An operation $\mathit{left} : H \times V \to V$. This operation must satisfy $\mathit{left}(h, v)g = h + vg$ for $v \in V$ and $g, h \in H$. Thanks to this axiom, we can without ambiguity write $h + v$ to denote the element $\mathit{left}(h, v)$. In the free forest algebra, $h + v$ is the context obtained from v by prepending the forest h (next to the root, not the hole). In a similar way we define $v + h$, in terms of an operation *right*.

As demonstrated above, the free forest algebra is a forest algebra. Clearly the free algebra depends on the leaf labels A and inner node labels B (and only on these); once these are given, the free algebra is denoted by $(A, B)^{\Delta}$. When describing a forest algebra, we usually only give names to the carrier sets H and V, leaving the operations implicit.

Let (H, V) and (G, W) be two forest algebras. A *forest algebra morphism*

$$\alpha : (H, V) \to (G, W)$$

is a pair of functions

$$\alpha = (\alpha_H, \alpha_V) \qquad \alpha_H : H \to G \qquad \alpha_V : V \to W$$

that preserve all operations in the signature, namely, composition $+$ in H, composition $\cdot$ in V, insertion, and the *left*, *right* operations. For instance, preserving insertion is:

$$\alpha_H(vh) = \alpha_V(v)(\alpha_H(h)) \ .$$

To avoid clutter, we omit the subscripts, and write $\alpha(h)$ instead of $\alpha_H(h)$, likewise for v.

If α is a morphism, then the *type under* α of a forest t is simply the value $\alpha(t)$. Whenever the morphism α is clear from the context, we omit the qualifier "under α".

In this paper, a forest algebra will either be a free forest algebra, or a finite forest algebra. In the first case, elements of the first sort will be called *forests* and denoted by s, t, while elements of the second sort will be called *contexts*, and denoted by p, q. In the second case, of a finite forest algebra, elements of the first sort will be called *forest types* and denoted by f, g, h, while elements of the second sort will be called *context types*, and denoted by u, v, w.

4.1. **Equivalence with regular languages.** In this section we show that forest algebras provide an equivalent description of regular tree languages. Although this has already been shown in [6], we present the proof here for two reasons. First, our definition is slightly different from the one in [6], where a neutral element was required in H. Second, the notion of semigroup automaton used in the equivalence will be used later on in the paper.

The point of forest algebras is to recognize forest languages. Let L be a set of forests over labels (A, B) and let (H, V) be a finite forest algebra. We say a morphism

$$\alpha : (A, B)^\Delta \to (H, V)$$

recognizes a forest language L if membership $t \in L$ depends only on the value $\alpha(t)$. In this case, we also say that the algebra (H, V) recognizes the language L. Note that this definition is for languages of forests, and not languages of trees, as in the logic $\mathsf{EF} + \mathsf{F}^{-1}$. We will deal with this discrepancy in Section 5.

Below we show that forest algebras recognize exactly the regular forest languages. What is a regular forest language? The definition used here, of a semigroup automaton, is chosen so that the translation to forest algebra is easiest. A *semigroup automaton* is a type of bottom-up finite automaton that can be used to recognize tree and forest languages. Let (A, B) be an alphabet. A semigroup automaton $\mathcal{A}$ over (A, B) is defined by a finite semigroup H, whose operation is denoted additively by $+$, along with two mappings (which describe the initial states and transitions, respectively):

$$\beta_A : A \to H \qquad \beta_B : B \to H^H$$

The purpose of the automaton is to uniquely associate a type $\beta(t) \in H$ to every forest t. This is done using the following rules:

$$\begin{aligned} \beta(a) &= \beta_A(a) \\ \beta(s_1 + \cdots + s_n) &= \beta(s_1) + \cdots + \beta(s_n) \\ \beta(bt) &= \beta_B(b)(\beta(t)) \ . \end{aligned}$$

Recall that in the last line above, bt is a tree that has b in the root and the forest t below.

An automaton recognizes a forest language L if membership $t \in L$ depends only on the value β. In other words, one can choose a set of accepting elements $F \subseteq H$ such that a forest t belongs to L if and only the value $\beta(t)$ belongs to F. The definition can be modified for recognizing tree languages by requiring the equivalence $t \in L \Leftrightarrow \beta(t) \in F$ to hold only for trees. Note that even when recognizing a tree language, a semigroup automaton is still obliged to assign a value from H to every forest.

It is not difficult to show that this definition is equivalent to other existing automata models for unranked trees, although there may be an exponential blowup when translating to semigroup automata.

Theorem 4.1. *A forest language is regular if and only if it is recognized by a finite forest algebra.*

Proof. Once we have a semigroup automaton, we can extend the mapping β so that contexts also get values, namely values in H^H. A context p is assigned the following mapping $\beta(p) \in H^H$:

$$h \mapsto \beta(pt) \ ,$$

where t is some forest with $\beta(t) = h$ (the choice of t does not change this value). It is easy to see that the mapping β (when seen as a mapping on both forests and contexts) is a forest algebra morphism

$$\beta : (A, B)^\Delta \to (H, H^H) \ .$$

This shows the harder direction in the proof of Theorem 4.1. The other direction, from a forest algebra to a semigroup automaton, is immediate. ☐

4.2. **Syntactic algebra.** The syntactic forest algebra of a forest language L is a canonical forest algebra that recognizes the language. It is defined using the following Myhill-Nerode equivalence over forests and contexts. Two forests s, t are considered equivalent if for every context p, either both or neither ps nor pt belongs to L. Two contexts p, q are considered equivalent if for every forest t, the forests pt and qt are equivalent in the above sense.

It turns out that the above defined equivalences are a congruence with respect to all operations in a forest algebra; therefore a quotient forest algebra can be defined, where elements of H are equivalence classes of forests, and elements of V are equivalence classes of contexts. This quotient forest algebra is called the *syntactic forest algebra* of L. The *syntactic morphism* is the morphism that assigns to each forest (resp. context) its equivalence class. The syntactic morphism recognizes L, furthermore it is optimal in the sense that the syntactic morphism factors through any morphism recognizing L, i.e. if β is a morphism recognizing L, and α is the syntactic morphism of L, then there is a (unique) morphism γ with $\alpha = \gamma \circ \beta$. In particular, the syntactic forest algebra is a morphic image of any forest algebra recognizing L, and a language has a finite syntactic algebra if and only if it is regular.

4.3. **Green's relations for trees.** Fix a forest algebra (H, V). In this section we introduce two preorders on V and H that will be used in the paper.

We say that context type $v \in V$ is *reachable* from a context type $w \in V$ if $v = wu$ holds for some context type $u \in V$. A *context component* is a maximal set of mutually reachable context types. Stated differently, two context types v, w are in the same context component if the ideals vV and wV are equal. Since reachability is transitive and reflexive, it induces an order (not necessarily linear) on context components.

We say a forest type $g \in H$ is *reachable* from a forest type $h \in H$ if $g = uh$ holds for some context type $u \in V$. A *forest component* is a maximal set of mutually reachable forests. Stated differently, two forest types g, h are in the same forest component if the ideals Vg and Vh are equal. As for context types, forest components are ordered by reachability. Note that $g + h$ is reachable from h, since we can take the context type u to be $g + \Box$.

These two preorders are related to Green's relations used in semigroup theory. Actually, reachability on contexts simply is the $\mathcal{R}$-order on the semigroup V. The reachability relation on H is not one of Green's relations, since its definition involves the two sorts H and V in the forest algebra.

5. Tree-Definable vs Forest-Definable

A tree language L is *tree-definable* if there is a formula of $\mathsf{EF} + \mathsf{F}^{-1}$ that is true exactly (in the root of) trees in L. In this paper, it will sometimes be convenient to talk about $\mathsf{EF} + \mathsf{F}^{-1}$ formulas defining properties of forests (and not only trees). We say a forest language L is *forest-definable* if L is a boolean combination of languages of the form "some tree in the forest satisfies φ", with φ a formula of $\mathsf{EF} + \mathsf{F}^{-1}$. Such a boolean combination will be called a *forest formula*. For instance, the following property of a forest $t_1 + \cdots + t_n$ is forest-definable: all trees $t_1, \ldots, t_n$ contain a leaf with label a, and at least one of these trees has root label b. Any nonempty tree language violates the following property, which is true for forest-definable languages:

$$t + t \in L \quad \text{iff} \quad t \in L \ ,$$

for the simple reason that $t+t$ is not a tree. Therefore no nonempty tree language is forest-definable. For the same reason, no nonempty forest-definable language is tree-definable.

In this paper, we will present a decidable characterization for forest-definable languages. Thanks to the following result, this will also give us a decidable characterization of tree-definable languages.

Proposition 5.1. *Let L be a tree language over (A, B). The following conditions are equivalent:*

- *L is tree-definable.*
- *For each inner node label $b \in B$, the forest language $\{t : bt \in L\}$ is forest-definable.*

Proof. We begin by showing that the first property implies the second. Assume then that L is tree-definable, and fix some $b \in B$. We need to show that the forest language $\{t : bt \in L\}$ is forest definable.

Let P be the set of contexts of the form $p = b(\Box + t)$, where t is a forest. Consider the following equivalence relation on trees:

$$s \sim t \qquad \text{iff} \qquad ps \in L \Leftrightarrow pt \in L \quad \text{holds for all } p \in P \ .$$

This equivalence relation has only finitely many classes, since it is coarser than the Myhill-Nerode equivalence relation used in the definition of syntactic algebra. Note that we would get the same equivalence relation by also considering contexts of the form $p = (s + \Box + t)$, since $\mathsf{EF} + \mathsf{F}^{-1}$ is invariant under reordering siblings. Furthermore, each of these equivalence classes is tree-definable, thanks to the following fact: if p is a context and L a tree-definable language then the set of trees t with $pt \in L$ is tree-definable. The standard proof of this fact is omitted here. For any forest $t = t_1 + \cdots + t_n$, membership $bt \in L$ only depends on the equivalence classes under $\sim$ of the trees $t_1, \ldots, t_n$ that the constitute the forest t. Since $\mathsf{EF} + \mathsf{F}^{-1}$ formulas are invariant under duplicating and reordering sibling trees, it is only the set of equivalence classes that counts, which can be described by a boolean combination of languages of the form required in forest-definable languages.

We now do the bottom-up implication. It suffices to show that if a forest language L is forest-definable, then for any inner node label $b \in B$, the tree language $\{bt : t \in L\}$ is tree-definable. The key step is that if a tree language K is tree-definable, then the following tree language:

$$\mathsf{X}K = \{b(t_1 + \cdots + t_n) : b \in B, \exists i.\ t_i \in K\}$$

is also tree-definable. Once we demonstrate how to write a formula for $\mathsf{X}K$, the formula tree-defining bL can be obtained from the formula forest-defining L.

Note that definability of the language $\mathsf{X}K$ does not mean we can add the child operator to the logic. This is because $\mathsf{X}K$ uses the child only at a fixed depth. For instance, the property "some node at depth 4 has the same label as its parent" is tree-definable, contrary to the property "some node has the same label as its parent".

The formula for $\mathsf{X}K$ can be obtained from the antichain composition principle, but we do a direct construction here. Let φ be the formula defining K. We define $\hat{\varphi}$ to be the formula obtained from φ by replacing every subformula ψ by $\psi \wedge \mathsf{F}^{-1}\mathit{true}$. This way, quantification in $\hat{\varphi}$ is relativized to non-root nodes. Finally, the formula for $\mathsf{X}K$ is

$$\mathsf{EF}\big((\mathsf{F}^{-1}\mathit{true}) \wedge (\neg\mathsf{F}^{-1}\mathsf{F}^{-1}\mathit{true}) \wedge \hat{\varphi}\big) \ .$$

The above formula nondeterministically picks a successor x of the root, and then tests if $\hat{\varphi}$ holds in x. Since $\hat{\varphi}$ is relativized to non-root nodes, evaluation of $\hat{\varphi}$ will never leave the subtree of x. $\square$

6. The identities and the main result

In this section we state our main result, the decidable characterization of the logic $\mathsf{EF} + \mathsf{F}^{-1}$.

The characterization uses a relation $\dashv$ over contexts in a forest algebra. The idea is that $u \dashv w$ holds if the context u can be obtained from the context w by removing forests that are siblings of the main path (recall that the main path contains ancestors of the hole). Let (H, V) be a forest algebra. For $u, w \in V$, we write $u \dashv w$ if u, w can be decomposed as

$$u = v_0 v_1 \cdots v_n \qquad w = v_0(h_1 + v_1) \cdots (h_n + v_n)$$

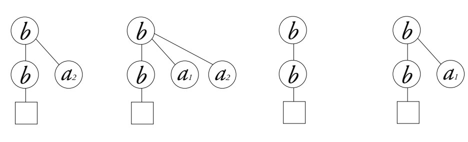
for some $v_0, \ldots, v_n \in V$ and $h_1, \ldots, h_n \in H$. The reason why we have v_0 above, and not $h_0 + v_0$, is that a context type can be empty, but there is no empty forest type. The following lemma shows that the relation $\dashv$ can be calculated in polynomial time using a least fixpoint algorithm:

Lemma 6.1. *The relation $\dashv$ is the least relation $R \subseteq V \times V$ such that:*

$$(v, v), (v, v + h), (v, h + v) \in R \qquad \text{for } v \in V,\ h \in H$$
$$(v, v'), (w, w') \in R \Rightarrow (vw, v'w') \in R \qquad \text{for } v, v', w, w' \in V\ .$$

Proof. The implication from $(v, w) \in R$ to $v \dashv w$ is proved by induction on the number of steps in the derivation. The converse implication is proved by induction on n in the definition of $\dashv$. $\square$

The relation $\dashv$ is transitive in some forest algebras, including all free forest algebras. However, in general it need not be transitive, as illustrated by the following example. Let the leaf alphabet A be $\{a_1, a_2\}$ and let the inner node alphabet B be $\{b\}$. Consider the forest language L: "the forest does not contain both labels a_1 and a_2 at the same time, and every node with label b has a sibling with label a_1 or a_2". Let α be the syntactic morphism of this language. Consider the following four contexts:



p_1 p_2 q_1 q_2

Clearly we have $\alpha(p_1) \dashv \alpha(p_2)$ and $\alpha(q_1) \dashv \alpha(q_2)$. We claim that $\alpha(p_2) = \alpha(q_1)$. Indeed, both contexts are "error" contexts, i.e. for any context r and forest t we have $rp_2t, rq_1t \notin L$. Therefore, if $\dashv$ were a transitive relation, we would have $\alpha(p_1) \dashv \alpha(q_2)$. This, however, cannot hold, since otherwise we could construct a tree in L with both a_1 and a_2 labels.

We are now ready to state the main theorem of this paper:

Theorem 6.2. *A language is forest-definable in $\mathsf{EF} + \mathsf{F}^{-1}$ if and only if its syntactic algebra satisfies the following identities:*

$$h + h = h \qquad g + h = h + g \tag{6.1}$$

$$(vw)^\omega = (vw)^\omega w(vw)^\omega \; . \tag{6.2}$$

$$(u_1w_1)^\omega(u_2w_2)^\omega = (u_1w_1)^\omega u_1w_2(u_2w_2)^\omega \qquad \textit{if } u_1 \dashv u_2, w_1 \dashv w_2 \; . \tag{6.3}$$

In the identities above, all variables are quantified universally. The identities in (6.1) say that children can be duplicated and reordered. This corresponds to bisimulation invariance in the following way: a forest language is bisimulation invariant if and only if its syntactic forest algebra satisfies (6.1). The identity (6.2) says that the vertical monoid belongs to the variety DA (although the commonly used identity is different). Only the last identity is new.

The exponent ω in properties (6.2) and (6.3) stands for "for almost all n". In particular, identity (6.2) should be read as:

$$\exists m \forall n \geq m \qquad (vw)^n = (vw)^n w(vw)^n \; .$$

Usually in semigroup theory, ω stands for "least idempotent power", but the above definition is equivalent for aperiodic monoids, which is the case here, thanks to (6.2).

An important corollary of the above theorem is that definability in $\mathsf{EF}{+}\mathsf{F}^{-1}$ is decidable:

Corollary 6.3. *It is decidable if a forest (resp. tree) language is forest-definable (resp. tree-definable) in* $\mathsf{EF} + \mathsf{F}^{-1}$*. The algorithm runs in polynomial time if the input is given as a forest algebra.*

Proof. To determine if a language is tree-definable, we calculate the languages $\{t : bt \in L\}$ and reduce to the characterization of forest-definable language thanks to Proposition 5.1. Therefore, we focus on deciding if a language is forest-definable.

We begin by finding the syntactic forest algebra. The syntactic forest algebra can be effectively calculated based on any representation of the tree language, be it a tree automaton, or a formula of some rich logic, such as MSO. In general, the syntactic forest algebra can be exponentially larger than a nondeterministic tree automaton, not to mention a formula of MSO.

Once the syntactic forest algebra has been calculated, the properties (6.1), (6.2) and (6.3) can be verified in polynomial time (with respect to the algebra). The relation $\dashv$ over V can be computed in polynomial time thanks to Lemma 6.1. The exponent ω is not a problem. Indeed, a consequence of (6.2) is that V is *aperiodic*, i.e. the identity $v^\omega = v^\omega v$ holds for all context types v. In particular, it is enough to test for $\omega = |V|$. □

The rest of this paper is devoted to showing Theorem 6.2. The "only if" implication in the above theorem is proved in Section 7 using a simple induction on formula size. The difficult part is the proof of the "if" implication, which is found in Section 8.

In the following fact, we show that property (6.3) in Theorem 6.2 is not redundant. In a similar way one can prove that neither (6.1) nor (6.2) is redundant.

Lemma 6.4. *There exists a forest algebra satisfying properties (6.1) and (6.2) but not (6.3).*

Proof. Let the leaf alphabet A be $\{a_1, a_2\}$ and let the inner node alphabet B be $\{b\}$. Consider the following language: "if a node has a child with label a_1, then it has an ancestor with a child with label a_2". The syntactic forest algebra of this language satisfies properties (6.1) and (6.2); but it does not satisfy (6.3), since for all $n \in \mathbb{N}$ we have

$$(bb)^n((b+a_2)(b+a_1))^n a_2 \in L \qquad (bb)^n b(b+a_1)((b+a_2)(b+a_1))^n a_2 \notin L \; . \qquad \square$$

7. Correctness

In this section we show that any language forest-definable in $\mathsf{EF} + \mathsf{F}^{-1}$ satisfies the identities from Theorem 6.2. For each of these identities we show that any formula of $\mathsf{EF} + \mathsf{F}^{-1}$ must, informally speaking, confuse the two trees described by the opposing sides of the identity. To show this confusion, we use an Ehrenfeucht-Fraïssé game. The plan of this section is as follows. First, in Section 7.1, we define the Ehrenfeucht-Fraïssé that characterizes $\mathsf{EF} + \mathsf{F}^{-1}$. Next, in Section 7.2, we use the game to show that languages defined in $\mathsf{EF} + \mathsf{F}^{-1}$ are closed under morphic preimages. Finally, in Section 7.3 we show that any language forest-definable in $\mathsf{EF} + \mathsf{F}^{-1}$ satisfies the identities from Theorem 6.2.

7.1. **Ehrenfeucht-Fraïsse Game.** In this section, we define an Ehrenfeucht-Fraïssé game that characterizes the logic $\mathsf{EF} + \mathsf{F}^{-1}$.

The game is played on two forests s_0 and s_1, with two distinguished nodes, x_0 in s_0 and x_1 in s_1. A configuration of the game is therefore a four-tuple (x_0, x_1, s_0, s_1). Finally, the game has a parameter $n \in \mathbb{N}$, which is called the number of *rounds*. The game is played by two players, Duplicator and Spoiler. The idea is that Duplicator claims that the same formulas of size at most n hold in x_0 and x_1.

The game is played as follows. Assume that there are $n \geq 0$ rounds left. If the labels of x_0, x_1 are different, then Spoiler wins the game immediately, and no further rounds are played. If the labels are the same, and $n = 0$, then Duplicator wins the game, and no further rounds are played. Finally, if the labels are the same and $n > 0$, a new round is played as follows.

First, Spoiler chooses one of the two nodes x_0, x_1, i.e. he chooses an index $i \in \{0, 1\}$. The idea is that Spoiler thinks that the node x_i has some property that the other node x_{1-i} does not have. He then chooses to make either a descendant move (in this case, Spoiler thinks that x_i has a descendant unlike all descendants of x_{1-i}) or an ancestor move (Spoiler thinks that x_i has an ancestor unlike all ancestors of x_{1-i}) . If Spoiler chooses a descendant (respectively, ancestor) move, then he must choose a proper descendant (respectively, proper ancestor) y_i of x_i in the forest s_i. To this, Duplicator must respond by choosing a proper descendant (respectively, proper ancestor) y_{1-i} of x_{1-i} in the other forest s_{1-i}. The idea is that Duplicator thinks that y_{1-i} is similar to y_i, at least as far as the remaining $n - 1$ rounds are concerned. Formally, the new configuration becomes (y_0, y_1, s_0, s_1) and the game continues with $n - 1$ rounds left.

We also define how the n-round game is played on two forests s_0, s_1 in case when the nodes x_0, x_1 are not specified. In this case, there is a special introductory round, where Spoiler chooses $i \in \{0, 1\}$ and a root node x_i in s_i; Duplicator responds with a root node x_{1-i} in the other forest. Then the standard n-round game continues from this configuration.

Proposition 7.1. *A forest language is forest-definable in* $\mathsf{EF}+\mathsf{F}^{-1}$ *if and only if for some* n*, Spoiler wins the* n*-round game for any pair of forests* $s_0 \in L$ *and* $s_1 \notin L$.

Proof. The proof is standard, and omitted here. The idea is that n is the nesting depth of the formulas used to forest-define L. The nesting depth counts the maximal nesting of EF and F^{-1} in a formula, while boolean operations are for free. □

7.2. **Morphic images.** In this section, we show that languages forest-definable in $\mathsf{EF}+\mathsf{F}^{-1}$ are closed under morphic preimages. Actually, we show a slightly more general result. The more general setting will be used in Section 9, where we show that our characterization also works for a different model of forest algebra, where empty forests are allowed.

We first describe the more general setting. The generalization is twofold. First, we allow empty forests. Second*[1], we consider forests over a single alphabet (unlike the two-sorted alphabet A, B considered before, with A allowed only in leaves and B allowed only in inner nodes). The new type of forests will be called *one-sorted forests*, to distinguish them from the *two-sorted forests* considered before. The one-sorted forests are more general in the following sense: the two-sorted forests over an alphabet (A, B) are a subset of the one-sorted forests over the alphabet $A \cup B$. Of course, the difference is not that big: the one-sorted forests over A are the two-sorted forests over (A, A), plus the empty forest.We also have an analogous concept of *one-sorted contexts*. A *one-sorted morphism*, with *source alphabet* A and *target alphabet* B is given by a function that assigns to each letter of A a one-sorted context, possibly empty, over B. A one-sorted morphism uniquely extends to one-sorted forests and one-sorted contexts. To avoid confusion, in this section we use the name *two-sorted morphism* for the morphisms introduced previously in the paper.

Theorem 7.2. *Let α be a one-sorted morphism. If a forest language L over the target alphabet B is forest-definable in $\mathsf{EF}+\mathsf{F}^{-1}$, then so is its inverse image $\alpha^{-1}(L)$.*

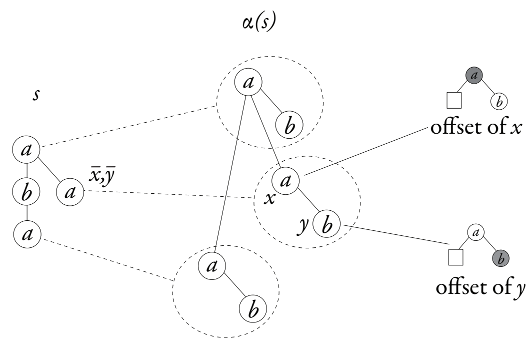
The version of this theorem for two-sorted morphisms is a special case of the one-sorted version, since every for two-sorted morphism there is a one-sorted morphism that gives the same results over all legal two-sorted forests.

To show this theorem, we will use the Ehrenfeucht-Fraïssé game. We fix the forest-language L and the (one-sorted) morphism α from the theorem for the rest of this section. Let n be the number of rounds obtained by applying Proposition 7.1 to the forest L in the statement of the theorem. By invoking Proposition 7.1 a second time, to establish that the inverse image $\alpha^{-1}(L)$ is forest-definable in $\mathsf{EF}+\mathsf{F}^{-1}$, it suffices to show that Spoiler can win the n-round game over any two preimages, one taken from the preimage $\alpha^{-1}(L)$, and the other taken from its complement. The proof will be by showing how a strategy of Duplicator over the preimage can be lifted to a strategy over the image, as stated in the following proposition.

Proposition 7.3. *If Duplicator wins the n-round game over s_0, s_1, then Duplicator also wins the n-round game over $\alpha(s_0), \alpha(s_1)$.*

To prove this transfer of strategies, we will be switching back and forth between the Ehrenfeucht-Fraïssé games on s_0, s_1 and on $\alpha(s_0), \alpha(s_1)$. To avoid confusion, we use the name *preimage game* for the former and we use the name *image game* for the latter. We will be comparing configurations of the two games in the following way. Every node x in a morphic image $\alpha(s)$ can be uniquely identified by two pieces of information: its *preimage* $\bar{x}$, which is a node in the preimage forest s, and its *offset*, which is a node of the context assigned by α to the label in $\bar{x}$. These concepts are illustrated below, in an example where both the source and target alphabets are $\{a, b\}$, and the one-sorted morphism is defined by $\alpha(a) = a(\Box + b)$ and $\alpha(b) = \Box$.

[1]* It turns out that in forest algebra, the first generalization entails the second.



Note that some nodes in the preimage forest s are not the preimage of any node in $\alpha(s)$, these are the nodes whose labels are mapped to an empty context by α.

Armed with the definitions of offset and preimage, we now prove the strategy transfer from Proposition 7.3. We only give the main invariant, which is described below. The missing part of the proof, for the introductory round of the game where the root nodes are chosen, is done in a similar way.

Lemma 7.4. *Let $m \leq n$. Let x_0, x_1 be nodes with the same offset such that $\bar{x}_0, \bar{x}_1$ have the same label. If Duplicator can win the n-round preimage game in configuration $(\bar{x}_0, \bar{x}_1, s_0, s_1)$, then he can also win the m-round image game in configuration $(x_0, x_1, \alpha(s_0), \alpha(s_1))$.*

Proof. The proof is by induction on n. Consider first the case of $n = 0$. By assumption on the preimage game, the nodes $\bar{x}_0$ and $\bar{x}_1$ have the same labels in s_0, s_1. Since the two nodes x_0, x_1 have the same offsets, they must also have the same labels in the images $\alpha(s_0), \alpha(s_1)$, and therefore Duplicator wins.

Consider now the induction step. We only do the case when Spoiler chooses a descendant move, the ancestor move is done the same way. Assume then that Spoiler chooses x_i and indicates a proper descendant y_i of x_i in $\alpha(s_i)$. How should Duplicator respond? There are two possible cases:

- The preimage $\bar{y}_i$ is a proper descendant of $\bar{x}_i$. We now go to the preimage game, and make Spoiler play a descendant move where he chooses $\bar{y}_i$. By assumption on Duplicator winning the preimage game, there is a proper descendant of $\bar{x}_{1-i}$, call it $\bar{y}_{1-i}$, such that Duplicator wins the $(n-1)$-round preimage game from configuration $(\bar{y}_0, \bar{y}_1, s_0, s_1)$. In particular, the nodes $\bar{y}_0, \bar{y}_1$ have the same labels in the preimage, and therefore the same possible offsets in the image. Therefore, there exists a node y_{1-i} in $\alpha(s_{1-i})$ such that its preimage is $\bar{y}_{1-i}$, and this node can be chosen to have the same offset as $\bar{y}_i$. We now use the induction assumption to show that Duplicator wins the rest of the image game from configuration $(y_0, y_1, \alpha(s_0), \alpha(s_1))$.
- If the preimage $\bar{y}_i$ is not a proper descendant of $\bar{x}_i$, then $\bar{y}_i = \bar{x}_i$ and the only difference between y_i and x_i is in the offset. Duplicator's response is to choose in the forest s_{1-i} a node y_{1-i} that has the same offset as y_i, and such that $\bar{y}_{1-i} = \bar{x}_{1-i}$. We then use the induction assumption to show that Duplicator wins the rest of the image game. □

7.3. **Correctness of the identities.** We are now ready to show the easier implication in Theorem 6.2, namely that the syntactic forest algebra of a language forest-definable in $\mathsf{EF} + \mathsf{F}^{-1}$ satisfies the three identities. Validity of (6.1) can easily be shown. We omit the proof of (6.2) for two reasons: first, it is the same as in the word case, see e.g. [13]; and second, it follows along similar lines as the proof of (6.3).

The rest of this section is devoted to showing the validity of identity (6.3). Let L be a forest language forest-definable in $\mathsf{EF} + \mathsf{F}^{-1}$. We need to show that the syntactic algebra of L satisfies identity (6.3). Recall that elements of the syntactic algebra are equivalence classes of the Myhill-Nerode equivalence relation. Therefore, in order to show the validity of (6.3), we have to show that for any formula φ of $\mathsf{EF} + \mathsf{F}^{-1}$, for all contexts $p_1 \dashv p_2$ and $q_1 \dashv q_2$, every context p and every nonempty forest t, for almost all $n \in \mathbb{N}$ the formula φ is true in some tree of either both or neither of the forests

$$s_0 = p(p_1q_1)^n(p_2q_2)^n t \qquad s_1 = p(p_1q_1)^n p_1q_2(p_2q_2)^n t \ . \tag{7.1}$$

We will use the Ehrenfeucht-Fraïssé game, and show that Duplicator can win the n-round game over the above two forests. To keep notation simple, we assume the following *simplifying assumptions* are met.

- The context p is a single node b (in particular, s_0 and s_1 are trees).
- The forest t is a single node a.
- The contexts p_1, p_2, q_1, q_2 are
$$\begin{aligned} p_1 &= b_1 \cdots b_k & p_2 &= b_1(a_1 + \Box) \cdots b_k(a_k + \Box) \\ q_1 &= b_{k+1} \cdots b_m & q_2 &= b_{k+1}(a_{k+1} + \Box) \cdots b_m(a_m + \Box) \end{aligned}$$
  for some $k < m$ and $b_1, \ldots, b_m \in B,\ a_1, \ldots, a_m \in A$.
- The labels $a, a_1, \ldots, a_m, b, b_1, \ldots, b_m$ and a are all distinct.

The trees s_0 and s_1 are shown in Figure 1. Why can we make these simplifying assumptions? The reason is that the general case follows from this special case by way of homomorphic images. More specifically, consider the two forests s_0, s_1 in the general case, as given in (7.1). We want to show that Duplicator wins the n-round game over these two forests. The key observation is that any two forests s_0, s_1 as in (7.1) can obtained as homomorphic images $s_0 = \alpha(t_0)$ and $s_1 = \alpha(t_1)$ from trees t_0, t_1 that satisfy the simplifying assumption, for some (two-sorted) morphism α. As long as we know how Duplicator can win the game over the simpler trees t_0, t_1, we can use Proposition 7.3 to transfer this result to the forests s_0, s_1.

We now proceed to describe a winning strategy for Duplicator over trees s_0, s_1 that satisfy the simplifying assumptions. We use the term *main path* for the ancestors of the node a. The *projection* of a node onto the main path is its closest ancestor (not necessarily proper) that is on the main path. For a node in either s_0 or s_1, the *ancestor block count* (respectively, *descendant block count*) is the number of ancestors with label b_m (respectively, descendants with b_1) of the node's projection onto the main path. For $m \leq n$, we say that two nodes x_0, x_1 in the trees s_0, s_1 are *m-similar* if their labels are the same and moreover one of the conditions in the following invariant holds:

(1) The trees s_0, s_1 agree on nodes in the subtrees of y_0, y_1; or
(2) The trees s_0, s_1 agree on nodes not in the subtrees of y_0, y_1; or
(3) The ancestor and descendant block counts of x_0, x_1 are both at least m.

Lemma 7.5. *Let $m \leq n$. If the nodes x_0, x_1 are m-similar, then Duplicator wins the m-round game from configuration (x_0, x_1, s_0, s_1).*

Proof. The proof is by induction on m. For the base case $m = 0$ we use the assumption that the labels are the same. Consider now the induction step. We only do one case, when Spoiler chooses a descendant move to go from x_1 to a node x_1' in the "new block" of s_1 (the

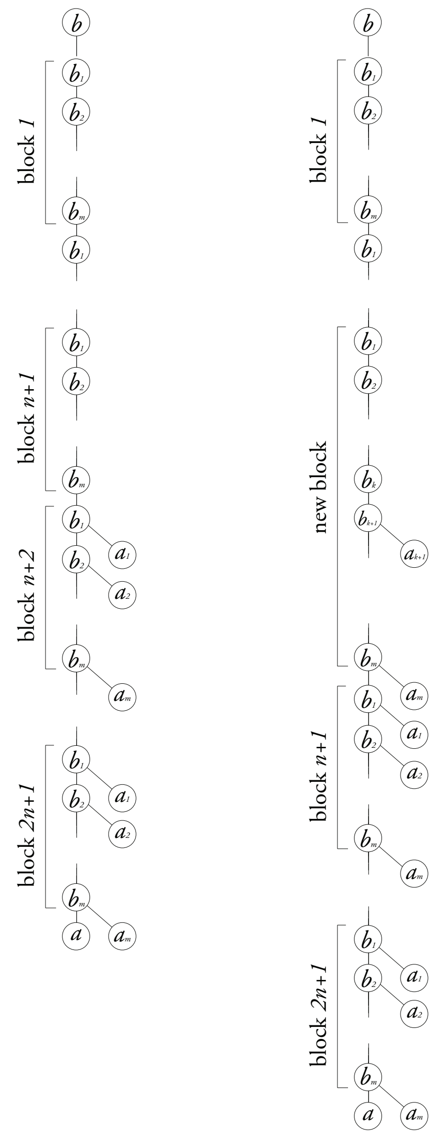


Figure 1: The trees s_0 and s_1

new block is the context $p_1 q_2$). This Spoiler move means that x_0, x_1 are m-similar for reason (2) or (3), since item (1) forbids a descendant of x_1 in the new block. What is Duplicator's response? Note that for all nodes in the new block, both the ancestor and descendant block counts are at least $n \geq m-1$. Duplicator goes to any node x_0' in the tree s_0 where the ancestor and descendant block counts are both at least $m-1$. This must be possible, since either one of items (2) or (3) of the invariant was true for x_0, x_1. The rest of the game is played according the induction assumption, since x_0' and x_1' are $(m-1)$-similar. □

By taking $m = n$ in the above lemma, we get the desired result. This is because the two roots of s_0, s_1 have the same (empty) prefixes, thus they are m-similar, and must therefore satisfy the same formulas of size $m = n$.

8. Completeness

This section is devoted to showing:

Proposition 8.1. *Any forest language recognized by a forest algebra satisfying (6.1), (6.2) and (6.3) can be forest-defined.*

The above statement immediately implies the more difficult "if" part of Theorem 6.2. Indeed, if L is recognized by an algebra satisfying (6.1), (6.2) and (6.3), then its syntactic algebra satisfies these identities. This is because the syntactic algebra is a morphic image of any algebra recognizing the language, and identities are preserved by morphic images.

Let $X \subseteq H$ be a set of forest types. We say a forest t is X*-trimmed* if the only subtrees of t that have a type in X are leaves. We say a tree language L is tree-definable *modulo* X if there is a formula φ such that

$$t \text{ satisfies } \varphi \qquad \text{iff} \qquad t \in L$$

holds for all X-trimmed trees (for other trees, φ may disagree with L). In a similar fashion, we define a forest language that is forest-definable modulo X.

Instead of Proposition 8.1, we show the slightly more general result below, which contains the induction parameters that appear in the proof.

Proposition 8.2. *Let $\alpha : (A, B)^\Delta \to (H, V)$ be a morphism, with (H, V) satisfying identities (6.1), (6.2) and (6.3). Let $X \subseteq H$ be a set of forest types, and let $v \in V$ be a context type. For each forest type $h \in H$ the following forest language is forest-definable modulo X:*

$$\{t : v(\alpha(t)) = h\} \ . \tag{8.1}$$

For the rest of Section 8, we fix $\alpha : (A, B)^\Delta \to (H, V)$, $h \in H$, $v \in V$ and $X \subseteq H$ from Proposition 8.2. Clearly Proposition 8.1 follows from the above result, taking $X = \emptyset$, v to be the empty context type $\square$, and doing a disjunction over all forest types $h \in \alpha(L)$. The rest of Section 8 is devoted to a proof of Proposition 8.2. The proof is by induction on four parameters:

(1) The size of H, i.e. the number of all forest types.
(2) The size of $H \setminus X$, i.e. the number of forest types that can be found outside leaves.
(3) The size of vV, i.e. the number of context types reachable from v.
(4) The size of B, i.e. the number of inner node labels.

The order of these parameters is important: first we try to minimize H, then the other three parameters (the order for the other three is not important). Note that the last parameter depends on the alphabet B, and the notion "modulo X" depends on the morphism.

We say a morphism α into (H, V) is *leaf saturated* if for every $h \in H$, there is a representative leaf label a whose type $\alpha(a)$ is h. In the rest of this section, we will only consider such morphisms. By adding leaf labels, any morphism can be extended to one that is leaf saturated, without affecting the target forest algebra.

We begin by outlining our proof strategy for Proposition 8.2. We will consider three possible cases. First, in Section 8.1, we see what happens when some inner node label $b \in B$ has the property that v cannot be reached from $v\alpha(b)$. Then, in Section 8.2, we see what happens if $H \setminus X$ intersects more than one forest component, i.e. contains at least two forest types that are not mutually reachable. Finally, in Section 8.3, we show that if neither of the above holds, then the formula φ in Proposition 8.2 can basically be replaced by either "true" or "false".

8.1. **For some inner node label $b \in B$, v is not reachable from $v\alpha(b)$.** We begin with this case, which is the easiest of the three. The basic idea is that we cut the forest into two parts, by looking at the first occurrence of b on each path, beginning at the root. Since after reading the label b, the context type v is no longer reachable, we can use the induction assumption to calculate the subtree below each such first b. These subtrees can then be squashed into single leafs using the antichain composition principle, and therefore the induction assumption can be used on a smaller alphabet of inner node labels, which now no longer contains b.

We say that two forest types $h, g \in H$ are *v-equivalent* if $vuh = vug$ holds whenever v is not reachable from vu.

Lemma 8.3. *For each h, the set of forests whose type is v-equivalent to h is forest-definable modulo X.*

Proof. Fix some context type u such that v is not reachable from vu. By induction assumption—the third parameter is decreased—the set of forests s satisfying $vu\alpha(s) = vuh$ is forest-definable modulo X. The set in the statement of the lemma is the intersection, over u, of all these sets. □

Lemma 8.4. *If $v, w, vu \in V$ are in the same context component, then so is wu.*

Proof. By assumption there must be context types v', w' with $vuw' = w$ and $wv' = v$. But then we have $wv'uw' = w$. In particular, $w(v'uw')^\omega v' = v$. Using identity (6.2), we get

$$v = w(v'uw')^\omega v' = w(v'uw')^\omega uw'(v'uw')^\omega v' = wuw'(v'uw')^\omega v' \ ,$$

which shows v can be reached from wu. □

Let $\gamma_1, \ldots, \gamma_n$ be all the equivalence classes of v-equivalence. For each such class γ_i, let L_i be the set of trees $\{bt : \alpha(t) \in \gamma_i\}$. Thanks to Lemma 8.3, each set L_i is tree-definable. For any $i = 1, \ldots, n$, let h_i be an arbitrarily chosen forest type in the class γ_i, and let a_i be a leaf label whose type is $\alpha(b)h_i$. The label a_i exists by assumption on leaf saturation. Note that a_i may have a different type than some of the trees in L_i, since h_i need not be the only forest type in γ_i. However, we will show that no information is lost by squashing subtree in L_i into a single leaf with label a_i, at least as long as the resulting forest is going to be an argument of v. More formally, we show:

Lemma 8.5. *Let $\varphi = b \wedge \neg \mathsf{F}^{-1} b$, i.e. "a b without b ancestors". For any forest t we have*

$$v\alpha(t) = v\alpha(t[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n]) \ .$$

Before we show this lemma, we show how it concludes the case considered in this section. Recall that we want to show that the following language is forest-definable modulo X:

$$L = \{t : v\alpha(t) = h\} \ .$$

By Lemma 8.5, this is the same language as

$$\{t : t[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n] \in L\} \ .$$

Since the substitution operation removes all letters b from the forest, we get

$$L = \{t : t[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n] \in K\} \ ,$$

where K is the set of trees in L that do not use the letter b. To K we can apply the induction assumption on a smaller alphabet, and then use the antichain composition principle to transfer definability from K to L.

We now resume with the proof of Lemma 8.5.

Proof. Note first that the tree on the right hand side of the equation is well defined, since the languages L_i are disjoint, and φ is an antichain formula. The proof is by induction on the number of b nodes in the forest t. The induction base, where there are no b's, is immediate since the substitution on the right hand side does not change the forest. Otherwise, let t be of the form pbs, with the context p not containing any b's on the main path, and let L_i be such that $bs \in L_i$. By induction assumption, we have

$$\alpha(pa_i[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n]) = \alpha(pa_i) \ .$$

By definition of the substitution we have

$$t[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n] = pa_i[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n] \ ,$$

it therefore remains to show that $v\alpha(pba_i) = v\alpha(pbs)$.

First, we claim that v is not reachable from $v\alpha(pb)$. Indeed, if v is not reachable from $v\alpha(p)$ then we are done. Otherwise, v and $v\alpha(p)$ are in the same context component. If this context component would also contain $v\alpha(pb)$, then by Lemma 8.4 it would also contain $v\alpha(b)$, a contradiction with the assumption on b.

Recall now the forest type h_i that represented the equivalence class $\gamma_i \ni \alpha(s)$. By assumption on $\alpha(s)$ and h_i being v-equivalent, we get

$$v\alpha(pbs) = v\alpha(pb)\alpha(s) = v\alpha(pb)h_i = v\alpha(p)\alpha(b)h_i = v\alpha(p)\alpha(a_i) = v\alpha(pa_i) \ . \qquad \square$$

8.2. **There is more than one forest component in $H \setminus X$.** We now turn to the second case in the proof of Proposition 8.2. Let $G \subseteq H$ be a forest component not included in X. We pick G so that no forest type in G can be reached from a forest type outside $X \cup G$. Intuitively speaking, forest types from G are close to the leaves. The essential idea in this section is that we will add G to X, by squashing each subtree of type g to a single leaf with the g written in its label. This is done by applying the antichain composition.

Let $W \subseteq V$ be the set of context types that preserve G, i.e. context types w such g is reachable from wg for some $g \in G$. The following lemma, proved the same way as Lemma 8.4, shows that "some" in the above definition can be replaced by "all".

Lemma 8.6. *If g, h, vg are in the same forest component, then so is vh.*

Let $F \subseteq H$ be the set of those forest types f from which a forest type in G can be reached. In particular, we have

$$G \subseteq F \subseteq H \ .$$

Note that all forest types in $F \setminus X$ are from G by choice of G. Furthermore, the inclusion $F \subseteq H$ is proper, since $H \setminus X$ contains more than one forest component by assumption. The inclusion $G \subseteq F$ may also be proper, however all forest types in the difference $F \setminus G$ are from X.

We say $f \in H$ is a *bad brother* if for all $g \in G$, we have $f + g \notin G$, i.e. g is not reachable from $f + g$. Likewise, we say $f \in H$ is a *good brother* if for all $g \in G$, we have $f + g \in G$, i.e. g is reachable from $f + g$. Note that by definition of F, all good brothers are in F. Clearly f is a bad brother if and only if the context type $f + \Box$ is outside W. Therefore by Lemma 8.6, every forest type in H is either a good brother or a bad brother. In particular, all forest types in G are good brothers, since they cannot be bad brothers by $g + g = g$.

Furthermore, since W is closed under context composition, good brothers are closed under forest concatenation, i.e. form a subsemigroup of H.

We fix the sets G, F and W for the rest of Section 8.2.

A *twig* is a tree of depth exactly two, i.e. a root and some leaves. A *twig node* is a node whose subtree is a twig.

Lemma 8.7. *There is a formula ψ such that in any X-trimmed tree, ψ holds in nodes with a subtree of type in G.*

Proof. Let t be an X-trimmed tree, and x a node in this tree. If the node is a leaf, then the type of its subtree can be read from the label. Otherwise, the type of the subtree must be either in G or outside F, by assumption on the tree being X-trimmed. We claim that the following condition is necessary and sufficient for the subtree of x to have a type outside F, and can furthermore be tested by a formula of $\mathsf{EF} + \mathsf{F}^{-1}$. The condition is that some descendant y of x, not necessarily proper, is either

(1) A leaf or twig node with a type outside F; or
(2) A non-twig inner node with a label $b \in B$ whose type $\alpha(b)$ is outside W; or
(3) An inner node whose brother has a leaf label a whose type $\alpha(a)$ is a bad brother.

We begin by showing that these conditions can be tested by an $\mathsf{EF} + \mathsf{F}^{-1}$ formula. Testing for 1) is simple. Using EF, we search for a candidate y for the node. If y is a leaf, we just test its label. Otherwise, we test if y is a twig node (no path of length at least two). Then we read the label of y and the set of labels in descendants of y, which uniquely determine the type of the subtree of y, thanks to idempotency and commutativity, i.e. identities (6.1). Condition 2 is tested in a similar way. For condition 3 we use EF to go into a leaf y with a label a whose type is a bad brother. We then test if y has a sibling that is an inner node (all ancestors of y have an inner node descendant).

We now show that these conditions are sufficient. The first one is clearly sufficient. For the other two, note that every inner node has a subtree with type outside X by assumption on the tree being X-trimmed. This type must then be either in $G \supseteq F \setminus X$ or outside F. For the second condition, let bs be the subtree of a non-twig inner node, with s a forest. Since s has depth at least two, its type must be outside X, and therefore either outside F, or in $G \supseteq F \setminus X$. In either case, the type of bs is outside F. The last condition is shown in a similar way.

It remains to show that the conditions are necessary. Indeed, assume that the subtree of x has a type outside G. Let s be a minimal subtree below x that has a type outside F. If s is a leaf or a twig, then item 1 must hold. Otherwise s is of the form $b(s_1 + \cdots s_n)$, for some label $b \in B$ and trees $s_1, \ldots, s_n$, with at least one tree s_i not being a leaf. By assumption on the tree being X-trimmed, the type of s_i is outside X. Since $F \setminus X \subseteq G$, the type of this s_i is in G. If all the types of s_j, for $j \neq i$, are good brothers, then the type of $s_1 + \cdots + s_n$ must belong to G by closure of good brothers under composition, and therefore case 2 must hold. Finally, we consider the case when the type of some tree s_j is a bad brother. Since all forest types from G are good brothers, the type of s_j is in $F \setminus G \subseteq X$. Since the tree is X-trimmed, s_j is a single leaf, and thus 3 holds. □

Lemma 8.8. *For each $g \in G$, the set of trees with type g is tree-definable modulo X.*

The general idea is that (G, W) is a (smaller) forest algebra, and therefore the induction assumption can be applied to languages recognized by (G, W). However, thanks to bad brothers and such, (G, W) does not recognize the language in the lemma. Before we solve this problem, we show how Lemmas 8.7 and 8.8 along with the antichain composition principle conclude the case considered in this section. The idea is that we add all forest types from G to X.

Let h, v be as in the statement of Proposition 8.2. We need to show that the language

$$L = \{t : v(\alpha(t)) = h\}$$

is forest-definable modulo X. By induction assumption, we know that this language is forest-definable modulo $X \cup G$. In other words, there is some forest-definable set of forests K that agrees with L over $(X \cup G)$-trimmed forests. To describe L modulo X, we will use the antichain composition principle.

Let ψ be the formula from Lemma 8.7. Let

$$\varphi = \psi \wedge \neg \mathsf{F}^{-1}\psi \ .$$

This formula holds in a node whose subtree has a type in G, and the node is closest to the root for this property. Thanks to the last clause, φ is an antichain formula. Let $G = \{g_1, \ldots, g_n\}$. By assumption that α is leaf saturated, for each g_i there is a leaf label $a_i \in A$ with $\alpha(a_i) = g_i$. For each g_i, let L_i be the set of trees with type g_i. Thanks to Lemma 8.8, each tree language L_i is tree-definable modulo X.

It is easy to see that squashing a subtree with type g_i into a single leaf with label a_i does not change the type of the whole tree. More precisely, a forest t has the same value as

$$t[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n] \ .$$

Furthermore, the above forest is $(X \cup G)$-trimmed, at least as long as t was X-trimmed. It follows that over X-trimmed forests, L agrees with

$$\{t : t[(L_1, \varphi) \to a_1, \ldots, (L_n, \varphi) \to a_n] \in K\} \ ,$$

which is forest-definable thanks to the antichain composition principle. It now remains to show Lemma 8.8, which we do in the next section.

8.2.1. *Trees with type in G.* Fix some forest type $g \in G$. Our goal is to show that the set of trees with type g is tree-definable modulo X.

Lemma 8.9. *Without loss of generality, we may assume that all forest types in F are good brothers and all inner node labels b satisfy $\alpha(b) \in W$.*

Proof. Recall that all forest types from G are good brothers. In particular, all bad brothers in F are from X, and can therefore only appear in leaves, as long as we are working over X-trimmed forests. Let $A' \subseteq A$ be the set of leaf labels that are mapped by α to a good brother in F. Let $B' \subseteq B$ be the set of inner node labels b with $\alpha(b) \in W$.

Let β be the restriction of α to this smaller alphabet:

$$\beta : (A', B')^{\Delta} \to (H, V) \ .$$

Note that over X-trimmed forests, the only forest types from F in the image of β are good brothers, and all inner node labels b satisfy $\alpha(b) \in W$. Assume now, that we have shown Lemma 8.8 for the morphism β, i.e. the set K of trees that have type g under β is tree-definable modulo X. We will use the antichain composition principle to extend this result to α. The idea is that we squash twig nodes into leaves, thus eliminating labels outside A', B'.

Let φ be a formula that is true in twig nodes (the node is not a leaf, but all of its proper descendants are leaves); this is clearly an antichain formula. Let $G = \{g_1, \dots, g_n\}$. By assumption that α is leaf saturated, for each g_i there is a leaf label $a_i \in A$ with $\alpha(a_i) = g_i$. For each g_i, let L_i be the set of twig trees with value g_i (under α). Each L_i is tree-definable, since the type of a twig tree is determined by its root label and the set of its leaf labels by (6.1). It is easy to see that a tree t over (A, B) has the same type under α as the tree

$$t[(L_1, \varphi) \to a_1, \dots, (L_n, \varphi) \to a_n] \ .$$

Furthermore, if the type of t under α is g, then the latter forest belongs to the domain of β, since all nodes with a label outside A' or B' are covered by φ. Therefore, we can use the antichain composition principle to conclude that the forests with value g under α can be defined in $\mathsf{EF} + \mathsf{F}^{-1}$. ☐

From now on, we use the assumptions stated in the previous lemma. Recall that good brothers are closed under concatenation, and therefore F is a subsemigroup of H. This allows us to define a semigroup automaton $\mathcal{A}$, whose semigroup is F. The input alphabet of this automaton is:

- The inner node labels are B
- The leaf labels are $A' = \{a \in A : \alpha(a) \in F\}$.

For $a \in A'$, we define $\beta_A(a)$ to be $\alpha(a)$. For $b \in B$, we would like the associated function $\beta_B(b)$ to be $\alpha(b)$. Even though Lemma 8.9 guarantees that $\alpha(b)$ belongs to W, this context type cannot be used since it need not generate a function $F \to F$. The reason is that $\alpha(b)h$ may be outside F for types h outside G. To solve this problem, we artificially redefine the function:

$$\beta_B(b)(h) = \begin{cases} \alpha(b)h & \text{if } \alpha(b)h \in F \\ g_0 & \text{otherwise.} \end{cases} \tag{8.2}$$

In the above, g_0 is an arbitrarily chosen forest type from G.

By the proof of Theorem 4.1, this automaton induces a forest algebra morphism

$$\beta : (A', B')^\Delta \to (F, F^F) \ .$$

This morphism is not the same as α, due to the second clause in (8.2). However, it agrees with α over the forests that are relevant to Lemma 8.8:

Lemma 8.10. *For any $g \in G$, and forest t, if $\alpha(t) = g$ then $\beta(t) = g$.*

Proof. If t has a type in G under α, then all of its leaf labels belong to A' by definition of F. Therefore, t belongs to the domain of β. The lemma is proved by induction on the size of t. If $\alpha(t) = g$, then the "bad" second case in (8.2) is never used while calculating $\beta(t)$. ☐

Lemma 8.11. *The image of β satisfies identities (6.1), (6.2) and (6.3).*

Proof. We only focus on identity (6.3), the others are easy to show. The key idea is that α and β only disagree in twig nodes, and these are not important for the identity (6.3).

Let then $p_1 \dashv p_2, q_1 \dashv q_2$ be contexts. We need to show that

$$\beta((p_1q_1)^\omega(p_2q_2)^\omega) = \beta((p_1q_1)^\omega p_1q_2(p_2q_2)^\omega) \ .$$

Thanks to the faithfulness of contexts in forest algebra, it suffices to show that both sides induce the same transformations on forests, i.e.

$$\beta((p_1q_1)^\omega(p_2q_2)^\omega t) = \beta((p_1q_1)^\omega p_1q_2(p_2q_2)^\omega t)$$

holds for every forest t.

Consider first the case when both p_2, q_2 have the hole in the root, and therefore so do p_1, q_2. In this case the equality above becomes:

$$\beta(\omega(s_1 + t_1) + \omega(s_2 + t_2) + t) = \beta(\omega(s_1 + t_1) + s_1 + t_2 + \omega(s_2 + t_2) + t) \ .$$

The above equality follows by commutativity of the horizontal monoid F, and aperiodicity of H, i.e. $\omega h + h = \omega h$. The latter is a consequence of aperiodicity of V, itself a consequence of (6.2), by iterating

$$\omega h + h = (h + \square)^\omega h = (h + \square)^\omega (h + \square)^\omega h = \omega h + h + h \ .$$

We can therefore now assume that in the context $p_2 q_2$, at least one inner node is an ancestor of the hole. Thanks to the assumption on leaf saturation, in the contexts p_1, q_1, p_2, q_2 every subtree that does not contain the hole can be squashed to a single node, without affecting the image under β. We therefore assume that in the contexts above, all nodes outside the main path are leaves. As remarked above, a consequence of equation (6.2) is that V is aperiodic, i.e. $v^\omega = v^\omega v$ holds for every context type v. Therefore, it is sufficient to show

$$\beta((p_1 q_1)^\omega (p_2 q_2)^\omega (p_2 q_2) t) = \beta((p_1 q_1)^\omega p_1 q_2 (p_2 q_2)^\omega (p_2 q_2) t) \ . \tag{8.3}$$

The only part where α and β disagree are twig nodes. Thanks to our assumption on the form of p_1, p_2, q_1, q_2, the only place where the forests in (8.3) contain twig nodes is $p_2 q_2 t$. Therefore, we have

$$\beta((p_1 q_1)^\omega (p_2 q_2)^\omega (p_2 q_2) t) = \alpha((p_1 q_1)^\omega (p_2 q_2)^\omega) \beta((p_2 q_2) t) \ .$$

In the same way we can decompose the right side of (8.3). Applying the assumption that the image of α satisfies (6.3), we get the desired result. □

Proof of Lemma 8.8. By Lemma 8.10, a tree has type g under α if and only if a) its type under α belongs to G; and b) it has type g under β. Condition a) can be tested by thanks to Lemma 8.7. Since F is a proper subset of H, we can use the induction assumption to test condition b). □

8.3. **The induction base.** In this section, we assume that the techniques from the previous two sections cannot be applied. That is:

- All forest types from $H \setminus X$ are in a single forest component.
- For all inner node labels $b \in B$, v is reachable from $v\alpha(b)$.

Note that the second assumption does not necessarily mean that any context type reachable from v is in the same context component. Indeed, it is possible that for some forest type g, the context type v is no longer reachable from $v(\square + g)$.

We will show

$$vf = vg \qquad \text{for all } f, g \in H \setminus X \ . \tag{8.4}$$

Before we do this, we show how Proposition 8.2 follows. For every every forest type $h \in H$, we need to show that the forest language

$$L = \{t : v(\alpha(t)) = h\}$$

is forest definable modulo X. By assumption (8.4), there is some forest type $h_0 \in H$ such that $vf = h_0$ holds for all $f \in H \setminus X$.

- If an X-trimmed forest t contains an inner node label—which can easily be tested by the logic—then $\alpha(t)$ must be in the single forest component $H \setminus X$. In particular, $v\alpha(t) = h_0$. So in this case, φ is either "true" or "false" depending on whether $h_0 = h$ or not.
- Otherwise, the forest t is the concatenation of some leaves $a_1 + \cdots + a_n$. In this case, the type of $v\alpha(t)$ can be calculated based on the set of leaf labels in t.

The rest of this section is devoted to showing (8.4). The following lemma is the key step in our proof (8.4). It says that not only any two forest types $h, g \in H \setminus X$ can be reached from each other—which is the assumption on there being one forest component—but they can also be reached from each other by only using contexts without any branching. Furthermore, the context type that goes from g to h can be chosen independently of g. However, all these statements are relative to context types from the context component of v.

Lemma 8.12. *Let $h \in H \setminus X$. There are inner node labels $b_1, \ldots, b_n \in B$ such that $wh = w\alpha(b_1 \cdots b_n)g$ holds for each forest type $g \in H \setminus X$ and context type w in the context component of v.*

Proof. Let h be a forest type outside X. We first show that there is a context type u_g such that $h = u_h f$ holds for every forest type $f \in H$. By assumption on there being only one forest component outside X, the forest type h can be reached from every forest type. In particular, there is some context type u such that $h = u(h_1 + \cdots + h_n)$, where $h_1, \ldots, h_n$ are all the forest types in H. Let

$$u_h = u(h_1 + \cdots + h_n + \Box) \ .$$

Thanks to idempotency and commutativity of H, i.e. identity (6.1),

$$h_1 + \cdots + h_n = h_1 + \cdots + h_n + f$$

holds for any forest type f, and therefore also $h = u_h f$.

We can decompose the context u_h as

$$u_h = (f_1 + \alpha(b_1)) \cdots (f_n + \alpha(b_n))$$

for some n and $f_1, \ldots, f_n \in H$ and $b_1, \ldots, b_n \in B$. (In general, some of the f_i may be empty; but the proof follows the same lines.) Let us denote $\alpha(b_i)$ by v_i. We will show that

$$wh = wv_1 \cdots v_n g$$

holds for any forest type g and any context type w in the context component of v, thus proving the lemma.

Let then g, w be as above. As for h, we can define a context type u_g such that $g = u_g f$ holds for any forest type f. This context can also be decomposed as

$$u_g = (f_{n+1} + \alpha(b_{n+1})) \cdots (f_m + \alpha(b_m))$$

for some $m \geq n+1$ and $f_{n+1}, \ldots, f_m \in H$ and $b_{n+1}, \ldots, b_m \in B$. As previously, we denote $\alpha(b_i)$ by v_i. By definition, we have

$$v_1 \cdots v_n \dashv u_h \qquad v_{n+1} \cdots v_m \dashv u_g \tag{8.5}$$

Let now $w \in V$ be in the same context component as v. By assumption on w and Lemma 8.4, also the context type $wv_1 \cdots v_m$ is in the same context component as v. In particular, there is some $\bar{w} \in V$ such that

$$wv_1 \cdots v_m \bar{w} = w \ .$$

By iterating the above ω times, and appending h, we get

$$wh = w(v_1 \cdots v_m \bar{w})^\omega h \ .$$

Since $u_h f = h$ holds for all forest types f, the above can be rewritten as

$$w(v_1 \cdots v_m \bar{w})^\omega (u_h u_g \bar{w})^\omega h \ .$$

Using the property from identity (6.3), we get

$$\begin{aligned} w(v_1 \cdots v_m \bar{w})^\omega (u_h u_g \bar{w})^\omega h &= w(v_1 \cdots v_m \bar{w})^\omega v_1 \cdots v_n u_g \bar{w} (u_h u_g v)^\omega h \\ &= w(v_1 \cdots v_m \bar{w})^\omega v_1 \cdots v_n g = \bar{w} v_1 \cdots v_n g \ , \end{aligned}$$

which concludes the proof of the lemma. □

We now use the above Lemma to conclude the proof of (8.4). Indeed, let f, g be forest types outside X. By the above lemma, there are inner node labels $b_1, \ldots, b_m \in B$ such that

$$f = w\alpha(b_1 \cdots b_n)h \qquad g = w\alpha(b_{n+1} \cdots b_m)h$$

holds for all w in the context component of v and all forest types h outside X. Let $v_i = \alpha(b_i)$. By assumption on the equivalence class of v and by Lemma 8.4, there must be some $v \in V$ such that

$$vv_1 \cdots v_m \bar{v} = v \ .$$

But then we have

$$vf = v(v_1 \cdots v_m \bar{v})^\omega f = v(v_1 \cdots v_m \bar{v})^\omega v_{n+1} \cdots v_m \bar{v} (v_1 \cdots v_m \bar{v})^\omega f = v(v_1 \cdots v_m \bar{v})^\omega g = vg \ .$$

The second equality follows from (6.2).

9. Empty forests

The forest algebra setting used in this paper does not allow empty forests. There is also a two-sorted alphabet (A, B), where letters from A are only allowed in leaves, and letters from B are only allowed in inner nodes. A different, and arguably more elegant, setting is considered in [6], where empty forests are allowed, and only one alphabet is used.

Why do we not use the forest algebra with empty forests here? The reason is that the completeness proof in Proposition 8.2 uses an induction on the size of the leaf alphabet, so it helps that the leaf alphabet is part of the definition of the forest algebra. The assumption on nonempty forests follows, since if we want a separate alphabet for leaves, there are algebraic reasons to consider forest algebras without the empty forest. A natural question emerges: does our characterization also work for forest algebra with empty forests? In this section, we give an informal argument that the answer to this question is yes.

We will not give a detailed discussion of forest algebra with empty forests here. We define only define the syntactic object. The interested reader is referred to [6]. Let A be an alphabet. We define A^Δ_H (respectively, A^Δ_V) to be the set of (possibly) empty forests (respectively, contexts) labeled by A, without any restriction on labels in leaves or inner nodes. We write A^Δ for the pair (A^Δ_H, A^Δ_V). The only difference between A^Δ and $(A, A)^\Delta$ is that the second does not allow the empty forest on its first coordinate. It is not hard to see that A^Δ is a forest algebra, as defined in Section 4. Given a set L of forests, possibly including the empty forest, the *syntactic forest algebra with empty forests of* L is defined to be the quotient of A^Δ under the two-sorted equivalence relation defined below.

$$\begin{array}{lll} t \simeq t' & \text{iff} & \forall p \in A^\Delta_V \;\; pt \in L \;\Leftrightarrow\; pt' \in L \\ p \simeq p' & \text{iff} & \forall q \in A^\Delta_V \forall s \in A^\Delta_H \;\; qps \in L \;\Leftrightarrow\; qp's \in L \end{array}$$

This equivalence relation is a refinement of the Myhill-Nerode equivalence introduced in Section 4 (for the case when $A = B$). It may possibly distinguish more contexts because the variable s can also quantify over the empty forest.

Theorem 9.1. *Let L be a forest language. Let (H, V) be its syntactic forest algebra, and let (H', V') be its syntactic forest algebra with empty forests. If (H, V) satisfies the identities from Theorem 6.2, then so does (H', V'), and vice versa.*

Proof. We begin with the right to left implication. Since $(A, A)^\Delta$ is a subalgebra of A^Δ, and since the equivalence relation defining (H', V') is a refinement of the equivalence relation defining (H, V), it follows that (H, V) is a subalgebra of (H', V'). In particular, any identities that hold in the latter must also hold in the former.

For the left to right implication, assume that (H, V) satisfies the identities from Theorem 6.2. By the theorem, the recognized language L is forest-definable in $\mathsf{EF} + \mathsf{F}^{-1}$. To conclude, we will show that if a language L is forest-definable in $\mathsf{EF} + \mathsf{F}^{-1}$, then its syntactic forest algebra with empty forests (H', V') satisfies the identities from Theorem 6.2. This follows by the correctness argument presented in Section 7. The reason why we can use that argument is that it relied on Proposition 7.3 to transfer Duplicator strategies, and this proposition also works for the more general one-sorted morphisms that are appropriate for forest algebras with empty forests. □

10. One quantifier alternation

In [13], it was shown that over words, the temporal logic $\mathsf{F} + \mathsf{F}^{-1}$ has the same expressive power as $\Sigma_2 \cap \Pi_2$, where

- Σ_2 are word properties definable by a first-order formula with quantifier prefix $\exists^*\forall^*$; the signature contains label tests and the left-to-right order on word positions.
- Π_2 are complements of Σ_2.

For instance, consider the word language b^*aA^* over the alphabet $A = \{a, b, c\}$. This language can be defined in $\mathsf{F} + \mathsf{F}^{-1}$ by the formula

$$\mathsf{F}(a \wedge \neg\mathsf{F}^{-1}\neg b) \; .$$

This language can also be defined both in Σ_2 and Π_2, as witnessed by the formulas:

$$\begin{array}{lll} \exists x \forall y & a(x) \;\wedge\; (y < x \;\Rightarrow\; b(y)) & \in \Sigma_2 \\ \forall x \exists y & c(x) \;\Rightarrow\; (y < x \;\wedge\; a(y)) & \in \Pi_2 \; . \end{array}$$

Both classes Σ_2 and Π_2 can be extended to trees using the descendant order on tree nodes. We show here that the result from [13] fails for trees:

Proposition 10.1. *Over trees, the classes $\mathsf{EF} + \mathsf{F}^{-1}$ and $\Sigma_2 \cap \Pi_2$ have incomparable expressive power. Likewise for forests.*

A mentioned in the introduction, the class $\Sigma_2 \cap \Pi_2$ was given an effective characterization in [3]. We prove the above proposition for forests, the case for trees is done the same way. The inequality

$$\mathsf{EF} + \mathsf{F}^{-1} \supsetneq \Sigma_2 \cap \Pi_2$$

is witnessed by the language "three nodes with label a", which cannot be defined in $\mathsf{EF}+\mathsf{F}^{-1}$ by virtue of (6.1). To show the remaining inequality

$$\mathsf{EF} + \mathsf{F}^{-1} \subsetneq \Sigma_2 \cap \Pi_2 \ ,$$

we will demonstrate in the following lemma that the forest property "no root node is a leaf" cannot be defined in Σ_2, although it is forest-definable in $\mathsf{EF} + \mathsf{F}^{-1}$.

Lemma 10.2. *Let a be a leaf label, b an inner node label, and φ be a formula of the form*

$$\exists x_1 \ldots x_i \forall y_1 \ldots y_j \psi(x_1 \ldots x_i, y_1 \ldots y_j) \quad \in \Sigma_2 \ ,$$

with ψ quantifier-free. Let $n > i + j$. If $n(ba)$ satisfies φ, then so does $n(ba) + a$.

Proof. Assume then that $n(ba)$ satisfies φ. We need to show that $n(ba) + a$ does too. For $x_1, \ldots, x_i$, we pick the same nodes in $n(ba) + a$ as the nodes in $n(ba)$ that witnessed φ. We need to show that for any assignment of the nodes $y_1, \ldots, y_j$ in $n(ba) + a$ that makes ψ false, we also can find an assignment in $n(ba)$ that makes ψ false. The key point is that any assignment of $x_1, \ldots, x_i, y_1, \ldots, y_j$ in $n(ba) + a$ must leave at least one copy of ba without any variables; this copy can be used in $n(ba)$ to simulate a. □

11. Closing remarks

The contribution of this paper is a characterization of languages definable in $\mathsf{EF} + \mathsf{F}^{-1}$. This characterization is expressed in terms of identities that must be satisfied in the syntactic algebra. A corollary of this characterization is an algorithm for deciding if a given regular language can be expressed in $\mathsf{EF} + \mathsf{F}^{-1}$. The algorithm runs in polynomial time if the input is given as a forest algebra.

As mentioned in the introduction, there are many open problems waiting to be solved in this field. Of those closely related to $\mathsf{EF} + \mathsf{F}^{-1}$, the following look interesting:

- What are the identities for two-variable first-order logic with the descendant relation? The question boils down to: what identity should replace idempotency $h + h = h$? Here is one candidate: $v(h + h) + vh = vh + vh$.
- What are the identities for an extension of $\mathsf{EF} + \mathsf{F}^{-1}$, where we allow operators of the form $\mathsf{EF}^k\varphi$, with the meaning: "the current node has k incomparable descendants where φ holds". This seems to be a reasonable extension of $\mathsf{EF} + \mathsf{F}^{-1}$ that is capable of counting in a proper way (recall that two-variable logic could express the property "there are two a's", but not the property "there are three a's").

It is conceivable that a modification of the techniques developed in this paper can be sufficient to solve the above two logics. For other logics mentioned in this paper, such as full first-order logic, or even variants of $\mathsf{EF}+\mathsf{F}^{-1}$ with horizontal order, new techniques need to be developed.

References

[1] M. Benedikt and L. Segoufin. Regular languages definable in FO. In *Symposium on Theoretical Aspects of Computer Science*, volume 3404 of *Lecture Notes in Computer Science*, pages 327 – 339, 2005.

[2] M. Bojańczyk. *Decidable Properties of Tree Languages.* PhD thesis, Warsaw University, 2004.

[3] M. Bojańczyk and L. Segoufin. Tree languages definable with one quantifier alternation. Submitted.

[4] M. Bojańczyk, L. Segoufin, and H. Straubing. Piecewise testable tree languages. Logic in Computer Science, 2008.

[5] M. Bojańczyk and I. Walukiewicz. Characterizing EF and EX tree logics. *Theoretical Computer Science*, 358(2-3):255–273, 2006.

[6] M. Bojańczyk and I. Walukiewicz. Forest algebras. In *Automata and Logic: History and Perspectives*, pages 107 – 132. Amsterdam University Press, 2007.

[7] U. Heuter. First-order properties of trees, star-free expressions, and aperiodicity. In *Symposium on Theoretical Aspects of Computer Science*, volume 294 of *Lecture Notes in Computer Science*, pages 136–148, 1988.

[8] R. McNaughton and S. Papert. *Counter-Free Automata*. MIT Press, 1971.

[9] J.-É. Pin. Logic, semigroups and automata on words. *Annals of Mathematics and Artificial Intelligence*, 16:343–384, 1996.

[10] A. Potthoff. First-order logic on finite trees. In *Theory and Practice of Software Development*, volume 915 of *Lecture Notes in Computer Science*, pages 125–139, 1995.

[11] M. P. Schützenberger. On finite monoids having only trivial subgroups. *Information and Control*, 8:190–194, 1965.

[12] P. Tesson and D. Thérien. Diamonds are forever: the variety da. In *Semigroups, Algorithms, Automata and Languages*, pages 475–500, 2002.

[13] D. Thérien and T. Wilke. Over words, two variables are as powerful as one quantifier alternation. In *ACM Symposium on the Theory of Computing*, pages 256–263, 1998.

[14] W. Thomas. On chain logic, path logic, and first-order logic over infinite trees. In *Logic in Computer Science*, pages 245–256, 1987.

[15] T. Wilke. Classifying discrete temporal properties. In *Symposium on Theoretical Aspects of Computer Science*, volume 1563 of *Lecture Notes in Computer Science*, pages 32–46, 1999.