Choose at most 3 of the following 5 exercises and solve them. When talking about completeness, we use polynomial time Karp reductions (the solutions would be the same for deterministic logarithmic space).

1. Show that if there is some language in NP which is coNP-hard, then NP = coNP.

2. Each of the following problems is complete for some complexity class, which is one of

   P NP PSPACE EXPTIME.

   In each case, find the class and prove completeness.

   (a) We are given a quantifier-free formula that uses variables $x, y, z, \ldots$, Boolean operations $\vee, \wedge, \neg$, order $<$, and equality $=$, e.g.

   $$(x < y) \wedge (y < z) \wedge (z < x).$$

   The question is: can we assign rational numbers to the variables so that the formula becomes true? In the given example, the answer is "no".

   (b) Consider formulas as in the previous exercise, but we also allow quantifiers, which range over rational numbers. For example, the following formula is true, because it says that the rational numbers are dense:

   $$\forall x \ \forall y \ \neg(x < y) \vee \exists z \ x < z < y.$$

   We are given such a formula, with no free variables, and the question is if it is true.

   (c) Consider a version of QBF (quantified Boolean formulas), where the quantifier-free part is given by a circuit. In other words, these are formulas of the form

   $$\underbrace{Q_1 x_1 \ Q_2 x_2 \ \cdots Q_n x_n}_{\text{each } Q_i \text{ is a quantifier}} \ \underbrace{\varphi(x_1, \ldots, x_n)}_{\text{a circuit}}.$$

   The question is deciding if such a formula is true.

   (d) We are given a circuit with n inputs and n outputs, which describes a function

   $$f : 2^n \to 2^n.$$

   The question is this: we start in the all-zeros vector from 2^n, and we keep on applying f, do we eventually reach the all-ones vector?