Two-Way Alternating Automata and Finite Models

Tedious proofs of irrelevant results

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Intuition on the automaton

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Some example properties recognized by alternating two-way automata:

- There is a vertex labelled by “a” in the graph
- There is an infinite path in the graph
- There is an infinite path in the graph and no vertex of this path is the starting point of some infinite backward path
The automaton $A$
An example: $\mathbb{N}$
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Parity condition

An infinite sequence $a_1, a_2, \ldots$ of elements from a finite set of natural numbers satisfies the *parity condition* if the lowest number occurring infinitely often is even.
A accepts only infinite graphs

**Fact 0** For any graph $G$, the automaton $A$ accepts in a vertex $v_1$ and state $q_1$ iff

1. No infinite backward path condition. $v_1$ is not the beginning of a sequence $v_1v_2\ldots$ where for all $i \in \{1, 2, \ldots\}$, $(v_{i+1}, v_i)$ is an edge in $G$.

2. Infinite forward path condition. $v_1$ is the beginning of a sequence $v_1v_2\ldots$ where for all $i \in \{1, 2, \ldots\}$, $(v_i, v_{i+1})$ is an edge in $G$ and $A$ accepts in $v_i$ and $q_1$.

**Cor:** $A$ accepts only infinite graphs.
Finite model problems

Automata
Instance: A two-way alternating automaton $\mathcal{A}$.
Question: Does $\mathcal{A}$ accept some finite graph?
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  Instance: A formula $\phi$ of the guarded fragment with fixed points
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All three are equivalent.
A strategy for the good player
Memoryless strategies

**Thm:** [Emmerson-Jutla/Mostowski] One of the players has a winning strategy and, moreover, it is a memoryless strategy.
The graph $\mathcal{N}$
Its unwinding
A strategy $s$ for the green player
Locally possible moves under $s$
Locally possible moves under $s$ with accessible positions
The graph $G_r(t, s)$
Parity length

The $i$-length of a sequence of numbers $a = a_1 a_2 \ldots a_n$ is the length of the longest sequence of $i$-s in the sequence $a'$ resulting from $a$ by taking out all numbers greater than $i$.

For example, the $1$-length of $131231$ is $3$. 
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The parity length of a sequence of numbers maximal $i$-length of the sequence for odd $i$. 
Parity length

- The *i-length* of a sequence of numbers $a = a_1 a_2 \ldots a_n$ is the length of the longest sequence of $i$-s in the sequence $a'$ resulting from $a$ by taking out all numbers greater than $i$. For example, the 1-length of 131231 is 3.

- The *parity length* of a sequence of numbers maximal $i$-length of the sequence for odd $i$.

- The *parity length* of a path labelled by priorities is the parity length of the corresponding sequence of priorities.
Properties of $Gr(t, s)$

$s$ is a winning strategy for the green player iff no infinite path in $Gr(t, s)$ violates the parity condition (the parity length of paths in $Gr(t, s)$ is finite).
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- $t$ can be wound back into a finite graph iff for some $s$, the parity length of paths in $\text{Gr}(t, s)$ is bounded, i.e., there is some $M \in \mathcal{N}$ such that all paths in $\text{Gr}(t, s)$ have parity length not greater than $M$. 
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- $t$ can be wound back into a finite graph iff for some $s$, the parity length of paths in $Gr(t, s)$ is bounded, i.e. there is some $M \in N$ such that all paths in $Gr(t, s)$ have parity length not greater than $M$.

- The finite graph question thus becomes: is there some tree $t$ and strategy $s$ such that the parity length of paths in $Gr(t, s)$ is bounded.
Regular trees and languages

A tree language is regular iff it is recognized by some finite automaton.

A tree is regular iff it contains a only finitely many non-isomorphomorphic subtrees.

Thm:[Rabin] Every regular tree language contains some regular tree.
Let $\mathcal{LB}$ be the set of graphs $\mathcal{G}_r(t,s)$ where the parity length of paths is bounded.
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Let $\mathbf{LF}$ be the set of graphs $G_r(t, s)$ where the parity length (both ways) of paths is finite.
Let $\mathbb{L}_B$ be the set of graphs $\mathcal{G}_r(t, s)$ where the parity length of paths is bounded.

Let $\mathbb{L}_F$ be the set of graphs $\mathcal{G}_r(t, s)$ where the parity length (both ways) of paths is finite.

$\mathbb{L}_B$ is not regular.
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Since $LF$ is regular and $LB$ is a sum of regular languages, we obtain:
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**Thm:** $LF$ is nonempty iff $LB$ is nonempty.
**Thm:** The finite graph problem is decidable
Signature
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Another graph
Another graph
Another graph
Tree unwinding

```
  a  b  c  a
  a  a  a  a
```

Two-Way Alternating Automata and Finite Models – p.18/18
Tree unwinding

![Diagram of tree unwinding](image-url)
Tree unwinding
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