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HOMOLOGICAL METHODS IN GEOMETRY AND TOPOLOGY

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Contents

1. Categories, functors, natural transformations

A *category* $\mathcal{C}$ is the data of a class of objects $\mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\mathrm{Mor}(\mathcal{C})$. Every morphism has a source and a target, we denote by $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ the collection of objects whose source is C_1 and target is C_2. We assume that there exists an associative composition $\mathrm{Hom}_{\mathcal{C}}(C_2, C_3) \times \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{C}}(C_1, C_3)$ and that every object has the identity $\mathrm{Id}_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$, $g \in \mathrm{Hom}_{\mathcal{C}}(C'', C)$, $f \circ \mathrm{Id}_C = f$ and $\mathrm{Id}_C \circ g = g$.

A *subcategory* $\mathcal{D} \subset \mathcal{C}$ is a category such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{D}) \subset \mathrm{Mor}(\mathcal{C})$.

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ is an *isomorphism* if there exists $g \in \mathrm{Hom}_{\mathcal{C}}(C_2, C_1)$ such that $f \circ g = \mathrm{Id}_{C_2}$ and $g \circ f = \mathrm{Id}_{C_1}$.

The opposite category $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ while $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C_1, C_2) = \mathrm{Hom}_{\mathcal{C}}(C_2, C_1)$.

Examples of categories include the categories of Sets, (pointed) topological spaces, (abelian) groups, the category Δ:

Objects of Δ are $[n]$, for $n = 0, 1, \ldots$. $\mathrm{Hom}_{\Delta}([m], [n])$ is the set of nonincreasing mappings from $\{0, \ldots, m\}$ to $\{0, \ldots, n\}$.

A (covariant) *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is the data of a mapping $\mathrm{Ob}\mathcal{C} \to \mathrm{Ob}\mathcal{D}$, $C \mapsto F(C)$ and a mapping $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, $\varphi \mapsto F(\varphi)$ such that $F(\psi\varphi) = F(\psi)F(\varphi)$ and $F(\mathrm{Id}_C) = \mathrm{Id}_{F(C)}$. A contravariant functor $\mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

A *natural transformation* $\eta\colon F \to G$ of functors $\mathcal{C} \to \mathcal{D}$ is the data of $\eta_C \in \mathrm{Hom}_{\mathcal{D}}(F(C), G(C))$, for any $C \in \mathrm{Ob}(\mathcal{C})$. Morphisms η_C induce commutative diagrams

$$\begin{array}{ccc} F(C_2) & \xrightarrow{\eta_{C_2}} & G(C_2) \\ {\scriptstyle F(\varphi)}\uparrow & & \uparrow{\scriptstyle G(\varphi)} \\ F(C_1) & \xrightarrow{\eta_{C_1}} & G(C_1) \end{array}$$

for any $\varphi \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$.

We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *faithful* if the map $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ is injective. F is *full* if the map is surjective. It is *essentially surjective* if every object in $\mathcal{D}$ is isomorphic to $F(C)$, for some $C \in \mathrm{Ob}(\mathcal{C})$.

A subcategory $\mathcal{D} \subset \mathcal{C}$ is *full* if the embedding functor $\mathcal{D} \to \mathcal{C}$ is fully faithful.

Functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists $G\colon \mathcal{D} \to \mathcal{C}$ and natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \to G \circ F$, $\nu\colon \mathrm{Id}_{\mathcal{D}} \to F \circ G$ such that η_C and ν_D are isomorphisms, for all $C \in \mathrm{Ob}(\mathcal{C})$, $D \in \mathrm{Ob}(\mathcal{D})$.

Exercise 1.1. Show that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is fully faithful and essentially surjective.

Examples of functors

- Any object $C \in \mathcal{C}$ defines functors $h^C\colon \mathcal{C} \to \mathrm{Set}$ and $h_C\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ via $h^C(C_1) = \mathrm{Hom}_{\mathcal{C}}(C, C_1)$, $h_C(C_1) = \mathrm{Hom}_{\mathcal{C}}(C_1, C)$. Functors h^C, h_C are *representable*.

- Let $\Delta^{\text{op}}\,\text{Set}$ be the category of *simplicial sets*. Its objects of $\Delta^{\text{op}}\,\text{Set}$ are functors $\Delta^{\text{op}} \to \text{Set}$. Morphisms are natural transformations of functors.

  The *geometric realisation* is a functor $|-|\colon \Delta^{\text{op}}\,\text{Set} \to \text{Top}$. To a simplicial set $X = \{X_n = X([n])\}$ it assigns $|X| = \bigsqcup_{n=0}^{\infty}(\Delta_n \times X_n)/R$ where Δ_n is the geometrical n-dimensional simplex

$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1,\, x_i \geq 0\}$$

  and the equivalence relation R is defined as follows: $(s, x) \in \Delta_n \times X_n$ is identified with $(t, y) \in \Delta_m \times X_m$ if there exists $f \in \text{Hom}_\Delta([m], [n])$ with $y = X(f)x$ and $s = \Delta_f t$. The topology on $|X|$ is the weakest for which $\bigsqcup_{n=0}^{\infty} X_n \times \Delta_n \to |X|$ is continuous. The map $\Delta_f\colon \Delta_m \to \Delta_n$ is the unique linear mapping which sends vertex $e_i \in \Delta_m$ to $e_{f(i)} \in \Delta_n$.
- Another example of a functor is the *Singular simplicial set* $\text{Sing}\colon \text{Top} \to \Delta^{\text{op}}\,\text{Set}$. For a topological space Y, $\text{Sing}(Y)(n)$ is the set of continuous maps $\Delta_n \to Y$ For $f \in \text{Hom}_\Delta([n], [m])$ the map $\text{Sing}(Y)(f)$ maps $\varphi\colon \Delta_m \to Y$ to $\varphi \circ \Delta_f\colon \Delta_n \to Y$.
- A *presheaf* of sets is a functor $(\text{Top}Y)^{\text{op}} \to \text{Set}$. Here, Y is a topological space and $\text{Top}Y$ is the category whose objects are open subsets of Y and morphisms are inclusions $U \to V$.

1.1. **Direct product, coproduct, fiber and cofiber product.**

Let X, Y be objects of a category $\mathcal{C}$.

The *direct product* $X \times Y$ is the object Z representing the functor

$$C \mapsto h_X(C) \times h_Y(C)$$

The *direct sum* $X \oplus Y$ is the object Z representing the functor

$$C \mapsto h^X(C) \cup h^Y(C).$$

Let S be an object of $\mathcal{C}$. Define category $\mathcal{C}_S$ whose objects are pairs (C, φ) of objects of $\mathcal{C}$ and morphisms $\varphi\colon C \to S$. Morphisms $\mathcal{C}_S$ $(C_1, \varphi_1) \to (C_2, \varphi_2)$ in $\mathcal{C}_S$ are such $f \in \text{Hom}_{\mathcal{C}}(C_1, C_2)$ that $\varphi_2 \circ f = \varphi_1$.

Let now X, Y be objects of $\mathcal{C}_S$, i.e. assume fixed $\varphi\colon X \to S$, $\psi\colon Y \to S$. The *fiber product* $X \times_S Y$ of X and Y over S is the direct product of (X, φ), (Y, ψ) in $\mathcal{C}_S$ considered as an object of $\mathcal{C}$.

Exercise 1.2. Write down the universal property of a fiber product.

Given morphisms $\alpha_X\colon S\to X$, $\alpha_Y\colon S\to Y$ in a category $\mathcal{C}$ the *cofiber product* $X\sqcup_S Y$ of X and Y along S is an object $Z\in\mathcal{C}$ together with $\beta_X\colon X\to Z$, $\beta_Y\colon Y\to Z$ satisfying the following universal property: given C in $\mathcal{C}$ and $\varphi_X\colon X\to C$, $\varphi_Y\colon Y\to C$ such that $\varphi_X\circ\alpha_X=\varphi_Y\circ\alpha_Y$, there exists unique $\varphi\colon Z\to C$ such that $\varphi\circ\beta_X=\varphi_X$ and $\varphi\circ\beta_Y=\varphi_Y$.

Exercise 1.3. Present cofiber product as a coproduct in an appropriate category.

1.2. **Adjoint functors.**

Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F\colon\mathcal{C}\to\mathcal{D}$, $G\colon\mathcal{D}\to\mathcal{C}$ functors. Functor F is *left adjoint* to G, $F\dashv G$ if there exist natural transformations $\varepsilon\colon FG\to\mathrm{Id}_{\mathcal{D}}$, $\eta\colon\ \mathrm{Id}_{\mathcal{C}}\to GF$, called the adjunction counit and unit, such that maps

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{C}}(C,G(D))\to\mathrm{Hom}_{\mathcal{D}}(F(C),D), &&\varphi\mapsto\varepsilon_D\circ F(\varphi),\\
&\mathrm{Hom}_{\mathcal{D}}(F(C),D)\to\mathrm{Hom}_{\mathcal{C}}(C,G(D)), &&\psi\mapsto G(\psi)\circ\eta_C
\end{aligned}$$

are inverse to each other.

Exercise 1.4. Show that $\varepsilon\colon FG\to\mathrm{Id}_{\mathcal{D}}$, $\eta\colon\ \mathrm{Id}_{\mathcal{C}}\to GF$ yield $F\dashv G$ if and only if the compositions $F\xrightarrow{F\eta}FGF\xrightarrow{\varepsilon F}F$, $G\xrightarrow{\eta G}GFG\xrightarrow{G\varepsilon}G$ are the identity transformations.

1.3. **Limits and colimits.**

Consider a category I and a functor $F\colon I\to\mathcal{C}$. A cone to F is an object N of $\mathcal{C}$ together with $\psi_i\colon N\to F(i)$, for any $i\in I$, such that for every $\alpha\in\mathrm{Hom}_I(i,j)$, $F(\alpha)\circ\psi_i=\psi_j$. A *limit* of $F\colon I\to J$ is a cone (L,φ_i) such that given any other cone (N,ψ_i) there exists a unique morphism $u\colon N\to L$ such that $\varphi_i\circ u=\psi_i$.

A cocone to F is an object W of $\mathcal{C}$ together with $\psi_i\colon F(i)\to W$, for any $i\in I$, such that for every $\alpha\in\mathrm{Hom}_I(i,j)$, $\psi_j\circ F(\alpha)=\psi_i$. A *colimit* of $F\colon I\to J$ is a cocone (T,φ_i) such that given any other cocone (W,ψ_i) there exists a unique morphism $u\colon T\to W$ such that $u\circ\varphi_i=\psi_i$.

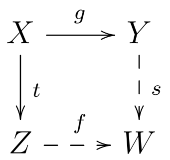
1.4. **Localisation in categories.**

The reference for this section is [Sta13, section 4.26].

Let $\mathcal{C}$ be a category. A set of arrows S in $\mathcal{C}$ is called a *left multiplicative system* if it has the following properties

LMS 1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements in S is in S;

LMS2 Every solid diagram



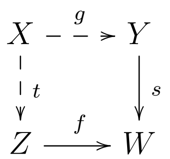
with $t \in S$ can be completed to a commutative dotted square with $s \in S$;

LMS3 For every $f, g\colon X \to Y$ and $t \in S$ such that $f \circ t = g \circ t$ there exists a $s \in S$ such that $s \circ f = s \circ g$.

A set of arrows S in $\mathcal{C}$ is called a *right multiplicative system* if it has the following properties

RMS 1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements in S is in S;

RMS2 Every solid diagram

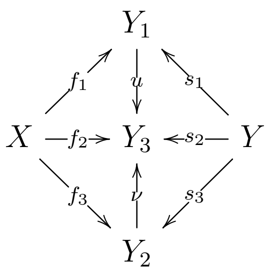


with $s \in S$ can be completed to a commutative dotted square with $t \in S$;

RMS3 For every $f, g\colon X \to Y$ and $s \in S$ such that$s \circ f = s \circ g$ there exists a $t \in S$ such that $f \circ t = g \circ t$.

A set of arrows is a *multiplicative system* if it is both left and right multiplicative system.

Let $\mathcal{C}$ be a category and S a left multiplicative system. We define a new category $S^{-1}\mathcal{C}$ of *left fractions* whose objects are objects of $\mathcal{C}$. Morphisms $X \to Y$ in $S^{-1}\mathcal{C}$ are equivalences classes of pairs $(f\colon X \to Y', s\colon Y \to Y')$ with $s \in S$. Two pairs $(f_1\colon X \to Y_1', s_1\colon Y \to Y_1')$, $(f_2\colon X \to Y_2', s_2\colon Y \to Y_2')$ are equivalent if there exists a third pair $(f_3\colon X \to Y_3', s_3\colon Y \to Y_3')$ and morphisms $u\colon Y_1 \to Y_3$, $\nu\colon Y_2 \to Y_3$ fitting into the commutative diagram



The composition of the equivalence classes of the pairs $(f\colon X \to Y', s\colon Y \to Y')$ and $(g\colon Y \to Z', t\colon Z \to Z')$ is defined as the equivalence class of a pair $(h \circ f, u \circ t)$ where h

and $u \in S$ are chosen to fit into a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z' \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle u} \\ Y' & \xrightarrow{h} & Z'' \end{array}$$

which exists by assumption. The identity morphism is the equivalence class of the pair $(\mathrm{Id}, \mathrm{Id})$.

Proposition 1.5. *Let $\mathcal{C}$ be a category and S a left multiplicative system of morphisms of $\mathcal{C}$.*

(1) The rules $X \mapsto X$, $(f\colon X \to Y) \mapsto (f\colon X \to Y, Id\colon Y \to Y)$ define a functor $Q\colon \mathcal{C} \to S^{-1}\mathcal{C}$ which commutes with finite colimits.
(2) For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.
(3) If $G\colon \mathcal{C} \to \mathcal{D}$ is a functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H\colon S^{-1}\mathcal{C} \to D$ such that $H \circ Q = G$.

Similarly, one defines the category $S^{-1}\mathcal{C}$ for a right multiplicative system S of morphisms of $\mathcal{C}$.

Lemma 1.6. *Let $\mathcal{C}$ be category and S a multiplicative system. The category of left fractions and the category of right fractions are canonically isomorphic.*

Proof. The universal property implies existence of mutually inverse functors $\mathcal{C}_{\text{left}} \to \mathcal{C}_{\text{right}}$ and $\mathcal{C}_{\text{right}} \to \mathcal{C}_{\text{left}}$. □

We say that a multiplicative system S is *saturated* if it satisfies

MS4 given three composable morphisms f, g, h, if $fg \in S$, $gh \in S$ then $g \in S$.

Lemma 1.7. *Let $\mathcal{C}$ be a category and S a left multiplicative system. Given any finite collection $g_i\colon X_i \to Y$ of morphisms of $S^{-1}\mathcal{C}$ we can find an element $s\colon Y \to Y'$ of S and a family of morphisms $f_i\colon X_i \to Y'$ such that each g_i is the equivalence class of the pair (f_i, s).*

Proof. Let $(X_i \to Y_i, s_i\colon Y \to Y_i)$ be a representative of g_i. The lemma follows if we can find $s\colon Y \to Y'$ in S such that for each i there is $a_i\colon Y_i \to Y'$ with $a_i \circ s_i = s$. If we have two indices $i = 1, 2$, we complete the square

$$\begin{array}{ccc} Y & \xrightarrow{s_2} & Y_2 \\ \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle t_2} \\ Y_1 & \xrightarrow{a_1} & Y' \end{array}$$

with $t_2 \in S$. Then $s = t_2 \circ s_2 \in S$ works. If we have $n > 2$ morphisms, we use the above trick to reduce to the case of $n - 1$ and proceed by induction. $\square$

2. Abelian categories

2.1. Additive and abelian categories.

Category $\mathcal{C}$ is *additive* if

(A1) Each set $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ is endowed with a structure of an abelian group, the composition of morphisms is bi-additive with respect to these structures,

(A2) There exists a zero object $0 \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathcal{C}}(0, 0)$ is the zero group.

(A3) For any pair of objects C_1, C_2 in $\mathcal{C}$ the direct sum and the direct product of X and Y exist and the canonical map $X \oplus Y \to X \times Y$ is an isomorphism.

Let k be a field (or a commutative ring). Category $\mathcal{C}$ is *k-linear* if $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ are endowed with a structure of a k-module and the composition factors via $\mathrm{Hom}_{\mathcal{C}}(C_2, C_3) \otimes_k \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{C}}(C_1, C_3)$.

Consider category $\mathcal{C}$ which satisfies axioms A1 and A2 (it is sometimes called preadditive). Let $\alpha\colon X \to Y$ be a morphism in $\mathcal{C}$.

The *kernel* of α is an object $K \in \mathcal{C}$ together with the map $k\colon K \to X$ such that, for any $C \in \mathcal{C}$ and any $\varphi\colon C \to X$ such that $\alpha \circ \varphi = 0$ there exists a unique $\overline{\varphi}\colon C \to K$ such that $k \circ \overline{\varphi} = \varphi$.

The *cokernel* of α is an object $Q \in \mathcal{C}$ together with the map $c\colon Y \to Q$ such that, for any $C \in \mathcal{C}$ and any $\psi\colon Y \to C$ such that $\psi \circ \alpha = 0$ there exists unique $\overline{\psi}\colon Q \to C$ such that $\overline{\psi} \circ c = \psi$.

We say that an additive category $\mathcal{A}$ is *abelian* if

(A4) for any morphism $\varphi \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ there exists a sequence

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} Q$$

such that

- $j \circ i = \varphi$,
- (K, k) is the kernel of φ, (Q, c) is the cokernel of φ,
- (I, i) is the cokernel of k and (I, j) is the kernel of c.

Consider diagrams

$$\begin{array}{ccc} S & \xrightarrow{\beta_X} & X \\ \downarrow{\scriptstyle \beta_Y} & & \\ Y & & \end{array} \qquad \begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle \gamma_X} \\ Y & \xrightarrow{\gamma_Y} & S \end{array}$$

in an abelian category $\mathcal{A}$. Then the cokernel of $S \to X \oplus Y$ is the cofiber product $X \sqcup_S Y$ and the kernel of $X \oplus Y \to S$ is the fiber product $X \times_S Y$.

2.2. **The category of modules over a quiver.**

A *quiver* Q consists of the set of vertices Q_0 and arrows Q_1 together with the source and target maps $s, t \colon Q_1 \to Q_0$.

Let k be a field. The path algebra $k[Q]$ of a quiver Q is a k-algebra whose k-basis consists of *paths* in Q, i.e. sequences of arrows $p = a_n \dots a_1$ such that $s(a_{i+1}) = t(a_i)$ for $i = 1, \dots, n-1$. The source of a path p is defined as the source of a_1 and the target of p is $t(a_n)$. Algebra $k[Q]$ contains also length-zero paths e_i at any vertex of $i \in Q_0$.

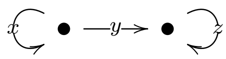
The composition in $k[Q]$ is defined as

$$p_1 \circ p_2 = \begin{cases} p_1 p_2 & \text{if } s(p_1) = t(p_2) \\ 0 & \text{otherwise.} \end{cases}$$

Let I be an ideal in $k[Q]$ generated by a finitely many linear combinations of paths with the same source and target. We say that (Q, I) is a *quiver with relations* and $k[Q]/I$ is the path algebra of (Q, I).

Example of a quiver with relations is

(1)



with

$$x^2 = 0, zy = y.$$

A *right module over a quiver* (Q, I) is a right module over the path algebra of (Q, I). Let M be a right module over (Q, I) and let $M_i = Me_i$. For an arrow a with $s(a) = i$ and $t(a) = j$ we have $e_j a e_i = a$, hence a can be thought of as a map $M_j \to M_i$.

The right module over the quiver (1) consists of V_1, V_2 and maps $x \colon V_1 \to V_1$, $y \colon V_2 \to V_1$ and $z \colon V_2 \to V_2$ such that $x^2 = 0$, $y \circ z = y$.

2.3. **Cohomology of a complex.**

Let $\mathcal{A}$ be an abelian category. A *(cohomological) complex* over $\mathcal{A}$ is a sequence

$$\dots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \dots \qquad (2)$$

of objects in $\mathcal{A}$ such that $d^i \circ d^{i-1} = 0$.

Given a complex (2) the fact that $d^i \circ d^{i-1} = 0$ implies that d^{i-1} uniquely factors via the kernel K^i of d^i. The *i-th cohomology* $H^i(A^\bullet)$ *of* (2) is the cokernel of the map $A^{i-1} \to K^i$.

We say that a complex (2) is *exact* if $H^i(A^\bullet) = 0$ for all i. In particular, an exact complex

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

is called a *short exact sequence.*

Exercise 2.1. Let $0 \to A_1 \xrightarrow{i} A_2 \to A_3 \to 0$ be a short exact sequence and let $\varphi\colon B_3 \to A_3$ be a morphism. Let $Q = A_2 \times_{A_3} B_3$ and let $\alpha\colon A_1 \to Q$ be induced by $i\colon A_1 \to A_2$ and $0\colon A_1 \to B_3$. Then $0 \to A_1 \xrightarrow{\alpha} Q \to B_3 \to 0$ is a short exact sequence in $\mathcal{A}$.

Exercise 2.2. Let $0 \to A_1 \xrightarrow{i} A_2 \xrightarrow{d} A_3 \to 0$ be a short exact sequence and let $\psi\colon A_1 \to B_1$ be a morphism. Let $B_2 = B_1 \sqcup_{A_1} A_2$ and $\theta\colon B_2 \to A_3$ be induced by $d\colon A_2 \to A_3$, $0\colon B_1 \to A_3$. Then $0 \to B_1 \to B_2 \xrightarrow{\theta} A_3 \to 0$ is a short exact sequence in $\mathcal{A}$.

2.4. **Left and right exact functors.**

Let now $\mathcal{A}$, $\mathcal{B}$ be abelian categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *left exact* if for any exact sequence $0 \to A_1 \to A_2 \to A_3$ in $\mathcal{A}$ the sequence $0 \to F(A_1) \to F(A_2) \to F(A_3)$ is exact in $\mathcal{B}$. Functor F is *right exact* if for any exact sequence $A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$ the sequence $F(A_1) \to F(A_2) \to F(A_3) \to 0$ is exact in $\mathcal{B}$. Functor F is *exact* if it is both left and right exact.

Exercise 2.3. Functor $F\colon \mathcal{A} \to \mathcal{B}$ is exact if and only if it maps any short exact sequence in $\mathcal{A}$ to a short exact sequence in $\mathcal{B}$.

2.5. **The category of sheaves.**

Let X be a topological space. We define the category $\mathcal{U}_X$ whose objects are open subsets $U \subset X$ and morphism $U \to V$ correspond to inclusions $U \to V$.

Let $\mathcal{A}$ be an abelian category. A *presheaf* on X with values in $\mathcal{A}$ is a functor $\mathcal{U}_X^{\mathrm{op}} \to \mathcal{A}$. Presheaves form a category $\mathrm{PSh}_{\mathcal{A},X}$ in which morphisms are natural transformations of functors.

A presheaf $\mathcal{F}$ is a *sheaf* if for any open $U \subset X$ and any open covering $U = \bigcup U_i$ the diagram

$$F(U) \longrightarrow \prod F(U_i) \rightrightarrows \prod F(U_i \cap U_j)$$

is an equalizer.

Sheaves form a full subcategory of presheaves denoted $\mathrm{Sh}_{\mathcal{A},X}$.

Assume that category $\mathcal{A}$ has colimits. Then the embedding functor $\iota\colon \mathrm{Sh}_{\mathcal{A},X} \to \mathrm{PSh}_{\mathcal{A},X}$ has left adjoint s. For a presheaf $\mathcal{F}$, $s(\mathcal{F}) = \mathcal{F}^{++}$, where $\mathcal{F}^{+}$ is a presheaf whose value on an open $U \subset X$ is define as

$$\mathcal{F}^{+}(U) = \varinjlim_{\mathcal{J}_U^{\mathrm{op}}} H^0(\mathcal{U}, \mathcal{F}).$$

Here, $\mathcal{J}_U^{\mathrm{op}}$ is the category whose objects are open covers of U and morphisms are given by inclusions of open covers. For $\mathcal{U} \in \mathcal{J}_U^{\mathrm{op}}$

$$H^0(\mathcal{U}, \mathcal{F}) = \{(f_i) \in \prod \mathcal{F}(U_i) \mid f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \text{ for all pairs } i, j\}.$$

Functor ι is left exact, hence kernels of sheaves are kernels of presheaves. A cokernel of a morphism φ of sheaves is $s(\mathrm{coker}\varphi)$, where coker φ is the cokernel in the category of presheaves.

2.6. **The long exact sequence of Ext-groups.**

An object $P \in \mathcal{A}$ is *projective* if the functor h^P is exact. An object $I \in \mathcal{A}$ is *injective* if h_I is exact.

Given objects A_1, A_2 of an abelian category $\mathcal{A}$ the group $\mathrm{Ext}^1_{\mathcal{A}}(A_2, A_1)$ is the group whose elements are classes of *extensions of* A_2 *by* A_1, i.e. short exact sequences $0 \to A_1 \to A \to A_2 \to 0$ modulo isomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & A & \longrightarrow & A_2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A' & \longrightarrow & A_2 & \longrightarrow & 0
\end{array}$$

The zero element of $\mathrm{Ext}^1_{\mathcal{A}}(A_2, A_1)$ is the trivial extension $0 \to A_1 \to A_1 \oplus A_2 \to A_2 \to 0$. Given $\zeta = \{0 \to A_1 \to A \to A_2 \to 0\}$, $\xi = \{0 \to A_1 \to B \to A_2 \to 0\}$ the sum $\zeta + \xi$ is the top row in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \longrightarrow & C & \longrightarrow & A_2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \Delta} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A_1 \oplus A_1 & \longrightarrow & Q & \longrightarrow & A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \\
0 & \longrightarrow & A_1 \oplus A_1 & \longrightarrow & A \oplus B & \longrightarrow & A_2 \oplus A_2 & \longrightarrow & 0
\end{array}$$

where Δ's are the morphisms induces by $\mathrm{Id}\colon A_i \to A_i$, $Q = A_2 \times_{A_2 \oplus A_2} (A \oplus B)$ and $C = A_1 \sqcup_{A_1 \oplus A_1} Q$.

Exercise 2.4. A short exact sequence $0 \to A_1 \xrightarrow{i} A \xrightarrow{d} A_2 \to 0$ defines a trivial element in $\mathrm{Ext}^1(A_2, A_1)$ if there exists $\tau\colon A \to A_1$ such that $\tau \circ i = \mathrm{Id}$.

Exercise 2.5. A short exact sequence $0 \to A_1 \xrightarrow{i} A \xrightarrow{d} A_2 \to 0$ defines a trivial element in $\mathrm{Ext}^1(A_2, A_1)$ if there exists $\sigma\colon A_2 \to A$ such that $d \circ \sigma = \mathrm{Id}$.

Proposition 2.6. *Let*

$$P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varphi} A_2 \to 0$$

be an exact sequence in an abelian category $\mathcal{A}$ with P_0, P_1, P_2 projective. Then the group $Ext^1(A_2, A_1)$ is isomorphic to the first cohomology H^1 of the complex

$$0 \to Hom(P_0, A_1) \to Hom(P_1, A_1) \to Hom(P_2, A_1) \tag{3}$$

Proof. We define maps $f\colon \mathrm{Ext}^1(A_2, A_1) \to H^1$ and $g\colon H^1 \to \mathrm{Ext}^1(A_2, A_1)$ and check that they are mutually inverse.

Let $\zeta = \{0 \to A_1 \xrightarrow{i} B \xrightarrow{d} A_2 \to 0\}$ be an element of $\mathrm{Ext}^1(A_2, A_1)$. As sequence $0 \to \mathrm{Hom}(P_0, A_1) \to \mathrm{Hom}(P_0, B) \to \mathrm{Hom}(P_0, A_2) \to 0$ is exact, there exists $\alpha\colon P_0 \to B$ such that $d \circ \alpha = \varphi$. The map α is unique up to $P_0 \to A_1$:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & A_1 & \xrightarrow{i} & B & \xrightarrow{d} & A_2 & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle\beta} & \nwarrow & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\mathrm{Id}} & & \\
 & & P_2 & \xrightarrow{d_1} & P_1 & \xrightarrow{d_0} & P_0 & \xrightarrow{\varphi} & A_2 & \longrightarrow & 0
\end{array}$$

The composite $d \circ \alpha \circ d_0 = \varphi \circ d_0$ is zero, hence there exists map β from P_1 to the kernel A_1 of d. The composite $i \circ \beta \circ d_1 = \alpha \circ d_0 \circ d_1 = 0$. As i is a monomorphism, it follows that $\beta \circ d_1 = 0$. We define $f(\zeta)$ as the class of β in the first cohomology of the complex (3).

Let now $\gamma\colon P_1 \to A_1$ be a morphism such that $\gamma \circ d_1\colon P_2 \to A_1$ is zero. Map γ factors via the cokernel of d_1, i.e. via the kernel I of d_0. Let $C = A_1 \sqcup_I P_0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{i} & C & \xrightarrow{d} & A_2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\mu} & & \uparrow{\scriptstyle\mathrm{Id}} & & \\
0 & \longrightarrow & I & \longrightarrow & P_0 & \xrightarrow{\varphi} & A_2 & \longrightarrow & 0
\end{array} \tag{4}$$

We define $g(\gamma)$ as the class of $\{0 \to A_1 \to C \to A_2 \to 0\}$ in $\mathrm{Ext}^1(A_2, A_1)$. To show that g is well-defined we need to check that if $\gamma = \theta \circ d_0$ for some $\theta\colon P_0 \to A_1$ then $g(\gamma) = 0$.

Assume $\gamma = \theta \circ d_0$. Maps $\theta\colon P_0 \to A_1$, $\mathrm{Id}\colon A_1 \to A_1$ yield $\tau\colon C \to A_1$ such that $\tau \circ i = \mathrm{Id}_{A_1}$. It follows that A_1 is a direct summand of C. As the quotient C/A_1 is isomorphic to A_2, we conclude that $C \simeq A_1 \oplus A_2$.

As diagram (4) commutes, we have $fg(\gamma) = \gamma$.

Let $\zeta = \{0 \to A_1 \xrightarrow{i} B \xrightarrow{d} A_2 \to 0\}$ be a class in $\mathrm{Ext}^1(A_2, A_1)$. Element $gf(\zeta)$ is the extension $0 \to A_1 \xrightarrow{\iota} C \xrightarrow{\delta} A_2 \to 0$ where C is the cofiber product $A_1 \sqcup_I P_0$. Morphisms $i\colon A_1 \to B$, $\alpha\colon P_0 \to B$ define $\xi\colon C \to B$. By definition of ξ, we have $\xi \circ \iota = i$. The composite $d \circ \xi \circ \mu$ equals $d \circ \alpha = \varphi$, hence the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{i} & B & \xrightarrow{d} & A_2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle \xi} & & \uparrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\iota} & C & \xrightarrow{\delta} & A_2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \mu} & & \uparrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & I & \longrightarrow & P_0 & \xrightarrow{\varphi} & A_2 & \longrightarrow & 0
\end{array}$$

commutes. It follows by the five lemma that ξ is an isomorphism, i.e. $\zeta = gf(\zeta)$. □

One can also consider an exact sequence

$$0 \to A_1 \to I_0 \to I_1 \to I_2$$

with injective I_0, I_1, I_2. Then $\mathrm{Ext}^1(A_2, A_1)$ is isomorphic to the first cohomology of the complex

$$0 \to \mathrm{Hom}(A_2, I_0) \to \mathrm{Hom}(A_2, I_1) \to \mathrm{Hom}(A_2, I_2)$$

2.7. **Exact categories.**

An *exact category* is an additive category $\mathcal{E}$ together with a fixed class $\mathcal{S}$ of *conflations*, i.e. pairs of composable morphisms

$$X \xrightarrow{i} Y \xrightarrow{d} Z \tag{5}$$

such that i is the kernel of d and d is the cokernel of i. We shall say that i is an *inflation* and d a *deflation*. The class $\mathcal{S}$ is closed under isomorphisms and the pair $(\mathcal{E}, \mathcal{S})$ is to satisfy the following axioms:

(Ex 0) $0 \to X \xrightarrow{\mathrm{Id}_X} X$ is a conflation,
(Ex 1) the composite of two deflations is a deflation,
(Ex 2) the pullback of a deflation against an arbitrary morphism exists and is a deflation,
(Ex 2') the pushout of an inflation along an arbitrary morphism exists and is an inflation.

Exercise 2.7. Let $\mathcal{A}$ be an abelian category and $\mathcal{E} \subset \mathcal{A}$ a full subcategory closed under extensions, i.e. given a short exact sequence $0 \to E_1 \to A \to E_2 \to 0$ in $\mathcal{A}$ with E_1, $E_2 \in \mathcal{E}$, object A also belongs to $\mathcal{E}$. Prove that $\mathcal{E}$ with conflations defined as exact sequences in $\mathcal{A}$ whose all terms lie in $\mathcal{E}$ is an exact category.

2.8. **Serre subcategory and quotient.**

The reference for this section is [Sta13, Sections 12.8 and 12.9].

Lemma 2.8. *Let $\mathcal{C}$ be a preadditive category and S a left or right multiplicative system. Then there exists a canonical additive structure on $S^{-1}\mathcal{C}$ such that the localisation functor $Q\colon \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.*

Proof. Let $\alpha, \beta\colon X \to Y$ be morphisms in $S^{-1}\mathcal{C}$. Lemma 1.7 implies that there exists $s \in S$ such that α is the equivalence class of a pair (f, s) and β is the equivalence class of a pair (g, s). Then $\alpha + \beta$ is defined as the equivalence class of $(f + g, s)$.

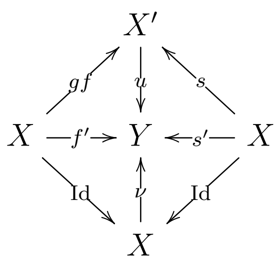
Functor Q commutes with finite (co)limits, hence $S^{-1}\mathcal{C}$ has a zero object and direct sums. □

Lemma 2.9. *Let $\mathcal{C}$ be an additive category and S a multiplicative system. Let X be an object of $\mathcal{C}$. The following are equivalent*

(1) *$Q(X) = 0$ in $S^{-1}\mathcal{C}$;*
(2) *there exists $Y \in Ob(\mathcal{C})$ such that $0\colon X \to Y$ is an element of S; and*
(3) *there exists $Z \in Ob(\mathcal{C})$ such that $0\colon Z \to X$ is an element of S.*

Proof. If (2) holds then $0 = Q(0)\colon Q(X) \to Q(Y)$ is an isomorphism. As $S^{-1}\mathcal{C}$ is additive, $Q(X) = 0$, i.e. (2) ⇒ (1). Similarly, (3) ⇒ (1).

Suppose that $Q(X) = 0$. Then $f\colon X \to 0$ is transformed into an isomorphism in $S^{-1}\mathcal{C}$. Let $s^{-1}g = ht^{-1}$ be the inverse morphism. $Id_X = s^{-1}gf$ means there exists a commutative diagram



Hence $ugf = f' = \nu = s' \in S$ and $ug\colon 0 \to Y$ is a morphism such that $X \to 0 \to Y$ is in S. It proves (1) ⇒ (2). The implication (1) ⇒ (3) is proved analogously. □

Proposition 2.10. *Let $\mathcal{A}$ be an abelian category.*

(1) *If S is a left multiplicative system then the category $S^{-1}\mathcal{A}$ has cokernels and the functor $Q\colon \mathcal{A} \to S^{-1}\mathcal{A}$ commutes with them.*
(2) *If S is a right multiplicative system then the category $S^{-1}\mathcal{A}$ has kernels and the functor $Q\colon \mathcal{A} \to S^{-1}\mathcal{A}$ commutes with them.*

(3) *If S is a multiplicative system then $S^{-1}\mathcal{A}$ is abelian and $Q\colon \mathcal{A} \to S^{-1}\mathcal{A}$ is exact.*

Proof. Assume S is a left multiplicative system and let $a\colon X \to Y$ be a morphism in $S^{-1}\mathcal{A}$. Then $a = (f\colon X \to Y', s\colon Y \to Y')$. Since $Q(s)$ is an isomorphism, existence of the cokernel of a is equivalent to the existence of the cokernel of $Q(f)$. Since Q commutes with finite colimits, $Q(\mathrm{coker}(f))$ is the sought cokernel.

The proof of (2) is similar.

By (1) and (2) we know that $S^{-1}\mathcal{A}$ has kernels and cokernels. It remains to check that $\mathrm{Coim} \simeq \mathrm{Im}$. As both are calculated in $\mathcal{A}$, the isomorphism follows from the isomorphism in $\mathcal{A}$. □

A full subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* if given a short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$ object A_2 lies in $\mathcal{X}$ if and only if A_1 and A_3 do.

We define

$$S = \{f \in \mathrm{Mor}(\mathcal{A}) \mid \ker(f), \mathrm{coker}(f) \in \mathcal{X}\}.$$

Lemma 2.11. *S is a multiplicative system.*

Proof. Clearly $\mathrm{Id}_X \in S$. Let now f, g be composable morphisms in S. Exact sequences

$$0 \to \ker(f) \to \ker(gf) \to \ker(g),$$
$$\mathrm{coker}(f) \to \mathrm{coker}(gf) \to \mathrm{coker}(g) \to 0$$

prove that $gf \in S$.

Consider a solid diagram

$$\begin{array}{ccc} A & \xrightarrow{\ g\ } & B \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ C & \xrightarrow{\ f\ } & C \sqcup_A B \end{array}$$

with $t \in S$. Then $\ker(t) \to \ker(s)$ is surjective and $\mathrm{coker}(t) \to \mathrm{coker}(s)$ is an isomorphism. This proves LMS2 and the proof of RMS2 is dual.

Finally, consider $f, g\colon B \to C$ and $s\colon A \to B$ such that $fs = gs$, i.e. $(f - g)s = 0$. Then $I = \mathrm{Im}(f - g)$ is the quotient of $\mathrm{coker} s$. Hence, $t\colon C \to C/I$ is an element of S and we have $t(f - g) = 0$. The proof of RMS3 is dual. □

Exercise 2.12. Prove that sequence $\mathrm{coker}(f) \to \mathrm{coker}(gf) \to \mathrm{coker}(g) \to 0$ is exact.

Let $\mathcal{X}$ be a Serre subcategory in an abelian category $\mathcal{A}$. We say that $\mathcal{X}$ is *localising* if the quotient functor $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{X}$ has a right adjoint $Q^!$. Then functor $Q^!$ is fully faithful and

$$Q^!\mathcal{A}/\mathcal{X} = \{A \in \mathcal{A} \,|\, \forall X \in \mathcal{X}, \mathrm{Hom}_{\mathcal{A}}(X, A) = 0 = \mathrm{Ext}^1(X, A)\}$$

is the category of $\mathcal{X}$-closed objects in $\mathcal{A}$, [Gab62].

3. Triangulated categories

We introduce derived categories following the exposition in [Kel96]. I've also used [Hap87] and `https://sites.math.washington.edu/~julia/teaching/581D_Fall2012/StableFrobIsTriang.pdf`

3.1. **Stable category of an exact category with enough injectives.**

Let $\mathcal{A}$ be an additive category. Let $C(\mathcal{A})$ be the category of complexes

$$\ldots \to A^n \xrightarrow{d^n} A^{n+1} \to \ldots$$

with morphisms $A^\bullet \to B^\bullet$ given by families $f^i \in \mathrm{Hom}_{\mathcal{A}}(A^i, B^i)$ such that $d^{i+1}f^i = f^{i+1}d^i$.

Exercise 3.1. Show that category $\mathcal{C}(\mathcal{A})$ with conflations defined as $(i^\bullet, p^\bullet)$ such that i^n, p^n is a split short exact sequence over $\mathcal{A}$ (i.e. $A^n \to A^n \oplus B^n \to B^n$) is an exact category.

Let $\mathcal{E}$ be an exact category. An object $I \in \mathcal{E}$ is *injective* if for any conflation $E_1 \to E \to E_2$ the sequence $0 \to \mathrm{Hom}(E_2, I) \to \mathrm{Hom}(E, I) \to \mathrm{Hom}(E_1, I) \to 0$ is exact. Similarly, $P \in \mathcal{E}$ is *projective*, if for any conlfation $E_1 \to E \to E_2$ the sequence $0 \to \mathrm{Hom}(P, E_1) \to \mathrm{Hom}(P, E) \to \mathrm{Hom}(P, E_2) \to 0$ is exact.

We say that an exact category $\mathcal{E}$ *has enough injectives* if any object $E \in \mathcal{E}$ fits into a conflation $E \to I \to E'$ with I injective.

Proposition 3.2. *Let $\mathcal{A}$ be an additive category. Complex $A^\bullet$ in $\mathcal{C}(\mathcal{A})$ is injective if and only if it is homotopic to zero, i.e. if there exists $(h^i\colon A^i \to A^{i-1})$ such that $d^i \circ h^i + h^{i+1} \circ d^{i+1} = Id_{A_i}$ for all i. The class of injectives and projectives objects in $\mathcal{C}(\mathcal{A})$ coincides and $\mathcal{C}(\mathcal{A})$ has enough injectives and projectives.*

Proof. Let $A^\bullet$ be any complex in $\mathcal{C}(\mathcal{A})$ and consider $(IA)^\bullet$ defined as $(IA)^n = A^n \oplus A^{n+1}$ with differential $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

A morphism $B^\cdot \to IA^\bullet$ is given by $\varphi^n\colon B^n \to A^n$. Indeed, such a collection defines a morphism of complexes $\psi\colon B^\bullet \to IA^\bullet$, $\psi^n = (\varphi_n, \varphi^{n+1}d^n)$.

We show that $IA^\bullet$ is injective. Any inflation in $\mathcal{C}(\mathcal{A})$ is of the form $\iota^n\colon B^n \to B^n \oplus C^n$ where the differential on $B^n \oplus C^n \to B^{n+1} \to C^{n+1}$ is $\begin{pmatrix} d_B & \alpha \\ 0 & d_C \end{pmatrix}$ (then $\iota^\bullet$ is a morphism of complexes). Given $\psi\colon B^\bullet \to IA^\bullet$ corresponding to $\varphi^n\colon B^n \to A^n$, morphism $\widetilde{\psi}\colon (B \oplus C)^\bullet \to IA^\bullet$ corresponding to $(\psi, 0)$ is such that $\widetilde{\psi} \circ \iota = \psi$.

The map $A^\bullet \to IA^\bullet$ corresponding to $\mathrm{Id}\colon A^n \to A^n$ is an inflation, hence $\mathcal{C}(\mathcal{A})$ has enough injectives.

If $A^\bullet$ is itself injective, $A^\bullet \to IA^\bullet$ splits, i.e. there exist $(d_A h^n, h^{n+1})\colon A^n \oplus A^{n+1} \to A^n$ such that $d_A h_n + h^{n+1} d_A = \mathrm{Id}_A$ (any morphism $IA^\bullet \to B^\bullet$ is determined by a family $\beta^{n+1}\colon A^{n+1} \to B^n$). Then h^n is a homotopy, i.e. $A^\bullet$ is null-homotopic.

Analogous argument shows that $IA^\bullet$ is projective, and that and projective complex is null-homotopic. □

Let $\mathcal{E}$ be an exact category with enough injectives. We write SE for the cokernel of the inflation $E \to I(E)$.

Exercise 3.3. Describe SA for a complex $A \in \mathcal{C}(\mathcal{A})$ of objects of an additive category $\mathcal{A}$.

Let $\mathcal{E}$ be an exact category with enough injective objects. The *stable category* $\underline{\mathcal{E}}$ has the same objects as $\mathcal{E}$ and morphisms in $\underline{\mathcal{E}}$ are morphisms in $\mathcal{E}$ modulo the subgroup of morphisms factoring through an injective object of $\mathcal{E}$. The composition in $\underline{\mathcal{E}}$ is induced from the composition in $\mathcal{E}$.

The main example Let $\mathcal{A}$ be an additive category. The *homotopy category* $\mathcal{H}(\mathcal{A})$ *of* $\mathcal{A}$ is the stable category of $\mathcal{C}(\mathcal{A})$. Often, the category $\mathcal{H}(\mathcal{A})$ is denoted by $\mathcal{K}(\mathcal{A})$.

Exercise 3.4. Show that morphisms in $\mathcal{H}(\mathcal{A})$ are morphisms of complexes modulo the homotopy relation.

Lemma 3.5. *Let E and E' be objects of an exact category $\mathcal{E}$ with enough injectives. If $E \oplus I \simeq E' \oplus I'$ for some injective objects $I, I' \in \mathcal{E}$ then $E \simeq E'$ in $\underline{\mathcal{E}}$.*

Proof. Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be an isomorphism $E \oplus I \to E' \oplus I'$ with inverse $\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$. Then $\alpha'\alpha + \beta'\gamma = \mathrm{Id}_E$. As $\beta'\gamma$ factors via injective I', we have $\alpha'\alpha = \mathrm{Id}_E$ in $\underline{\mathcal{E}}$. □

Proposition 3.6. *Let E be an object of an exact category $\mathcal{E}$ with enough projectives. Then the cokernel of an inflation $E \to I$, for $I \in \mathcal{E}$ injective, is unique up to isomorphism in $\underline{\mathcal{E}}$.*

Proof. Consider $E \to I_1 \to C_1$, $E \to I_2 \to C_2$ and the pushout

$$\begin{array}{ccccc} E & \longrightarrow & I_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ I_2 & \longrightarrow & D & \longrightarrow & C_1 \\ \downarrow & & \downarrow & & \\ C_2 & \longrightarrow & C_2 & & \end{array}$$

As I_1 and I_2 are injective, the middle row and column split, hence $C_1 \oplus I_2 \simeq C_2 \oplus I_1$. We conclude by above Lemma. □

For $E \in \mathcal{E}$ we denote $T(E)$ the cokernel of an inflation $E \to I$ with injective I. By the above Proposition $T(E) \in \underline{\mathcal{E}}$ is well-defined. We extend $E \mapsto T(E)$ to a functor $T \colon \underline{\mathcal{E}} \to \underline{\mathcal{E}}$ by mapping $f \colon E \to E'$ in $\mathcal{E}$ to a morphism $T(E) \to T(E')$ which makes the following diagram commutative:

$$\begin{array}{ccccc} E' & \xrightarrow{i'} & I' & \xrightarrow{p'} & T(E') \\ f \uparrow & & g \uparrow & & h \uparrow \\ E & \xrightarrow{i} & I & \xrightarrow{p} & T(E) \end{array}$$

Morphism g exists because I' is injective and it determines h uniquely. Assume that $g' \colon I \to I'$ is another morphism such that $g' \circ i = i' \circ f$ and let h' be such that $p' \circ g' = h' \circ p$. Then $(g - g')i = i'f - i'f = 0$, hence there exits $s \colon T(E) \to I'$ such that $g - g' = s \circ p$. Then $(h - h')p = p'(g - g') = p' \circ s \circ p$, hence $h - h' = p' \circ s$, as p is an epimorphism. It follows that morphism h is uniquely defined in $\underline{\mathcal{E}}$.

Let $f \colon A \to B$ be a morphism in $\mathcal{E}$ and $A \to I \to T(A)$ a conflation with injective I. Consider push-out of f along $A \to I$ and complete it to a conflation:

$$\begin{array}{ccccc} B & \xrightarrow{g} & C & \xrightarrow{h} & T(A) \\ f \uparrow & & \uparrow & & \simeq \uparrow \\ A & \longrightarrow & I & \longrightarrow & T(A) \end{array} \tag{6}$$

Then $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ is a *standard triangle* in $\underline{\mathcal{E}}$.

We define *distinguished triangles* in $\underline{\mathcal{E}}$ as sequences $A \to B \to C \to T(A)$ isomorphic to standard triangles.

Theorem 3.7. *Let $\mathcal{E}$ be an exact category with enough injectives. Then $\underline{\mathcal{E}}$ is a* suspended category, *i.e. it satisfies:*

(ST1) *Every triangle isomorphic to a distinguished triangle is distinguished. Every morphism $f\colon A \to B$ can be embedded into a distinguished triangle $A \to B \to C \to T(A)$. For any object A triangle $A \xrightarrow{Id} A \to 0 \to T(A)$ is distinguished.*

(ST2) *If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ is a distinguished triangle then $B \xrightarrow{g} C \xrightarrow{h} T(A) \xrightarrow{T(f)} T(B)$ is distinguished.*

(ST3) *Every solid diagram*

$$\begin{array}{ccccccc}
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & T(A) \\
\varphi\uparrow & & \psi\uparrow & & \theta\,\vdots & & T(\varphi)\uparrow \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & T(A)
\end{array}$$

can be completed to a dashed diagram

(ST4) *Given $f\colon A \to B$, $g\colon B \to C$ there exists*

$$\begin{array}{ccccccc}
 & & T(B) & \xrightarrow{T(f')} & T(D) & & \\
 & & g''\uparrow & & j''\uparrow & & \\
 & & E & \xrightarrow{Id} & E & & \\
 & & g'\uparrow & & j'\uparrow & & \\
A & \xrightarrow{h=gf} & C & \xrightarrow{h'} & F & \xrightarrow{h''} & T(A) \\
Id\uparrow & & g\uparrow & & j\uparrow & & Id\uparrow \\
A & \xrightarrow{f} & B & \xrightarrow{f'} & D & \xrightarrow{f''} & T(A)
\end{array}$$

with distinguished middle columns and middle rows.

Proof. A triangle isomorphic to a distinguished triangle is isomorphic to a standard triangle, hence it is distinguished. We have constructed a standard triangle for any $f\colon A \to B$. From the commutative diagram

$$\begin{array}{ccccc}
A & \longrightarrow & I(A) & \longrightarrow & T(A) \\
\mathrm{Id}\uparrow & & \uparrow & & \uparrow \\
A & \longrightarrow & I(A) & \longrightarrow & T(A)
\end{array}$$

and the fact that $I(A) \simeq 0$ in $\underline{\mathcal{E}}$ we conclude that $A \xrightarrow{\mathrm{Id}} A \to 0 \to T(A)$ is distinguished. We have thus proved (TR1).

To prove (TR2) we first note that any morphism in $\underline{\mathcal{E}}$ is a class of an inflation. Indeed, let $f\colon A \to B$ be any map. Then diagram (6) implies that $A \to B \oplus I(A) \to C$ is a conflation. (We can present $\mathcal{E}$ as an extension-closed subcategory of an abelian category

$\mathcal{A}$ in such a way that conflations in $\mathcal{E}$ are short exact sequences in $\mathcal{A}$. Then it it enough to show that $0 \to A \to B \oplus I(A) \to C \to 0$ is a short exact sequence.) Then the composite $A \to B \to I(B)$ is an inflation and we have a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & I(B) & \longrightarrow & T(A) \\ {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow & & \uparrow \\ A & \xrightarrow{f} & B & \longrightarrow & C \end{array}$$

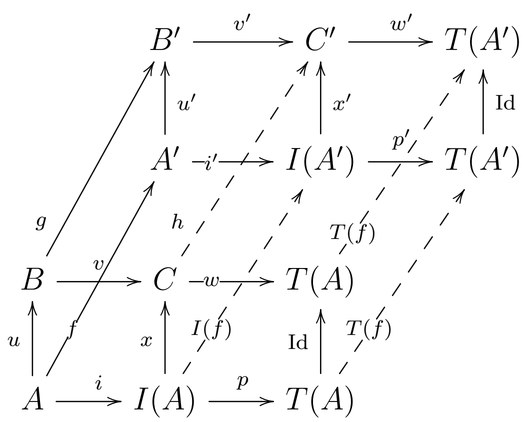
with conflations in rows. Then

$$\begin{array}{ccccc} C & \longrightarrow & T(A) & \xrightarrow{T(f)} & T(B) \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{Id}} \\ B & \longrightarrow & I(B) & \longrightarrow & T(B) \\ \uparrow & & \uparrow & & \\ A & \xrightarrow{\mathrm{Id}} & A & & \end{array}$$

has conflations in rows and columns. It follows that $B \to C \to T(A) \to T(B)$ is distinguished.

In particular, the composition of two consecutive morphism in a distinguished triangle is zero: $A \to B \to C$ factors via $I(A)$; for $B \to C \to T(A)$ we can 'rotate' the triangle.

Consider a solid diagram:



which commutes in $\underline{\mathcal{E}}$, i.e. $u'f - gu$ factors via an injective object. We can assume that there exists $a \colon I(A) \to B'$ (not in the picture) such that $gu - u'f = ai$. Morphism f induces $I(f)$ (non-unique) and $T(f)$ (unique in $\underline{\mathcal{E}}$).

Object C is a pushout, hence $x'I(f) + v'a \colon I(A) \to C'$ and $v'g \colon B \to C'$ (they agree on A: $(x'I(f) + v'a)i = x'i'f + v'gu - v'u'f = v'u'f - v'gu - v'u'f = v'gu$) define $h \colon C \to C'$. To check that it defines a morphism of triangles we need to check that $w'h = T(f)w$. As C is a pushout, it suffices to check that $w'hv = T(f)wv$ and $w'hx = T(f)wx$ in $\underline{\mathcal{E}}$. The first equality is clear as $w'hv = w'v'g = 0 = T(f)wv$ (composition of 2 consecutive morphisms in a distinguished triangle is zero). Finally $w'hx = w'x'I(f) = p'I(f) = T(f)p = T(f)wx$ which finishes the proof of (TR3).

To prove (TR4) we consider standard triangles:

$$\begin{array}{ccccc} B & \xrightarrow{f'} & D & \xrightarrow{f''} & T(A) \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \mathrm{Id}} \\ A & \xrightarrow{i_A} & I(A) & \xrightarrow{p_A} & T(A) \end{array} \qquad \begin{array}{ccccc} C & \xrightarrow{h'} & F & \xrightarrow{h''} & T(A) \\ {\scriptstyle gf}\uparrow & & \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \mathrm{Id}} \\ A & \xrightarrow{i_A} & I(A) & \xrightarrow{p_A} & I(A) \end{array}$$

$$\begin{array}{ccccc} C & \xrightarrow{g'} & E & \xrightarrow{g''} & T(B) \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \mathrm{Id}} \\ B & \xrightarrow{i_D f'} & I(D) & \xrightarrow{T(f')^{-1}p_D} & T(B) \end{array}$$

We can choose f' to be an inflation and consider $i_D \circ f'$ and an injective envelope of B.

We need a morphism $D \to F$. As D is a pushout a pair $h'g\colon B \to F$, $\gamma\colon I(A) \to F$ define unique $j\colon D \to E$. Then $jf' = h'g$ and $j\alpha = \gamma$.

Next, we want $F \to E$. Again a pair $g'\colon C \to E$ and $\beta i_D \alpha\colon I(A) \to E$ define unique $j'\colon F \to E$ such that $j'\gamma = \beta i_D \alpha$ and $j'h' = g'$.

We have a diagram

$$\begin{array}{ccccc} C & \xrightarrow{h'} & F & \xrightarrow{j'} & E \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle \beta} \\ B & \xrightarrow{f'} & D & \xrightarrow{i_D} & I(D) \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \alpha} & & \\ A & \xrightarrow{i_A} & I(A) & & \end{array}$$

The bottom left square and the big left rectangle are pushouts, hence so is the top left square. The top rectangle is a pushout, too, hence the top right square is a pushout. We thus get $j''\colon E \to T(D)$ such that $j''\beta = p_D\colon I(D) \to D$.

To get a commutative diagram

$$\begin{array}{ccccccc}
 & & T(B) & \xrightarrow{T(f')} & T(D) & & \\
 & & {\scriptstyle g''}\uparrow & & {\scriptstyle j''}\uparrow & & \\
 & & E & \xrightarrow{\mathrm{Id}} & E & & \\
 & & {\scriptstyle g'}\uparrow & & {\scriptstyle j'}\uparrow & & \\
A & \xrightarrow{h=gf} & C & \xrightarrow{h'} & F & \xrightarrow{h''} & T(A) \\
{\scriptstyle \mathrm{Id}}\uparrow & & {\scriptstyle g}\uparrow & & {\scriptstyle j}\uparrow & & {\scriptstyle \mathrm{Id}}\uparrow \\
A & \xrightarrow{f} & B & \xrightarrow{f'} & D & \xrightarrow{f''} & T(A)
\end{array}$$

it remains to check that $T(f')g'' = j''$ and $h''j = f''$. It is a straightforward verification using the fact that E and D are pushouts. □

3.2. **Triangulated categories.**

Definition 3.8. A suspended category is *triangulated* if the functor T is an equivalence.

An exact category $\mathcal{E}$ with enough injectives and projectives is *Frobenius* if the class of projective objects coincides with the class of injective objects.

Exercise 3.9. Show that stable category of a Frobenius category is triangulated.

We shall usually denote the shift functor in a triangulated category by [1].

P. May in `http://www.math.uchicago.edu/~may/MISC/Triangulate.pdf` proved that the octahedron axiom is equivalent to

(ST4) A commutative square

$$\begin{array}{ccc}
C & \longrightarrow & D \\
\uparrow & & \uparrow \\
A & \longrightarrow & B
\end{array}$$

can be completed to a diagram

$$\begin{array}{ccccccc}
A[1] & \longrightarrow & B[1] & \longrightarrow & E[1] & \longrightarrow & A[2] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & G[1] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
C & \longrightarrow & D & \longrightarrow & I & \longrightarrow & C[1] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
A & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A[1]
\end{array}$$

with distinguished rows and columns in which all squares except to upper right one commute and the upper right square anti-commutes.

Lemma 3.10. *Let $\mathcal{D}$ be a triangulated category. The composition of two consecutive morphisms in a distinguished triangle is zero.*

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ be a distinguished triangle. By TR1 triangle $A \xrightarrow{\mathrm{Id}} A \xrightarrow{0} 0 \xrightarrow{0} A[1]$ is distinguished. By TR2 we have morphisms of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ {\scriptstyle \mathrm{Id}}\uparrow & & {\scriptstyle f}\uparrow & & \uparrow & & {\scriptstyle \mathrm{Id}}\uparrow \\ A & \xrightarrow{\mathrm{Id}} & A & \longrightarrow & 0 & \longrightarrow & A[1] \end{array}$$

In particular, the second square commutes, i.e. $gf = 0$. □

Lemma 3.11. *Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$. Then, for any $D \in \mathcal{D}$ complexes*

$$Hom(D, A) \to Hom(D, B) \to Hom(D, C)$$

$$Hom(C, D) \to Hom(B, D) \to Hom(A, D)$$

are exact.

Proof. We already know that the composition of two consecutive morphisms in a triangle is zero. It suffices to show that given $\varphi \colon D \to B$ such that $v\varphi = 0$ there exists $\psi \colon D \to A$ such that $u\psi = \varphi$.

Consider a solid morphism of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & Z & \xrightarrow{w} & A[1] \\ {\scriptstyle \psi}\uparrow & & {\scriptstyle \varphi}\uparrow & & \uparrow & & \uparrow \\ D & \xrightarrow{\mathrm{Id}} & D & \longrightarrow & 0 & \longrightarrow & D[1] \end{array}$$

By (ST3) morphism ψ such that $u\psi = \varphi$ exists.

Analogously one proves exactness of the second sequence. □

Let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ a triangulated category. An additive functor $F \colon \mathcal{D} \to \mathcal{A}$ is *cohomological* if for any distinguished triangle $A \to B \to C \to A[1]$ the sequence $F(A) \to F(B) \to F(C)$ is exact.

Let F be a cohomological functor and $F^k := F \circ [k]$. Then for any distinguished triangle $A \to B \to C \to A[1]$ the sequence

$$\ldots \to F^i(A) \to F^i(B) \to F^i(C) \to F^{i+1}(A) \to \ldots$$

is exact.

Lemma 3.12. *Let $\mathcal{D}$ be a triangulated category. If in a morphism of triangles two arrows are isomorphisms, then so is the third.*

Proof. Let

$$\begin{array}{ccccccc} A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \\ {\scriptstyle a}\uparrow & & {\scriptstyle b}\uparrow & & {\scriptstyle c}\uparrow & & {\scriptstyle a[1]}\uparrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \end{array}$$

be a morphism of triangles. Assume that a and c are isomorphism. Take and $D \in \mathcal{D}$. Applying $\mathrm{Hom}(D, -)$ to the above diagram gives a commutative 5×2 diagram of abelian groups. By the 5-lemma, the morphism $\mathrm{Hom}(D, B) \to \mathrm{Hom}(D, B')$ given by the composition with b is an isomorphism. By Yoneda, b is an isomorphism. □

Lemma 3.13. *Let $f\colon A \to B$ be a morphism in a triangulated category $\mathcal{D}$. The following are equivalent:*

(1) f is an isomorphism,
(2) $A \xrightarrow{f} B \to 0 \to A[1]$ is a distinguished triangle,
(3) For any distinguished triangle $A \xrightarrow{f} B \to C \to A[1]$, C is isomorphic to 0.

Proof. We know that $A \xrightarrow{\mathrm{Id}} A \to 0 \to A[1]$ is distinguished. Triangle $A \xrightarrow{f} B \to 0 \to A[1]$ is isomorphic with it, hence it is also distinguished. It proves (1) $\Rightarrow$ (2). Lemma 3.12 implies that (2) $\Rightarrow$ (3), triangle $A \to B \to C \to A[1]$ has a map to $A \to B \to 0 \to A[1]$ which is an isomorphism on A and B, hence it is on C. Finally,

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ {\scriptstyle =}\uparrow & & {\scriptstyle \mathrm{Id}}\uparrow & & {\scriptstyle f}\uparrow & & \uparrow \\ 0 & \longrightarrow & A & \xrightarrow{\mathrm{Id}} & A & \longrightarrow & 0 \end{array}$$

is a morphism of triangles in which two morphisms are isomorphisms. Hence, by Lemma 3.12, f is an isomorphism, i.e. (3) $\Rightarrow$ (1). □

Proposition 3.14. *Let $\mathcal{A}$ be an abelian category and $\mathcal{H}(\mathcal{A})$ the stable category of the category of complexes over $\mathcal{A}$. Then functor $H^0(-)\colon \mathcal{C}(\mathcal{A}) \to \mathcal{A}$ induces a cohomological functor $\mathcal{H}(\mathcal{A}) \to \mathcal{A}$.*

Proof. Any distinguished triangle is isomorphic to a standard triangle and to a rotation of a standard triangle, hence it is enough to show that given a morphism $f\colon A^\bullet \to B^\bullet$ in $\mathcal{C}(\mathcal{A})$ the sequence $H^0(B^\bullet) \to H^0(C_f^\bullet) \to H^0(A^\bullet[1])$ is exact.

The cone C_f is of the form $C_f^n = B^n \oplus A^{n+1}$ with differential $\begin{pmatrix} d_B & f^{n+1} \\ 0 & -d_A \end{pmatrix}$. Hence $B^\bullet \to C_f^\bullet \to A^\bullet[1]$ is a conflation in $\mathcal{C}(\mathcal{A})$.

Hence, we prove that given a conflation $A^\bullet \to B^\bullet \to C^\bullet$ in $\mathcal{C}(\mathcal{A})$ the sequence $H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet)$ is exact. Snake lemma for the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{n-1} & \longrightarrow & B^{n-1} & \longrightarrow & C^{n-1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_A^n} & & \uparrow{\scriptstyle d_B^n} & & \uparrow{\scriptstyle d_C^n} & & \\
0 & \longrightarrow & A^n & \longrightarrow & B^n & \longrightarrow & C^n & \longrightarrow & 0
\end{array}$$

proves that $0 \to \ker d_A^n \to \ker d_B^n \to \ker d_C^n$ is exact. Similarly, $\operatorname{coker} d_A^{n-1} \to \operatorname{coker} d_B^{n-1} \to \operatorname{coker} d_C^{n-1} \to 0$ is exact. Then the snake lemma for

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker d_A^n & \longrightarrow & \ker d_B^n & \longrightarrow & \ker d_C^n & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \operatorname{coker} d_A^{n-1} & \longrightarrow & \operatorname{coker} d_B^{n-1} & \longrightarrow & \operatorname{coker} d_C^{n-1} & \longrightarrow & 0
\end{array}$$

proves that $H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet)$ is exact. $\square$

Let $\mathcal{D}_1$, $\mathcal{D}_2$ be triangulated categories. An *exact functor* $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ is an additive functor which commutes with the shift functor and maps distinguished triangles to distinguished triangles.

Exercise 3.15. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be Frobenius exact categories. A functor $F\colon \mathcal{E}_1 \to \mathcal{E}_2$ which maps injective objects in $\mathcal{E}_1$ to injective objects in $\mathcal{E}_2$ induces an exact functor $\underline{F}\colon \underline{\mathcal{E}}_1 \to \underline{\mathcal{E}}_2$.

3.3. **Localization of triangulated categories.**

Let $\mathcal{A}$ be an abelian category. While studying derived functors we have discovered that sometimes it is convenient to replace objects of $\mathcal{A}$ by their resolutions.

We want to consider the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$. It shall be a triangulated category whose objects will be complexes of objects in $\mathcal{A}$.We want to identify any two resolutions of an object of $\mathcal{A}$. More generally, we would like to say that complexes are isomorphic if there is a morphism which induces an isomorphism of all cohomology objects (such a morphism will be called a quasi-isomorphism).

To construct $\mathcal{D}(\mathcal{A})$ we want to consider the stable category of complexes $\mathcal{H}(\mathcal{A})$ and invert some morphisms in it (precisely the quasi-isomorphisms). We have already seen this kind of construction, it is a quotient of a category by a multiplicative system S. To ensure that the quotient category is still triangulated, we put extra conditions on S.

Let $\mathcal{D}$ be a triangulated category. We say that a multiplicative system S of arrows in $\mathcal{D}$ is *compatible with the triangulated structure* if the following conditions hold:

MS5 For $s \in S$ and $n \in \mathbb{Z}$, $s[n] \in S$,

MS6 Given a solid commutative square

$$\begin{array}{ccccccc} X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & X'[1] \\ {\scriptstyle s}\uparrow & & {\scriptstyle s'}\uparrow & & {\scriptstyle s''}\uparrow & & {\scriptstyle s[1]}\uparrow \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

whose rows are distinguished and $s, s' \in S$, there exists $s'' \in S$ such that the diagram is a morphism of triangles.

Lemma 3.16. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be an exact functor of triangulated categories. Let*

$$S = \{f \in Mor(\mathcal{D}) \mid F(f) \text{ is an isomorphism}\}.$$

Then S is a saturated multiplicative system compatible with the triangulated structure.

Proof. Identity morphisms lie in S and a composition of two morphisms in S lie in S, so MS1 is satisfied. Isomorphisms satisfy 2-out-of-3 property, so MS4 is satisfied. Clearly MS5 also holds.

Property MS6 follows from Lemma 3.12.

Next, we check that MS2 holds. Let $f\colon A \to B$ be a morphism and $t\colon A \to C$ a morphism in S. Then, we have a morphism of triangles

$$\begin{array}{ccccccc} C & \longrightarrow & B' & \longrightarrow & D & \longrightarrow & C[1] \\ {\scriptstyle t}\uparrow & & \uparrow & & {\scriptstyle \mathrm{Id}}\uparrow & & {\scriptstyle t[1]}\uparrow \\ A & \xrightarrow{f} & B & \longrightarrow & D & \longrightarrow & A[1] \end{array}$$

Since S satisfies MS6, morphism $B \to B'$ lies in S, i.e. LMS2 holds. The proof of RMS2 is dual.

It remains to show MS3. Let $f, g\colon A \to B$ be morphism, such that, for $t\colon C \to A$ in S, $ft = gt$. In other words, $at = 0$, for $a = g - t$. Consider diagram

$$\begin{array}{ccccc} & & & & D \\ & & & & {\scriptstyle j}\uparrow \\ & & A & \xrightarrow{a} & B \\ & & {\scriptstyle \mathrm{Id}}\uparrow & & {\scriptstyle b}\uparrow \\ C & \xrightarrow{t} & A & \xrightarrow{q} & E \end{array}$$

in which the bottom row is distinguished, morphism b is any morphism such that $bq = a$ (it exists by Lemma 3.11) and the right column is distinguished. Then $ja = jbq = 0$.

If we apply F to this diagram, Lemma 3.13 implies that $F(E) \simeq 0$, hence $F(j)$ is an isomorphism. By definition, $j \in S$, which finishes the proof of LMS3. The proof of RMS3 is analogous. □

Remark 3.17. In the proof of Lemma 3.16 we have in fact proved that if a multiplicative system satisfies MS1, MS5 and MS6, then it satisfies MS2.

Similarly, we have

Lemma 3.18. *Let $H\colon \mathcal{D} \to \mathcal{A}$ be a cohomological functor between a triangulated category and an abelian category. Let*

$$S = \{f \in Mor(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbb{Z}\}.$$

Then S is a saturated multiplicative system compatible with the triangulated structure.

Exercise 3.19. Prove the Lemma.

Proposition 3.20. *Let $\mathcal{D}$ be a triangulated category and S a multiplicative system compatible with the triangulate structure. Then there exists unique structure of triangulated category on $S^{-1}\mathcal{D}$ such that $Q\colon \mathcal{D} \to S^{-1}\mathcal{D}$ is exact.*

Proof. We already know that $S^{-1}\mathcal{D}$ is additive. We define $Q(D)[1] = Q(D[1])$ and we say that a triangle in $S^{-1}\mathcal{D}$ is distinguished if it is an image of a distinguished triangle in $\mathcal{D}$. □

Exercise 3.21. Check that axioms TR1-TR4 hold, *cf.* [Sta13, Proposition 13.5.5].

The quotient $S^{-1}\mathcal{D}$ has universal property with respect to exact functors $\mathcal{D} \to \mathcal{D}'$ and homological functors $\mathcal{D} \to \mathcal{A}$.

Now, we are ready to define the derived category $\mathcal{D}(\mathcal{A})$ of an abelian category. Let $\mathcal{C}(\mathcal{A})$ be the category of unbounded complexes of objects of $\mathcal{A}$. We consider the functor

$$H^0(-)\colon \mathcal{C}(\mathcal{A}) \to \mathcal{A}.$$

By Proposition 3.14, it is cohomological. Hence, by Lemma 3.18

$$S = \{f \in \mathrm{Mor}(\mathcal{H}(\mathcal{A})) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbb{Z}\}$$

is a multiplicative system compatible with triangulated structure. We define

$$\mathcal{D}(\mathcal{A}) = S^{-1}\mathcal{H}(\mathcal{A}).$$

We could also start from the stable category $\mathcal{H}^b(\mathcal{A})$ of *bounded complexes*, the stable category $\mathcal{H}^-(\mathcal{A})$ of *bounded above complexes* or the stable category $\mathcal{H}^+(\mathcal{A})$ of *bounded below complexes*. Considering analogous quotients, we get

$$\mathcal{D}^b(\mathcal{A}) = S^{-1}\mathcal{H}^b(\mathcal{A}), \qquad \mathcal{D}^-(\mathcal{A}) = S^{-1}\mathcal{H}^-(\mathcal{A}), \qquad \mathcal{D}^+(\mathcal{A}) = S^{-1}\mathcal{H}^+(\mathcal{A}),$$

the *bounded, bounded above* and *bounded below* derived category of $\mathcal{A}$.

3.4. **Derived category as a quotient by acyclic complexes.**

The construction of derived category $\mathcal{D}(\mathcal{A})$ required the category $\mathcal{A}$ to be abelian, as we considered cohomology of complexes of objects of $\mathcal{A}$. We shall now describe another definition of $\mathcal{D}(\mathcal{A})$ which admits an immediate generalisation to the case when $\mathcal{A}$ is only exact.

Let $\mathcal{D}$ be a triangulated category. We say that full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is *saturated* if whenever $X \oplus Y$ is isomorphic to an object of $\mathcal{D}'$ then X and Y are.

Lemma 3.22. *Let $F\colon \mathcal{D} \to \mathcal{D}_1$ be an exact functor of triangulated categories. Let $\mathcal{D}'$ be the full subcategory with objects*

$$Ob(\mathcal{D}') = \{D \in \mathcal{D} \mid F(D) \simeq 0\}.$$

Then $\mathcal{D}'$ is a saturated triangulated subcategory of $\mathcal{D}$.

Proof. Clear. □

Let $f\colon D_1 \to D_2$ be a morphism in a triangulated category. By TR1, there exists a distinguished triangle $D_1 \xrightarrow{f} D_2 \to D_3 \to D_1[1]$. We call D_3 the *cone* of f. It is unique up to a non-unique isomorphism.

Proposition 3.23. *Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory of a triangulated category. Set*

$$S = \{f \in Mor(\mathcal{D}) \,|\, \text{the cone of } f \in \mathcal{D}'\}.$$

Then the following are equivalent

(1) S is a saturated multiplicative system.
(2) $\mathcal{D}'$ is a saturated subcategory.

Let $\mathcal{D}' \subset \mathcal{D}$ be a saturated full triangulated subcategory of $\mathcal{D}$. The *Verdier quotient* $\mathcal{D}/\mathcal{D}'$ is $S^{-1}\mathcal{D}$, for the multiplicative system S defined as in Proposition 3.23. From the universal property of $S^{-1}\mathcal{D}$ (see Proposition 1.5) we conclude:

Proposition 3.24. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{D}' \subset \mathcal{D}$ a saturated full triangulated subcategory. Let $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{D}'$ be the quotient functor.*

(1) *If $H\colon \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category such that $H(D') \simeq 0$, for all $D' \in \mathcal{D}'$, then there exists a unique factorization $\overline{H}\colon \mathcal{D}/\mathcal{D}' \to \mathcal{A}$ such that $\overline{H} \circ Q = H$ and $\overline{H}$ is a homological functor too.*

(2) *If $F\colon \mathcal{D} \to \mathcal{D}_1$ is an exact functor into a triangulated category such that $F(D') \simeq 0$, for all $D' \in \mathcal{D}'$, then there exists a unique factorization $\overline{F}\colon \mathcal{D}/\mathcal{D}' \to \mathcal{D}_1$ such that $\overline{F} \circ Q = F$ and $\overline{F}$ is an exact functor too.*

Let $\mathcal{A}$ be an abelian category and $\mathrm{Ac}(A) \subset \mathcal{H}(\mathcal{A})$ be the category of *acyclic* complexes, i.e. complexes $A^\bullet$ such that $H^i(A^\bullet) = 0$, for all i. Then

$$\begin{aligned} S &= \{f \in \mathrm{Mor}(\mathcal{H}(\mathcal{A})) \mid H^i(f) \textit{ is an isomorphism for all } i \in \mathbb{Z}\} \\ &= \{f \in \mathrm{Mor}(\mathcal{H}(\mathcal{A})) \mid \textit{ the cone of } f \in \mathrm{Ac}(\mathcal{A})\}. \end{aligned}$$

It follows that the derived category $\mathcal{D}(\mathcal{A})$ is the quotient of $\mathcal{H}(\mathcal{A})$ by the subcategory of acyclic complexes. Similarly, $\mathcal{D}^b(\mathcal{A}) \simeq \mathcal{H}^b(\mathcal{A})/\mathrm{Ac}^b(\mathcal{A})$, $\mathcal{D}^-(\mathcal{A}) \simeq \mathcal{H}^-(\mathcal{A})/\mathrm{Ac}^-(\mathcal{A})$, $\mathcal{D}^+(\mathcal{A}) \simeq \mathcal{H}^+(\mathcal{A})/\mathrm{Ac}^+(\mathcal{A})$, where $\mathrm{Ac}^*(\mathcal{A})$, for $* \in \{b, +, -\}$ is defined as $\mathrm{Ac}(\mathcal{A}) \cap \mathcal{H}^*(\mathcal{A})$.

Lemma 3.25. *Let A, B be objects of an abelian category $\mathcal{A}$. Then $Ext^1_{\mathcal{A}}(A, C) \simeq Hom_{\mathcal{D}(\mathcal{A})}(A, C[1])$.*

Proof. For simplicity let us assume that $\mathcal{A}$ has enough projective objects. Let $P^\bullet$ be a projective resolution of A. Then $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, C[1]) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(P^\bullet, C[1]) \simeq \mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(P^\bullet, C[1])$ is the 1'st cohomology group of the complex $\mathrm{Hom}(P^0, C) \to \mathrm{Hom}(P^1, C) \to \dots$. (We will show on tutorials that $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(P^\bullet, C[1]) \simeq \mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(P^\bullet, C[1])$.) ☐

Proposition 3.26. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$ and ζ the corresponding class in $Ext^1(C, A)$. Then $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\zeta} A[1]$ is a distinguished triangle in $\mathcal{D}(\mathcal{A})$.*

Proof. Let us calculate the cone of $A \to B$ in $\mathcal{H}(\mathcal{A})$. The injective envelope of A is complex $A \xrightarrow{Id} A$ concentrated in degrees -1 and 0. The pushout of $A \xrightarrow{f} B$ and $A \to IA$ is the complex $A \xrightarrow{f} B$. Indeed, a morphism $(IA)^\bullet \to D^\bullet$ is a map $A \to D^{-1}$ while a morphism $B \to D^\bullet$ is a map $B \to \ker(D^0 \to D^{-1})$. If these agree on A, they define a unique map of complexes $\{A \xrightarrow{f} B\} \to D^\bullet$.

By construction of triangulated structure $A \to B \to \{A \xrightarrow{f} B\} \to A[1]$ is a distinguished triangle in $\mathcal{H}(\mathcal{A})$. Morphism $\{A \xrightarrow{f} B\} \to C$ induced by g is a quasi-isomorphism, hence $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\xi} A[1]$ is distinguished in $\mathcal{D}(\mathcal{A})$.

Let $P^\bullet$ be a projective resolution of C. Then by Proposition 2.6 the element ζ is a class of a map $P^{-1} \to A$. The construction is such that $P^\bullet \to \{A \xrightarrow{f} B\}$ is a morphism of complexes (we construct $\varphi\colon P^{-1} \to A$ by first lifting $P^0 \to C$ to $P^0 \to B$ and then noticing that $P^{-1} \to P^0 \to B \to C$ is zero, so it factors via φ). We get a commuting triangle of quasi-isomorphisms

$$\begin{array}{ccc} P^\bullet & \longrightarrow & C \\ & \searrow \quad \nearrow_{g} & \\ & \{A \xrightarrow{f} B\} & \end{array}$$

The map $C \xrightarrow{\xi} A[1]$ in the distinguished triangle is such that the composite $\{A \xrightarrow{f} B\} \to C \xrightarrow{\xi} A[1]$ is the projection to $A[1]$. Then $P^\bullet \to C \xrightarrow{\xi} A[1]$ is the map φ. It follows that $\xi = \zeta$. □

Proposition 3.26 can be generalized, *cf.* [Kel96, Part 11]:

Proposition 3.27. *Let $\mathcal{A}$ be an abelian category and let $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet$ be a complex in $\mathcal{C}(\mathcal{A})$ such that $A^n \to B^n \to C^n$ is a short exact sequence, for any n. Then there exists $\zeta\colon C^\bullet \to A^\bullet[1]$ such that $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{\zeta} A^\bullet[1]$ is a distinguished triangle in $\mathcal{D}(\mathcal{A})$.*

4. t-structures

4.1. **The motivating example.** Let $\mathcal{A}$ be an abelian category and $\mathcal{D}^*(\mathcal{A})$ its derived category. We consider the functor

$$\iota_0\colon \mathcal{A} \to \mathcal{C}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}) \tag{7}$$

which maps an object $A \in \mathcal{A}$ to the class of a complex $\{\ldots \to 0 \to A \to 0 \to \ldots\}$ with A in degree zero.

Exercise 4.1. Let $A^\bullet$ be a complex in $\mathcal{D}(\mathcal{A})$ such that $H^n(A^\bullet) = 0$, for $n \geq N$. Then $A^\bullet$ is isomorphic to a complex concentrated in degrees $\leq N$.

Proposition 4.2. *Functor ι is fully faithful and induces an equivalence of $\mathcal{A}$ with the full subcategory of $\mathcal{D}(\mathcal{A})$ of complexes $A^\bullet$ such that $H^i(A) = 0$, for $i \neq 0$.*

Proof. Clearly, $\mathrm{Hom}_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(\iota_0(A), \iota_0(B))$. As there cannot be a homotopy between two morphisms $\iota_0(A) \to \iota_0(B)$, we further have $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(\iota_0(A), \iota_0(B)) \simeq \mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(\iota_0(A), \iota_0(B))$. We check that the last space is isomorphic to the space of morphisms in the derived category. A morphism $i_0(A) \to i_0(B)$ is

$$\begin{array}{ccccc} & & C & & \\ & \nearrow^{f} & & \nwarrow^{g}_{\simeq} & \\ \iota_0(A) & & & & \iota_0(B) \end{array}$$

where g is a quasi-isomorphism. As $\iota_0(A) \to C$ is a morphism of complexes, the image of $A \to C^0$ is contained in the kernel of d^0. Hence, we can assume that $C^i = 0$, for $i > 0$. As g is a quasi-isomorphism, we know that coker $d^{-1} \simeq B$. The canonical map $\pi\colon C^0 \to \text{coker } d^{-1}$ yields a commutative diagram

$$\begin{array}{ccccc} & & C & & \\ & \nearrow^{f} & \downarrow{\scriptstyle \pi} & \nwarrow^{g} & \\ \iota_0(A) & \xrightarrow{\pi f} & \iota_0(B) & \xleftarrow{\mathrm{Id}} & \iota_0(B) \end{array}$$

which proves that ι_0 is full.

Let now $f, g\colon A \to B$ be two morphisms whose images in $\mathcal{D}(\mathcal{A})$ are equal, i.e. we have a commutative diagram

$$\begin{array}{ccccc} & & \iota_0(B) & & \\ & \nearrow^{f} & \downarrow{\scriptstyle \alpha} & \nwarrow^{\mathrm{Id}} & \\ \iota_0(A) & \xrightarrow{h} & C & \xleftarrow[\alpha]{} & \iota_0(B) \\ & \searrow_{g} & \uparrow{\scriptstyle \alpha} & \swarrow_{\mathrm{Id}} & \\ & & \iota_0(B) & & \end{array}$$

As α is an isomorphism, as before, we can assume that $C^i = 0$, for $i > 0$. Then the composition $B \xrightarrow{\alpha} C^0 \xrightarrow{\pi} \text{coker } d^{-1}$ is as isomorphism. Hence, $\alpha f = \alpha g$ implies $f = \pi\alpha f = \pi\alpha g = g$, i.e. ι_0 is faithful.

It remains to show that ι_0 is essentially surjective. Let $A^\bullet$ be a complex such that $H^i(A^\bullet) = 0$, for $i \neq 0$. Morphism of complexes

$$\begin{array}{ccccccccc} \dots & \longrightarrow & A^{-1} & \xrightarrow{d^{-1}} & A^0 & \xrightarrow{d^0} & A^1 & \longrightarrow & \dots \\ & & {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & A^{-1} & \longrightarrow & \ker d^0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

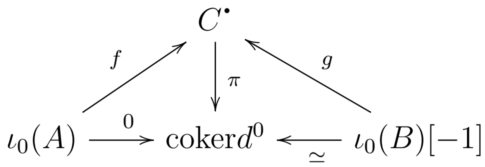
is a quasi-isomorphism, hence we can assume that $A^i = 0$, for $i > 0$. Similarly,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \text{coker } d^{-1} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

is a quasi-isomorphism, hence $A^\bullet$ is isomorphic (in $\mathcal{D}(\mathcal{A})$) to a complex $i_0(H^0(A^\bullet))$. $\square$

Lemma 4.3. *Let $\mathcal{A}$ be an abelian category. Consider functor $\iota_0\colon \mathcal{A} \to \mathcal{D}(\mathcal{A})$ as in (7). Then $Hom_{\mathcal{D}(\mathcal{A})}(i_0(A), i_0(B)[-1]) = 0$, for all $(A, B) \in \mathcal{A}$.*

Proof. A morphism in $\mathcal{D}(\mathcal{A})$ is of the form



with a quasi-isomorphism g. We can assume that $C^i = 0$, for $i > 1$. Then the map $B \to C^1 \to$ coker d^0 is an isomorphism, which proves that (f, C, g) is equivalent to $(0, B, \mathrm{Id})$, i.e. $\mathrm{Hom}(\iota_0(A), \iota_0(B)[-1]) = 0$. $\square$

Proposition 4.4. *Let $\mathcal{A}$ be an abelian category and consider functor $\iota_0\colon \mathcal{A} \to \mathcal{D}(\mathcal{A})$ as above. Then a complex $A \to B \to C$ of objects of $\mathcal{A}$ is a short exact sequence if and only if $\iota_0(A) \to \iota_0(B) \to \iota_0(C) \to \iota_0(A)[1]$ is a distinguished triangle.*

Proof. With Proposition 3.26 we have proved one implication, that a short exact sequence yields a distinguished triangle.

Let now $A \xrightarrow{f} B \xrightarrow{g} C$ be a complex in $\mathcal{A}$ such that $\iota_0(A) \to \iota_0(B) \to \iota_0(C) \to \iota_0(A)[1]$ is a distinguished triangle. We show that f is the kernel of g and g the cokernel of f.

Let $D \in \mathcal{A}$ be any object and $\varphi\colon D \to B$ a morphism such that $g\varphi = 0$. Then, as ι_0 is fully faithful, $\iota_0(g) \circ \iota_0(\varphi) = 0$. Exact sequence

$$\mathrm{Hom}(\iota_0(D), \iota_0(C)[-1]) \to \mathrm{Hom}(\iota_0(D), \iota_0(A)) \to \mathrm{Hom}(\iota_0(D), \iota_0(B)) \to \mathrm{Hom}(\iota_0(D), \iota_0(C))$$

together with vanishing of $\mathrm{Hom}(\iota_0(D), \iota_0(C)[-1])$ implies that there exists unique ψ such that $f \circ \psi = \varphi$. It follows that f is the kernel of g.

The proof that g is the cokernel of f is analogous. $\square$

It follows that $\mathcal{A} \subset \mathcal{D}(\mathcal{A})$ is a full subcategory and we can read off the short exact sequences in $\mathcal{A}$ from the triangulation of $\mathcal{D}(\mathcal{A})$.

Proposition 4.5. *Any complex $A^\bullet \in \mathcal{D}(\mathcal{A})$ fits into a distinguished triangle*

$$\tau_{\leq 0}A^\bullet \to A^\bullet \to \tau_{\geq 1}A^\bullet \to \tau_{\leq 0}A^\bullet[1]$$

with

$$H^i(\tau_{\leq 0}A^\bullet) = \begin{cases} H^i(A) & \text{for } i \leq 0 \\ 0 & \text{for } i \geq 1. \end{cases} \qquad H^i(\tau_{\geq 1}A^\bullet) = \begin{cases} H^i(A) & \text{for } i \geq 1 \\ 0 & \text{for } i \leq 0. \end{cases}$$

Proof. We define $\tau_{\leq 0}A^\bullet$ as the bottom row of the diagram of a morphism of complexes

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & A^{-1} & \xrightarrow{d^{-1}} & A^0 & \xrightarrow{d^0} & A^1 & \longrightarrow & \dots \\
 & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & A^{-1} & \longrightarrow & \ker d^0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

We define $\tau_{\geq 1}A^\bullet$ as the top row of the diagram of a morphism of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \operatorname{coker} d^0 & \longrightarrow & A^2 & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & 0
\end{array}$$

The conditions on cohomology are clearly satisfied.

Next we prove that these two maps fit into a distinguished triangle. By Proposition 3.27 the cone of the map $\tau_{\leq 0}A^\bullet \to A^\bullet$ is a complex $0 \to A^0/\ker d^0 \to A^1 \to \dots$. Then the cokernel of d^0 is isomorphic to the cokernel of $A^0/\ker d^0 \to A^1$, hence the cone is quasi-isomorphic to $\tau_{\geq 1}A^\bullet$. $\square$

Analogously as Lemma 4.3 one proves

Lemma 4.6. *Let $\mathcal{A}$ be an abelian category. Consider complexes $A^\bullet, B^\bullet \in \mathcal{D}(\mathcal{A})$ such that $H^i(A^\bullet) = 0$ for $i \geq 1$ and $H^j(B^\bullet) = 0$, for $j \leq 0$. Then $Hom_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet) = 0$.*

We define

$$\mathcal{D}(\mathcal{A})^{\leq 0} = \{A^\bullet \mid H^i(A) = 0 \text{ for } i \geq 1\} \tag{8}$$

$$\mathcal{D}(\mathcal{A})^{\geq 1} = \{A^\bullet \mid H^j(A) = 0 \text{ for } j \leq 0\}. \tag{9}$$

Theorem 4.7. *Consider an abelian category $\mathcal{A}$.*

- (t1) *The subcategory $\mathcal{D}(\mathcal{A})^{\leq 0}$ is closed under the shift by one, $\mathcal{D}(\mathcal{A})^{\leq 0}[1] \subset \mathcal{D}(\mathcal{A})^{\leq 0}$. The subcategory $\mathcal{D}(\mathcal{A})^{\geq 1}$ is closed under the shift by minus one, $\mathcal{D}(\mathcal{A})^{\geq 1}[-1] \subset \mathcal{D}(\mathcal{A})^{\geq 1}$.*
- (t2) *For any $A^\bullet \in \mathcal{D}(\mathcal{A})^{\leq 0}$ and $B^\bullet \in \mathcal{D}(\mathcal{B})^{\geq 1}$, $Hom_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet) = 0$.*
- (t3) *Any $A^\bullet$ fits into a distinguished triangle $\tau_{\leq 0}A^\bullet \to A^\bullet \to \tau_{\geq 1}A^\bullet \to \tau_{\leq 0}A^\bullet[1]$, with $\tau_{\leq 0}A^\bullet \in \mathcal{D}(\mathcal{A})^{\leq 0}$, $\tau_{\geq 1}A^\bullet \in \mathcal{D}(\mathcal{A})^{\geq 1}$.*

Proof. As the shift is a shift to the left, $H^i(A^\bullet[1]) = H^{i+1}(A^\bullet)$. If $H^i(A^\bullet) = 0$, for $i \geq 1$, then $H^i(A^\bullet[1]) = 0$, for $i \geq 0$.

Property $(t2)$ follows from Lemma 4.6. Property $(t3)$ follows from Proposition 4.5. $\square$

4.2. **Definition and first properties.**

A t-structure on a triangulated category $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ which satisfy (t1)-(t3) above. The t-structure describe in Theorem 4.7 is the *standard t-structure* on $\mathcal{D}(\mathcal{A})$.

For a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$, we define

$$\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n], \qquad \mathcal{D}^{\geq n} = \mathcal{D}^{\geq 1}[-n+1], \qquad \mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}.$$

The category $\mathcal{A}$ is the *heart* of the t-structure.

Lemma 4.8. *Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ be a t-structure on a triangulated category $\mathcal{D}$. Then $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n+1})$ is a t-structure on $\mathcal{D}$, for any $n \in \mathbb{Z}$.*

Proof. Follows immediately from the fact that $[-n]$ is an equivalence of $\mathcal{D}$. □

Lemma 4.9. *Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ be a t-structure on a triangulated category $\mathcal{D}$. Then the triangle in (t3) is unique in $A \in \mathcal{D}$. Moreover, $\tau_{\leq 0}$ defines a functor $\mathcal{D} \to \mathcal{D}^{\leq 0}$, left adjoint to the inclusion $\mathcal{D}^{\leq 0} \to \mathcal{D}$ and $\tau_{\geq 1}$ defines a functor $\mathcal{D} \to \mathcal{D}^{\geq 1}$ right adjoint to the inclusion $\mathcal{D}^{\geq 1} \to \mathcal{D}$.*

Exercise 4.10. Prove the Lemma.

Using appropriate shifts we define functors $\tau_{\leq n} \colon \mathcal{D} \to \mathcal{D}^{\leq n}$ and $\tau_{\geq n} \colon \mathcal{D} \to \mathcal{D}^{\geq n}$.

Example: Let $\mathcal{A}$ be the category of right modules over a quiver

$$1 \longrightarrow 2$$

Objects of $\mathcal{A}$ are vector spaces V_2, V_1 and a morphism $V_2 \to V_1$. We denote it by $(V_2 \xrightarrow{f} V_0)$. Let

$$\mathcal{T} = \{(0 \xrightarrow{0} V_1)\}, \qquad \mathcal{F} = \{(V_2 \xrightarrow{0} 0)\}$$

be full subcategories of $\mathcal{A}$ with modules supported on one of the vertex. Then $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and any $(V_2 \xrightarrow{f} V_0) \in \mathcal{A}$ fits into a short exact sequence

$$0 \to (0 \xrightarrow{0} V_1) \to (V_2 \xrightarrow{f} V_1) \to (V_2 \xrightarrow{0} 0) \to 0.$$

The subcategories $(\mathcal{T}, \mathcal{F})$ yield an example of a *torsion pair*, i.e. a pair of subcategories $\mathcal{T}$, $\mathcal{F}$ of an abelian category $\mathcal{A}$ such that $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and any $A \in \mathcal{A}$ fits into a short exact sequence $0 \to T \to A \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Put $\mathcal{D} = \mathcal{D}(\mathcal{A})$ and define

$$(10) \qquad {}^p\mathcal{D}^{\leq 0} = \{A^\bullet \in \mathcal{D} \,|\, H^0(A^\bullet) \in \mathcal{T}, H^i(A^\bullet) = 0 \text{ for } i \geq 1\},$$

$$(11) \qquad {}^p\mathcal{D}^{\geq 1} = \{A^\bullet \in \mathcal{D} \,|\, H^0(A^\bullet) \in \mathcal{F}, H^j(A^\bullet) = 0 \text{ for } j \leq -1\}.$$

Proposition 4.11. *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair on an abelian category* $\mathcal{A}$. *Then* $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 1})$ *is a torsion pair on the category* $\mathcal{D}$. *It is the* tilt *of the standard* t*-structure in the torsion pair* $(\mathcal{T}, \mathcal{F})$.

Proof. Condition (t1) is clearly satisfied. Let now $A \in {}^p\mathcal{D}^{\leq 0}$ and $B \in {}^p\mathcal{D}^{\geq 1}$. We can decompose A with respect to an appropriate shift of the standard t-structure:

$$\tau_{\leq -1}A \to A \to \tau_{\geq 0}A \to \tau_{\leq -1}A[1].$$

Then in the long exact sequence

$$\mathrm{Hom}(\tau_{\leq -1}A[1], B) \to \mathrm{Hom}(\tau_{\geq 0}A, B) \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(\tau_{\leq -1}A, B)$$

the first an the last groups vanish (because $\tau_{\leq -1}A[1] \in \mathcal{D}^{\leq -1}$) so $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(\tau_{\geq 0}A, B)$. Similarly, we have the decomposition of B:

$$\tau_{\leq 0}B \to B \to \tau_{\geq 1}B \to \tau_{\leq 0}B[1].$$

Again, $\tau_{\geq 0}A \in \mathcal{D}^{\leq 0}$ so in the exact sequence

$$\mathrm{Hom}(\tau_{\geq 0}A, \tau_{\leq 0}B[1]) \to \mathrm{Hom}(\tau_{\geq 0}A, \tau_{\leq 0}B) \to \mathrm{Hom}(\tau_{\geq 0}A, B) \to \mathrm{Hom}(\tau_{\geq 0}A, \tau_{\geq 1}B)$$

the first and the last group vanish. It follows that $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(\tau_{\geq 0}A, \tau_{\leq 0}A) = \mathrm{Hom}(H^0(A), H^0(B)) = 0$ (we use Propositions 4.2 and 4.5 and the fact that to know that cohomology objects of the truncations of A and B and to know that we can calculate morphisms in the category $\mathcal{A}$).

It remains to prove (t3). Let $A^\bullet$ be a complex in $\mathcal{D}$. We have a distinguished triangle

$$\tau_{\leq -1}A \to \tau_{\leq 0}A \to H^0(A)$$

and a decomposition of $H^0(A)$ with respect to the torsion pair $0 \to T \to H^0(A) \to F \to 0$. By Proposition 3.26 we can view the latter as a distinguished triangle too.

Consider diagrams with distinguished rows and columns:

$$\begin{array}{ccccc}
T & \longrightarrow & \tau_{\leq -1}A[1] & \longrightarrow & {}^p\tau_{\leq 0}A[1] \\
\uparrow{\scriptstyle \mathrm{Id}} & & \uparrow & & \uparrow \\
T & \longrightarrow & H^0(A) & \longrightarrow & F \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \tau_{\leq 0}A & \longrightarrow & \tau_{\leq 0}A
\end{array}$$

$$\begin{array}{ccccc} 0 & \longrightarrow & \tau_{\geq 1}A & \longrightarrow & \tau_{\geq 1}A \\ \uparrow & & \uparrow & & \uparrow \\ {}^p\tau_{\leq 0}A & \longrightarrow & A & \longrightarrow & {}^p\tau_{\geq 1}A \\ \mathrm{Id}\uparrow & & \uparrow & & \uparrow \\ {}^p\tau_{\leq 0}A & \longrightarrow & \tau_{\leq 0}A & \longrightarrow & F \end{array}$$

We claim that the middle row of the second diagram is the triangle we were looking for. We use the top row of the first diagram to calculate cohomology of ${}^p\tau_{\leq 0}A$. We get that $H^0({}^p\tau_{\leq 0}A) = T$ and higher cohomology vanish. The right column of the right diagram implies that $H^0({}^p\tau_{\geq 1}A) = F$ and all lower cohomology vanish. □

The heart $\mathcal{A}^\sharp$ of the new t-structure is

$$\mathcal{A}^\sharp = \{A^\bullet \in \mathcal{D} \,|\, H^{-1}(A^\bullet) \in \mathcal{F}, H^0(A^\bullet) \in \mathcal{T}, H^i(A^\bullet) = 0 \text{ for } i \neq -1, 0\}.$$

Let B be an object of $\mathcal{A}^\sharp$. The decomposition of B with respect to the standard t-structure is

$$H^{-1}(B)[1] \to B \to H^0(B) \to H^{-1}(B)[2],$$

hence B is determined by a class in $\mathrm{Ext}^2(H^0(B), H^{-1}B)$.

In the example we considered $\mathrm{Ext}^2((0 \to V_1), (V_2 \to 0)) = 0$, hence any $B \in \mathcal{A}^\sharp$ is isomorphic to $H^{-1}(B)[1] \oplus H^0(B)$. It follows that $\mathcal{A}^\sharp \simeq \text{mod–}k \oplus k$.

In fact, the heart of a t-structure is always an abelian category. However, the derived category of the heart is not necessarily equivalent to the original triangulated category. (To see it on the example we note that any distinguished triangle in $\mathcal{D}(k \oplus k)$ splits while in the original category it is not the case).

Before we prove this fact we discuss the cohomology functors associated with a t-structure. In the following I use the presentation in [HTT08, Chapter 8].

Lemma 4.12. *The following conditions on $D \in \mathcal{D}$ are equivalent*

(1) $D \in \mathcal{D}^{\leq n}$ (resp. $D \in \mathcal{D}^{\geq n}$)
(2) The canonical morphism $\tau_{\leq n}D \to D$, (resp. $D \to \tau_{\geq n}D$) is an isomorphism
(3) $\tau_{>n}D = 0$ (resp. $\tau_{<n}D = 0$).

Lemma 4.13. *Let $D' \to D \to D''$ be a distinguished triangle in $\mathcal{D}$. If $D', D'' \in \mathcal{D}^{\leq 0}$ then $D \in \mathcal{D}^{\leq 0}$. If $D', D'' \in \mathcal{D}^{\geq 0}$ then $D \in \mathcal{D}^{\geq 0}$.*

Proof. We prove that $\tau_{>0}D = 0$. To do it, we show that $\mathrm{Hom}(D, E) = 0$ for any $E \in \mathcal{D}^{>0}$. It follows from the exact sequence $\mathrm{Hom}(D'', E) \to \mathrm{Hom}(D, E) \to \mathrm{Hom}(D', E)$. □

Proposition 4.14. *Let a, b be two integers.*

(1) If $b \geq a$ *then* $\tau_{\leq b} \circ \tau_{\leq a} \simeq \tau_{\leq a} \circ \tau_{\leq b} \simeq \tau_{\leq a}$, $\tau_{\geq b} \circ \tau_{\geq a} \simeq \tau_{\geq a} \circ \tau_{\geq b} \simeq \tau_{\geq b}$.
(2) If $a > b$ *then* $\tau_{\leq b} \circ \tau_{\geq a} \simeq \tau_{\geq a} \circ \tau_{\leq b} \simeq 0$.
(3) $\tau_{\geq a} \circ \tau_{\leq b} \simeq \tau_{\leq b} \circ \tau_{\geq a}$.

Proof. For $b \geq a$ canonical morphism $\tau_{\leq b}\tau_{\leq a} \to \tau_{\leq a}$ is an isomorphism as $\tau_{>b}\tau_{\leq a} = 0$. $\tau_{\leq a} \circ \tau_{\leq b} \simeq \tau_{\leq a}$ as the composition of two adjoint functors is the adjoint to the composition. The remaining isomorphisms are proved similarly.

For $a > b$ part (2) is clear.

In part (3) we might assume that $b \geq a$. Let $D \in \mathcal{D}$. We first construct morphism $\varphi \colon \tau_{\geq a}\tau_{\leq b}D \to \tau_{\leq b}\tau_{\geq a}D$. We have a distinguished triangle

$$\tau_{\leq b}\tau_{\geq a}D \to \tau_{\geq a}D \to \tau_{>b}\tau_{\geq a}D \simeq \tau_{>b}D$$

It follows that $\tau_{\leq b}\tau_{\geq a}D \in \mathcal{D}^{\geq a}$ (see the previous lemma). Then we get isomorphisms $\mathrm{Hom}(\tau_{\leq b}D, \tau_{\geq a}D) \simeq \mathrm{Hom}(\tau_{\leq b}D, \tau_{\leq b}\tau_{\geq a}D) \simeq \mathrm{Hom}(\tau_{\geq a}\tau_{\leq b}D, \tau_{\leq b}\tau_{\geq a}D)$. We consider φ which corresponds to $\tau_{\leq b}D \to D \to \tau_{\geq a}D$.

We have a distinguished triangle

$$\tau_{<a}D \simeq \tau_{<a}\tau_{\leq b}D \to \tau_{\leq b}D \to \tau_{\geq a}\tau_{\leq b}D$$

which implies that $\tau_{\geq a}\tau_{\leq b}D \in \mathcal{D}^{\leq b}$. Consider octahedron

$$\begin{array}{ccccc}
0 & \longrightarrow & \tau_{>b}D & \longrightarrow & \tau_{>b}D \\
\uparrow & & \uparrow & & \uparrow \\
\tau_{<a}D & \longrightarrow & D & \longrightarrow & \tau_{\geq a}D \\
\uparrow & & \uparrow & & \uparrow \\
\tau_{<a}D & \longrightarrow & \tau_{\leq b}D & \longrightarrow & \tau_{\geq a}\tau_{\leq b}D
\end{array}$$

In the distinguished triangle

$$\tau_{\geq a}\tau_{\leq b}D \to \tau_{\geq a}D \to \tau_{>b}D$$

the first term lies in $\mathcal{D}^{\leq b}$ and the last in $\mathcal{D}^{>b}$. Hence, it is the canonical decomposition of $\tau_{\leq a}D$, i.e.$\tau_{\geq a}\tau_{\leq b}D \simeq \tau_{\leq b}\tau_{\geq a}D$. □

Proposition 4.15. *Let* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ *be a t-structure on a triangulated category* $\mathcal{D}$. *The heart* $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ *is an abelian category.*

Proof. Let $A, B \in \mathcal{A}$. Then $A \oplus B \in \mathcal{A}$ as $\tau_{<0}$ and $\tau_{>0}$ vanish.

Let now $f \colon A \to B$ be a morphism in $\mathcal{A}$. It fits into a distinguished triangle $A \to B \to C$. We shall show that the kernel of f is $H^1(C) = \tau_{\leq -1}C[1]$ and the cokernel is $\tau_{\geq 0}C$. For any D in $\mathcal{A}$ we have exact sequences

$$0 \simeq \operatorname{Hom}(A[1], D) \to \operatorname{Hom}(C, D) \to \operatorname{Hom}(B, D) \to \operatorname{Hom}(A, D)$$
$$0 \simeq \operatorname{Hom}(D, B[-1]) \to \operatorname{Hom}(D, C[-1]) \to \operatorname{Hom}(D, A) \to \operatorname{Hom}(D, B)$$

where the isomorphisms with zero follow from $\operatorname{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$. Moreover, $\operatorname{Hom}(C, D) \simeq \operatorname{Hom}(\tau_{\geq 0}C, D)$ and $\operatorname{Hom}(D, C[-1]) \simeq \operatorname{Hom}(D[1], C) \simeq \operatorname{Hom}(D[1], \tau_{\leq -1}C)$, as functors $\tau_{\geq 0}$ and $\tau_{\leq -1}$ are adjoint to appropriate inclusions.

It remains to show that the $\operatorname{Coim} f \to \Im f$ is an isomorphism. Octahedron

$$\begin{array}{ccccc}
0 & \longrightarrow & H^{-1}C[2] & \longrightarrow & H^{-1}(C)[2] \\
\uparrow & & \uparrow & & \uparrow \\
B & \longrightarrow & H^0(C) & \longrightarrow & I[1] \\
\uparrow & & \uparrow & & \uparrow \\
B & \longrightarrow & C & \longrightarrow & A
\end{array}$$

gives distinguished triangles $I \to B \to H^0(C)$ and $H^{-1}(C) \to A \to I$ which imply that $I \in \mathcal{A}$. Hence, I is the kernel of $B \to H^0(C) = \operatorname{coker} f$ and I is the cokernel of $\ker f = H^{-1}(C) \to A$. □

Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ be a t-structure on a triangulated category $\mathcal{D}$ with heart $\mathcal{A}$. Define

$$H^0_{\mathcal{A}} = \tau_{\geq 0}\tau_{\leq 0} \colon \mathcal{D} \to \mathcal{A}.$$

Proposition 4.16. *Functor H^0 is cohomological.*

Proof. Consider a distinguished triangle $A \to B \to C \to A[1]$. We shall show that $H^0(A) \to H^0(B) \to H^0(C)$ is exact. We proceed in few steps.

- When $A, B, C \in \mathcal{D}^{\leq 0}$ then $H^0(A) \to H^0(B) \to H^0(C) \to 0$ is exact.

  Let D be an object of $\mathcal{A}$. We check the universal property of the cokernel and we use the fact that $\operatorname{Hom}(H^0(A), D) \simeq \operatorname{Hom}(A, \iota_0(D))$ (because there are no morphisms $\tau_{<0}A \to D$). We have an exact sequence

  $$\operatorname{Hom}(A[1], \iota_0 D) \to \operatorname{Hom}(C, \iota_0 D) \to \operatorname{Hom}(B, \iota_0 D) \to \operatorname{Hom}(A, \iota_0 D).$$

  As $A[1] \in \mathcal{D}^{<0}$ the first space is isomorphic to zero, hence $H^0(C)$ is the cokernel of $H^0(A) \to H^0(B)$.
- When $A \in \mathcal{D}^{\leq 0}$ then $H^0(A) \to H^0(B) \to H^0(C) \to 0$ is exact.

  First we claim that $\tau_{>0}B \simeq \tau_{>0}C$. Let $D \in \mathcal{D}^{>0}$. Then $\operatorname{Hom}(A, D) = 0$ and $\operatorname{Hom}(A[1], D) = 0$, so $\operatorname{Hom}(C, D) \to \operatorname{Hom}(B, D)$ are isomorphic. As $\tau_{>0}$ is left adjoint to the inclusion of $\mathcal{D}^{>0}$, we conclude that $\tau_{>0}B \simeq \tau_{>0}C$.

The map $A \to B$ factors via $A \to \tau_{\leq 0}B \to B$. From the octahedron

$$\begin{array}{ccccc} 0 & \longrightarrow & \tau_{>0}B & \longrightarrow & \tau_{>0}C \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & B & \longrightarrow & C \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & \tau_{\leq 0}B & \longrightarrow & \tau_{\leq 0}C \end{array}$$

we conclude that $A \to \tau_{\leq 0}B \to \tau_{\leq 0}C$ is a distinguished triangle, hence by the previous point $H^0(A) \to H^0(B) \to H^0(C) \to 0$ is exact.

- For $C \in \mathcal{D}^{\geq 0}$ sequence $0 \to H^0(A) \to H^0(B) \to H^0(C)$ is exact.

  It is proved analogously or by considering the opposite categories.
- The general case.

  Octahedrons

$$\begin{array}{ccccc} 0 & \longrightarrow & C & \longrightarrow & C \\ \uparrow & & \uparrow & & \uparrow \\ \tau_{\leq 0}A & \longrightarrow & B & \longrightarrow & Q \\ \uparrow & & \uparrow & & \uparrow \\ \tau_{\leq 0}A & \longrightarrow & A & \longrightarrow & \tau_{>0}A \end{array} \qquad \begin{array}{ccccc} \tau_{>0}A & \longrightarrow & 0 & \longrightarrow & \tau_{>0}A[1] \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & B & \longrightarrow & C \\ \uparrow & & \uparrow & & \uparrow \\ \tau_{\leq 0}A & \longrightarrow & B & \longrightarrow & Q \end{array}$$

give triangles $\tau_{\leq 0}A \to B \to Q$ and $Q \to C \to \tau_{>0}A[1]$. By previous points

$$H^0(A) \to H^0(B) \to H^0(Q) \to 0, \qquad 0 \to H^0(Q) \to H^0(C) \to H^1(A)$$

are exact, hence $H^0(A) \to H^0(B) \to H^0(C)$ is exact.

□

We say that a t-structure is *bounded* if it is *non-degenerate*, i.e. $\bigcap_n \mathcal{D}^{\leq n} = 0 = \bigcap_n \mathcal{D}^{\geq n}$ and for any $D \in \mathcal{D}$ all but finitely many cohomology objects $H(D)$ vanish.

Theorem 4.17. *Let $\mathcal{A}$ be the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, $F\colon \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}$ a t-exact functor. The functor F is an equivalence of categories if and only if $Ext^*_{\mathcal{D}}$ is generated by $Ext^1_{\mathcal{D}}$, i.e. for any $X, Y \in \mathcal{A}$ any class in $Ext^n(X,Y) = Hom(X, Y[n])$ is a linear combination of monomials $\beta_1 \dots \beta_n$ with $\beta_j \in Ext^1_{\mathcal{D}}(X_j, X_{j+1})$, for someq $X_j \in \mathcal{A}$.*

Let $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$, $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$ be t-structures on triangulated categories $\mathcal{D}_1$, $\mathcal{D}_2$. An exact functor $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ (i.e. $F(-[1]) \simeq F(-)[1]$, F maps distinguished triangles to distinguished triangles) is *left t-exact* if $F(\mathcal{D}_1^{\geq 1}) \subset \mathcal{D}_2^{\geq 1}$. F is *right t-exact*, if $F(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\geq 0}$ and F is *t-exact* if it is both left and right t-exact.

Exercise 4.18. Let $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ be left adjoint to $G\colon \mathcal{D}_2 \to \mathcal{D}_1$. Assume that G is right t-exact, for some t-structures on $\mathcal{D}_1$ and $\mathcal{D}_2$. Prove that F is left t-exact.

4.3. **Semi-orthogonal decompositions and recollements.**

A *semi-orthogonal decomposition* $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ of a triangulated category $\mathcal{D}$ is a pair of full triangulated subcategories of $\mathcal{D}$ such that $(\mathcal{D}_2, \mathcal{D}_1)$ is a t-structure on $\mathcal{D}$.

Let us write down explicitly what it means. Category $\mathcal{D}_2$ is $\mathcal{D}^{\leq n}$, for all n, while $\mathcal{D}_1 = \mathcal{D}^{\geq n}$, for all n. In particular, $\mathrm{Hom}(D_2, D_1) = 0$ and, any object $D \in \mathcal{D}$ fits into a distinguished triangle

$$D_2 \to D \to D_1 \to D_2[1]$$

with $D_1 \in \mathcal{D}_1$, $D_2 \in \mathcal{D}_2$. Moreover, $j_*\colon \mathcal{D}_1 \to \mathcal{D}$ has left adjoint $j^*\colon \mathcal{D} \to \mathcal{D}_1$ and $i_*\colon \mathcal{D}_2 \to \mathcal{D}$ has right adjoint $i^!\colon \mathcal{D} \to \mathcal{D}_2$.

A full subcategory $\mathcal{D}'$ of a triangulated category $\mathcal{D}$ is *right admissible* if the inclusion functor $i_*\colon \mathcal{D}' \to \mathcal{D}$ has right adjoint $i^!\colon \mathcal{D} \to \mathcal{D}'$.

A full subcategory $\mathcal{D}'$ of a triangulated category $\mathcal{D}$ is *left admissible* if the inclusion functor $j_*\colon \mathcal{D}' \to \mathcal{D}$ has left adjoint $j^*\colon \mathcal{D} \to \mathcal{D}'$.

Proposition 4.19. [Bon89] *Let $i_*\colon \mathcal{D}' \to \mathcal{D}$ be the inclusion of a full triangulated subcategory. The following conditions are equivalent:*

1. *Category $\mathcal{D}'$ is right admissible*
2. *Category $\mathcal{D}$ admits a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}' \rangle$ where $\mathcal{D}_1 = \{D \in \mathcal{D} \,|\, Hom(\mathcal{D}', D) = 0\}$.*

Proof. Assume that $\mathcal{D}'$ is right admissible. We check that conditions (t1)-(t3) are satisfied. The first two are clear. For $D \in \mathcal{D}$ let D_1 be such that

$$D_1 \to D \to i_* i^* D \to D_1[1]$$

is distinguished, where $D \to i_* i^* D$ is the adjunction unit. Then $D_1 \in \mathcal{D}_1$, hence (t3) holds.

In the opposite direction, if $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}' \rangle$ then $\mathcal{D}'$ is right admissible. □

We say that a full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is *admissible* if it is left and right admissible.

Example 4.20. Let $E \in \mathcal{D}$ be an exceptional object in a k-linear Ext-finite (i.e. $\dim_k \bigoplus_i \mathrm{Hom}(D_1, D_2[i]) < \infty$ for any pair D_1, D_2 of objects of $\mathcal{D}$) triangulated category $\mathcal{D}$, i.e. assume that $\mathrm{Hom}(E,E) = k$ and $\mathrm{Hom}(E, E[i]) = 0$, for all $i \neq 0$. Then functor $i_*\colon \mathcal{D}^b(k) \to \mathcal{D}$ such that $i_*(k) = E$ is fully faithful and has left and right adjionts:

$$i^!(D) = \bigoplus_i \mathrm{Hom}(E, D[i])[-i] \qquad\qquad i^*(D) = \bigoplus_i \mathrm{Hom}(D, E[i])^\vee[i].$$

Example 4.21. Let X be a regular projective scheme. Then $\mathcal{D}^b(\mathrm{Coh}(X))$ is a k-linear Ext-finite triangulated category. Assume that $H^i(\mathcal{O}_X) = 0$, for $i > 0$. Let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L} \in \mathcal{D}^b(\mathrm{Coh}(X))$ is an exceptional object.

An admissible subcategory $i_*\colon \mathcal{D}_2 \to \mathcal{D}$ yields two semi-orthogonal decompositions

$$\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle = \langle \mathcal{D}_2, \mathcal{D}_3 \rangle.$$

Exercise 4.22. Let $k_*\colon \mathcal{D}_3 \to \mathcal{D}$ be the inclusion functor and $k^!$ its right adjoint. Prove that *the mutation functor* $k^! j_*\colon \mathcal{D}_1 \to \mathcal{D}_3$ is an equivalence with quasi-inverse $j^* k_*\colon \mathcal{D}_3 \to \mathcal{D}_1$.

Exercise 4.23. For semi-orthogonal decompositions $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle = \langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ show that functor $i_1^*\colon \mathcal{D} \to \mathcal{D}_1$ left adjoint to the inclusion, is right adjoint to the inlusion of $\mathcal{D}_3 \to \mathcal{D}$.

Classically, a admissible subcategory is known as *recollement*, i.e. triangulated categories $\mathcal{D}_1$, $\mathcal{D}$, $\mathcal{D}_2$ together with 6 functors:

$$\mathcal{D}_1 \;\begin{smallmatrix} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{smallmatrix}\; \mathcal{D} \;\begin{smallmatrix} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{smallmatrix}\; \mathcal{D}_2 \tag{12}$$

such that

(1) Functors $i^* \dashv i_* \dashv i^!$, $j_! \dashv j^* \dashv j_*$ are adjoint and i_*, $j_!$, j_* are fully faithful,
(2) $i_*\mathcal{D}_1$ is the kernel of j^*
(3) Every object $D \in \mathcal{D}$ fits into distinguished triangles

$$i_*i^!D \to D \to j_*j^*D \to i_*i^!D[1], \qquad j_!j^*D \to D \to i_*i^*D \to j_!j^*D[1].$$

Exercise 4.24. Prove that the data of a recollement is equivalent to the data of an admissible subcategory $i_*\mathcal{D}_1$.

Exercise 4.25. Given a recollement (12), prove that $\mathcal{D}_2$ is the quotient of $\mathcal{D}$ by $\mathcal{D}_1$.

4.4. **Gluing of t-structures.**

Consider a recollement (12) and t-structures $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$, $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$.

Theorem 4.26. *There exists a unique t-structure on $\mathcal{D}$ such that functors i_*, j^* are t-exact. The t-structure is defined as*

$$\begin{aligned}\mathcal{D}^{\leq 0} &= \{D \in \mathcal{D} \mid j^*D \in \mathcal{D}_2^{\leq 0}, i^*D \in \mathcal{D}_1^{\leq 0}\},\\ \mathcal{D}^{\geq 1} &= \{D \in \mathcal{D} \mid j^*D \in \mathcal{D}_2^{\geq 1}, i^!D \in \mathcal{D}_1^{\geq 1}\}.\end{aligned}$$

Proof. Condition (t1) is clearly satisfied. Let now $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$. We consider the following diagram with exact rows and columns

$$\begin{array}{ccccc}
\mathrm{Hom}(j_!j^*X, i_*i^!Y) & \longrightarrow & \mathrm{Hom}(j_!j^*X, Y) & \longrightarrow & \mathrm{Hom}(j_!j^*X, j_*j^*Y)\\
\uparrow & & \uparrow & & \uparrow\\
\mathrm{Hom}(X, i_*i^!Y) & \longrightarrow & \mathrm{Hom}(X, Y) & \longrightarrow & \mathrm{Hom}(X, j_*j^*Y)\\
\uparrow & & \uparrow & & \uparrow\\
\mathrm{Hom}(i_*i^*X, i_*i^!Y) & \longrightarrow & \mathrm{Hom}(i_*i^*X, Y) & \longrightarrow & \mathrm{Hom}(i_*i^*X, j_*j^*Y)
\end{array}$$

given by distinguished triangles $j_!j^*X \to X \to i_*i^*X \to j_!j^*X[1]$, $i_*i^!Y \to Y \to j_*j^*Y$.

The bottom left space in the diagram vanishes because $i^*X \in \mathcal{D}_1^{\leq 0}$, $i^!Y \in \mathcal{D}_1^{\geq 1}$ and i_* is fully faithful.

The top left space is zero because $j^*i_* = 0$. For the same reason the bottom right space vanishes. Finally, the top right is isomorphic to $\mathrm{Hom}(j^*X, j^*Y)$, as $j^*j_* \simeq \mathrm{Id}$, hence it is also zero because $j^*X \in \mathcal{D}_2^{\leq 0}$, $j^*Y \in \mathcal{D}_2^{\geq 1}$.

It follows that $\mathrm{Hom}(X, i_*i^!Y) = 0 = \mathrm{Hom}(X, j_*j^*Y)$, hence $\mathrm{Hom}(X, Y) = 0$ and (t2) holds.

Now, let X be any object of $\mathcal{D}$. Consider octahedrons:

$$\begin{array}{ccccc}
Y & \longrightarrow & X & \longrightarrow & j_*\tau_{\geq 1}j^*X\\
\uparrow & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow\\
i_*i^!X & \longrightarrow & X & \longrightarrow & j_*j^*X\\
\uparrow & & \uparrow & & \uparrow\\
j_*\tau_{\leq 0}j^*X[-1] & \longrightarrow & 0 & \longrightarrow & j_*\tau_{\leq 0}j^*D
\end{array}$$

$$\begin{array}{ccccc}
A & \longrightarrow & Y & \longrightarrow & i_*\tau_{\geq 1}i^*Y \\
\uparrow & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow \\
j_!j^*Y & \longrightarrow & Y & \longrightarrow & i_*i^*Y \\
\uparrow & & \uparrow & & \uparrow \\
i_*\tau_{\leq 0}i^*Y[-1] & \longrightarrow & 0 & \longrightarrow & i_*\tau_{\leq 0}i^*Y
\end{array}$$

$$\begin{array}{ccccc}
i_*\tau_{\geq 1}i^*Y & \longrightarrow & B & \longrightarrow & j_*\tau_{\geq 1}j^*X \\
\uparrow & & \uparrow & & \uparrow \\
Y & \longrightarrow & X & \longrightarrow & j_*\tau_{\geq 1}j^*X \\
\uparrow & & \uparrow & & \uparrow \\
A & \xrightarrow{\mathrm{Id}} & A & \longrightarrow & 0
\end{array}$$

We claim that $A \to X \to B$ is a distinguished triangle with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

From the top row of the third diagram we conclude that $j^*B \simeq \tau_{\geq 1}j^*X \in \mathcal{D}_2^{\geq 1}$ (because $j^*i_* = 0$).

The top row of the second diagram implies that $j^*A \simeq j^*Y$, which, by the left column of the first diagram, is isomorphic to $\tau_{\leq 0}j^*X$.

As $j^*i_* = 0$ so are its right $i^!j_*$ and left $i^*j_!$ adjoints. The vanishing of $i^*j_!$ and the left column of the second diagram implies that $i^*A \simeq \tau_{\leq 0}i^*Y$. Finally, as $i^!j_* = 0$, the top row of the third diagram implies that $i^!B \simeq \tau_{\geq 1}i^*Y$. □

4.5. **Intermediate extension.**

Consider a recollement (12) and non-degenerate t-structures $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$, $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$ with hearts $\mathcal{A}_1$, respectively $\mathcal{A}_2$. Let $\mathcal{A} \subset \mathcal{D}$ be the heart of the glued t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$.

The six functors in (12) induce functors between abelian categories: $i_* \colon \mathcal{A}_1 \to \mathcal{A}$ and $j^* \colon \mathcal{A} \to \mathcal{A}_2$ are the restrictions to the hearts of the t-exact functors. ${}^p i^*$ and ${}^p i^!$ are defined as $\mathcal{H}^0 \circ i^*|_{\mathcal{A}}$ and $\mathcal{H}^0 \circ i^!|_{\mathcal{A}}$. Since i^* is right t-exact and $i^!$ is left t-exact, functor ${}^p i^*$ is right exact while ${}^p i^!$ is left exact. Functors ${}^p j_!, {}^p j_* \colon \mathcal{A}_2 \to \mathcal{A}$ are defined analogously, as $\mathcal{H}^0 \circ j_!|_{\mathcal{A}_2}$ and $\mathcal{H}^0 \circ j_*|_{\mathcal{A}_2}$. As before, ${}^p j_!$ is right t-exact and ${}^p j_*$ is left t-exact.

As functor $j_!$ is fully faithful, the adjunction unit $\mathrm{Id} \to j^*j_!$ is an isomorphism. Its inverse corresponds under the $j^* \dashv j_*$ adjunction to a morphism

$$j_! \to j_*.$$

By considering the 0'th cohomology we obtain a morphism

$$^pj_! \to {}^pj_*.$$

Its image defines the *intermediate extension*

$$j_{!*}\colon \mathcal{A}_2 \to \mathcal{A}.$$

5. Perverse sheaves

Let M be an n-dimensional oriented closed manifold. Then *Poincare duality* states that the k'th cohomology group of M is isomorphic to the $n-k$'th homology group of M:

$$H^k(M,\mathbb{Z}) \simeq H_{n-k}(M,\mathbb{Z}).$$

If, however, M is singular, Poincare duality does not hold. One needs to replace the cohomology with the intersection cohomology groups of a variety with coefficients in a local system.

Originally, the intersection cohomology was introduced to study the failure, due to presence of singularities, of Poincare duality. Later, it has been noticed that the intersection complexes of irreducible subvarieties of M are the simple *perverse sheaves.*

The category of perverse sheaves will be introduced as a heart of an appropriate t-structure.

We shall consider a topological space X and a sheaf $\mathcal{R}$ of rings with unity on X. We look at the category $\mathrm{Sh}_{\mathcal{R}}(X)$ of sheaves of $\mathcal{R}$-modules and its derived category $\mathcal{D}(\mathrm{Sh}_{\mathcal{R}}(X))$.

Under some additional conditions, given a closed subset $Z \subset X$ with open complement U, we shall construct a recollement

$$\mathcal{D}(\mathrm{Sh}_{\mathcal{R}}(Z)) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \mathcal{D}(\mathrm{Sh}_{\mathcal{R}}(X)) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \mathcal{D}(\mathrm{Sh}_{\mathcal{R}}(U))$$

To define functors in the recollement, we first discuss derived functors.

5.1. Derived functors.

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left (resp. right) exact functor between abelian categories. We define its extension $RF\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ (resp. $LF\colon \mathcal{D}^-(\mathcal{A}) \to \mathcal{D}^-(\mathcal{B})$) which is the *right* (resp. *left*) *derived functor of* F.

The functors RF, LF are *exact*, i.e. they map distinguished triangles to distinguished triangles.

If functor F is exact, construction of the derived functor is easy:

Proposition 5.1. *Assume that* $F\colon \mathcal{A} \to \mathcal{B}$ *is exact. Then, for* $* \in \{, -, +, b\}$*, functor* $\mathcal{H}^*(F)\colon \mathcal{H}^*(\mathcal{A}) \to \mathcal{H}^*(\mathcal{B})$ *maps acyclic complexes to acyclic complexes, hence it induces an exact functor* $\mathcal{D}^*(F)\colon \mathcal{D}^*(\mathcal{A}) \to \mathcal{D}^*(\mathcal{B})$.

The main idea of the construction of derived functors is to apply F componentwise to some selected representatives in equivalence classes of quasi-isomorphic complexes.

A class $\mathcal{R} \subset \mathcal{A}$ is *adapted* to a left (resp. right) exact functor F if it is stable under direct sums and satisfies

(1) F maps any acyclic (no cohomology in $\mathcal{A}$) complex from $\mathrm{Kom}^+(\mathcal{R})$ (resp. $\mathrm{Kom}^-(\mathcal{R})$) into an acyclic complex.
(2) Any object of $\mathcal{A}$ is a subobject (resp. a quotient object) of an object of $\mathcal{R}$.

Proposition 5.2. *Let* $\mathcal{R}$ *be a class of objects adapted to a left exact functor* $F\colon \mathcal{A} \to \mathcal{B}$ *and* $S_{\mathcal{R}}$ *the class of quasi-isomorphisms in* $K^+(\mathcal{R})$*. Then* $S_{\mathcal{R}}$ *is a localising class of morphisms in* $K^+(\mathcal{R})$ *and the canonical functor*

$$K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to \mathcal{D}^+(\mathcal{A})$$

is an equivalence.

We define the right derived functor RF on objects of $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ termwise. Fixing the inverse of the equivalence in the above proposition, we get a functor

$$RF\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B}).$$

To prove the independence of the choice of $\mathcal{R}$ we need to define the derived functor using a universal property.

Definition 5.3. The *derived functor* of an additive left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ is a pair consisting of an exact functor $\mathcal{D}^+(F)\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and a morphism of functors $\varepsilon_F\colon Q_B \circ K^+(F) \to \mathcal{D}^+(F) \circ Q_A$:

$$\begin{array}{ccc} \mathcal{D}^+(\mathcal{A}) & \xrightarrow{\mathcal{D}^+(F)} & \mathcal{D}^+(\mathcal{B}) \\ {\scriptstyle Q_A}\uparrow & & \uparrow{\scriptstyle Q_B} \\ K^+(\mathcal{A}) & \xrightarrow{K^+(F)} & K^+(\mathcal{B}) \end{array}$$

satisfying the following universal property: for any exact functor $G\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and a morphism of functors $\varepsilon\colon Q_B \circ K^+(F) \to G \circ Q_A$ there exists a unique morphism of functors $\eta\colon \mathcal{D}^+(F) \to G$ such that $(\eta \circ Q_A) \circ \varepsilon_F = \varepsilon$.

The derived functor of a right exact functor is a pair consisting of a functor $\mathcal{D}^-(F)\colon \mathcal{D}^-(\mathcal{A}) \to \mathcal{D}^-(\mathcal{B})$ and a morphism $\varepsilon_F\colon \mathcal{D}^-(F) \circ Q_A \to Q_B \circ K^+(F)$ satisfying the universal property with a morphism $\eta\colon G \to \mathcal{D}^-(F)$.

The universal property implies that the derived functor is unique. One checks that the functor defined using an adapted class satisfies the universal property.

Theorem 5.4. *If $\mathcal{A}$ contains sufficiently many injective (resp. projective) objects then the class of all these objects is adapted to any left (resp. right) exact functor F.*

Proof. Let $I^\bullet$ be a bounded below acyclic complex of injective objects. Then the identity morphism is homotopic to the zero morphism (we checked on tutorials that there are no morphisms in the homotopy category from a complex with projective terms to an acyclic complex. Here we use the dual argument). The image under F of the homotopy shows that the identity on $F(I^\bullet)$ is homotopic to the zero morphism, hence $F(I^\bullet)$ is acyclic. □

To define derived functors on unbounded derived categories, one uses K-projective complexes. A complex $P^\bullet$ is *K-projective* if for any acyclic complex $A^\bullet$ the complex $\mathrm{Hom}(P^\bullet, A^\bullet)$ is acyclic (we consider all morphisms $f^i \colon P^i \to A^{d+i}$, differential is given by composition with differentials of $P^\bullet$ and $A^\bullet$. Then the morphism in the category $C(\mathcal{A})$ of complexes are the closed degree-zero morphisms, while morphisms in $\mathcal{H}(\mathcal{A})$ are zero'th cohomology of the complex $\mathrm{Hom}(P^\bullet, A^\bullet)$).

A K-projective left resolution of a complex $A^\bullet$ is a quasi-isomorphism $P^\bullet \to A^\bullet$ with K-projective $P^\bullet$. We can use K-projective or K-injective resolutions to calculate the right and left derived functors.

In [Spa88] Spaltenstein showed that:

(1) Let R be an associative ring with unit and $\mathcal{A}$ the category of right R modules. Then any complex of objects of $\mathcal{A}$ admits a K-projective and K-injective resolution.
(2) Let $\mathcal{O}$ be a sheaf of rings on a topological space X and $\mathcal{A}$ the category of sheaves of $\mathcal{O}$-modules. Then any complex of objects of $\mathcal{A}$ admits a K-injective resolution.

5.2. **The six functors formalism.**

Let $(X, \mathcal{R}_X)$ be a ringed space and $\mathcal{F}_1, \mathcal{F}_2$ sheaves on X. Then $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$ are sheaves of abelian groups on X.

Proposition 5.5. *Let X be a topological space, $\mathcal{R}$ a sheaf of rings with unit on X. Then any sheaf of $\mathcal{R}$-modules can be embedded into an injective sheaf of $\mathcal{R}$-modules.*

Proof. The proof is based on Godement's method. Let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules. For any $x \in X$ we can construct a monomorphism $\mathcal{F}_X \to I(x)$ where $I(x)$ is injective over $\mathcal{R}_x$. We define sheaf

$$\mathcal{I}(U) := \prod_{x \in U} I(x).$$

It is injective and $\mathcal{F}$ is its subsheaf. $\square$

A sheaf $\mathcal{F}_1$ is *flat* if functor $\mathcal{F}_1 \otimes (-)$ is exact. $\mathcal{F}_1$ is flat if and only if the stalk of $\mathcal{F}_1$ at any point of X is flat. As any sheaf on X is a quotient of a flat sheaf, we have functor

$$\mathcal{F}_1 \otimes^L (-)\colon \mathcal{D}^-(\mathrm{Sh}_{\mathcal{R}_X}) \to \mathcal{D}^-(\mathrm{Sh}_{\mathbb{Z}_X}).$$

As any sheaf on X is a subsheaf of an injective sheaf, we also get

$$R\,\mathcal{H}\mathrm{om}(\mathcal{F}_1, -)\colon \mathcal{D}^+(\mathrm{Sh}_{\mathcal{R}_X}) \to \mathcal{D}^+(\mathrm{Sh}_{\mathbb{Z}_X}).$$

Let $f\colon X \to Y$ be a morphism of topological spaces and let $\mathcal{F}$ be a sheaf on X. The *direct image* of $\mathcal{F}$, i.e the sheaf $f_\bullet(\mathcal{F})$ on Y is defined as

$$f_\bullet(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)).$$

Let $(f,\varphi)\colon (X,\mathcal{R}_X) \to (Y,\mathcal{R}_Y)$ be a morphism of ringed spaces, so that $\varphi\colon \mathcal{R}_Y \to f_\bullet(\mathcal{R}_X)$ is a morphism of sheaves of modules. Then for any $\mathcal{F} \in \mathrm{Sh}_{\mathcal{R}_X}$ the sheaf $f_\bullet(\mathcal{F})$ inherits via φ a structure of $\mathcal{R}_Y$-module so we have a functor $f_\bullet\colon \mathrm{Sh}_{\mathcal{R}_X} \to \mathrm{Sh}_{\mathcal{R}_Y}$. It is left exact.

To construct its right derived functor we use Proposition 5.5 and get

$$Rf_\bullet\colon \mathcal{D}^+(\mathrm{Sh}_{\mathcal{R}_X}) \to \mathcal{D}^+(\mathrm{Sh}_{\mathcal{R}_Y}).$$

When Y is a point and $\mathcal{R}_Y = \mathbb{Z}$, we get the derived functor

$$R\Gamma\colon \mathcal{D}^+(\mathrm{Sh}_{\mathcal{R}_X}) \to \mathcal{D}^+(\mathrm{Ab}).$$

Theorem 5.6. *Let $\Phi\colon Sh_{\mathcal{R}_X} \to Sh_{\mathbb{Z}}$ be the forgetful functor (of the structure of $\mathcal{R}_X$-module). Then the functors $R\Gamma$ and $R(\Gamma\circ\Phi)$ are naturally isomorphic. If $X = \bigcup U_i$ is an open covering with $H^q(U_{i_1} \cap \ldots \cap U_{i_p}, \mathcal{F}) = 0$ for all $q > 0$, $p \geq 1$ then $H^i(X,\mathcal{F})$ can be computed via the Čech complex.*

Functor $f_\bullet\colon \mathrm{Sh}_{\mathcal{R}_X} \to \mathrm{Sh}_{\mathcal{R}_Y}$ has left adjoint, the *inverse image* functor $f^\bullet$. For a sheaf $\mathcal{G}$ on Y $f^\bullet(\mathcal{G})$ is a sheaf associated to the presheaf

$$U \mapsto \lim_{f(U)\subset V} \mathcal{G}(V)$$

where the limit is taken over all open subsets $V \subset Y$ containing $f(U)$.

If $\mathcal{G}$ was a sheaf of modules, not just abelian groups, then $f^\bullet(\mathcal{G})$ has a structure of $f^\bullet(\mathcal{R}_Y)$-module. The morphism $\mathcal{R}_Y \to f_\bullet\mathcal{R}_X$ given by the map (f,φ) of ringed spaces yields $f^\bullet(\mathcal{R}_Y) \to \mathcal{R}_X$. It allows us to define

$$f^*(\mathcal{G}) = \mathcal{R}_X \otimes_{f^\bullet(\mathcal{R}_Y)} f^\bullet(\mathcal{G}).$$

Functor $f^\bullet$ is exact (the stalk of $f^\bullet\mathcal{F}$ at y is the stalk of $\mathcal{F}$ at $f(y)$), while f^* is right exact.

Let now $f\colon X \to Y$ be a morphism of locally compact topological spaces and $\mathcal{F}$ a sheaf on X. For $V \subset Y$ open put

$$f_!(\mathcal{F})(V) = \{s \in \mathcal{F}(f^{-1}(U)) \,|\, \text{supp }(s) \to V \text{ is proper}\}.$$

Recall, that morphism is proper if the preimage of any compact set is compact. Then $f_!\mathcal{F}$ is a sheaf on Y, the *direct image with compact support* of $\mathcal{F}$. It is a subsheaf of $f_\bullet\mathcal{F}$ and functor

$$f_!\colon \operatorname{Sh}_{\mathcal{R}_X} \to \operatorname{Sh}_{\mathcal{R}_Y}$$

is left exact. Indeed, let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be an exact sequence of sheaves on X. We need to show that $f_!\mathcal{F}_1$ is the kernel of $f_!\mathcal{F}_2 \to f_!\mathcal{F}_3$. It is clear, as any section of $f_\bullet\mathcal{F}_2$ which becomes zero as a section of $f_\bullet\mathcal{F}_3$ is a section of $f_\bullet\mathcal{F}_1$.

Considering injective resolutions, we obtain functor

$$Rf_!\colon \mathcal{D}^+(\operatorname{Sh}_{\mathcal{R}_X}) \to \mathcal{D}^+(\operatorname{Sh}_{\mathcal{R}_Y}).$$

When Y is a point, we recover $\Gamma_c(X,\mathcal{F})$, global sections with compact support.

Let us now assume that $h\colon W \to X$ is an embedding of a locally closed subset. For a sheaf $\mathcal{E}$ on W and $U \subset X$ open, we have

$$h_!\mathcal{E}(U) = \{s \in \mathcal{E}(U \cap W) \,|\, \text{supp }(s) \subset U \text{ is closed }\}.$$

Lemma 5.7. *Let $x \in X$ be a point which does not lie in W. Then the stalk $h_!\mathcal{E}_x$ is zero.*

Proof. Let $U \subset X$ be an open neighbourhood of x and let $s \in h_!\mathcal{E}(U)$ be a section. Then s is a section of $\mathcal{E}(W \cap U)$ and supp $s \subset U$ is closed. Hence, there exists $U' \subset U$ such that $x \in U'$ and $U' \cap \text{supp } s = \emptyset$. Hence, any germ is zero at x. □

As the stalks of $h_!\mathcal{E}$ at $x \in W$ are equal to the stalks of $\mathcal{E}$, $h_!$ is an exact functor.

Functor $h_!$ induces an equivalence of $\operatorname{Sh}_{\mathcal{R}_W}$ with the full subcategory $\operatorname{Sh}^W_{\mathcal{R}_X}$ of $\operatorname{Sh}_{\mathcal{R}_X}$ consisting of sheaves whose stalks outside W are zero. The inverse of this equivalence is given by h^*.

For a sheaf $\mathcal{F}$ on X we define a sheaf $\mathcal{F}^W$ which, for any $U \subset X$ is

$$\mathcal{F}^W(U) = \{s \in \mathcal{F}(U) \,|\, \text{supp }(s) \subset W\}.$$

Then $\mathcal{F}^W \in \operatorname{Sh}^W_{\mathcal{R}_X}$ and we put

$$h^!\mathcal{F} := h^*\mathcal{F}^W.$$

Lemma 5.8. *Functor $h^!$ is right adjoint to $h^!$.*

Proof. We fix a map $h_!h^!\mathcal{F} \to \mathcal{F}$ and show that the induced morphism

$$\mathrm{Hom}_W(\mathcal{E}, h^!\mathcal{F}) \to \mathrm{Hom}_X(h_!\mathcal{E}, h_!h^!\mathcal{F}) \to \mathrm{Hom}_X(h_!\mathcal{E}, \mathcal{F}) \tag{13}$$

is an isomorphism for any $\mathcal{E} \in \mathrm{Sh}_{\mathcal{R}_W}$, $\mathcal{F} \in \mathrm{Sh}_{\mathcal{R}_X}$.

By definition

$$h_!h^!\mathcal{F}(U) = \{s \in \mathcal{F}(U) \,|\, \mathrm{supp}\ (s) \subset W\}$$

so we have a canonical map $h_!h^!\mathcal{F} \to \mathcal{F}$. Moreover, it follows that for any sheaf G whose stalks outside W are zero, any morphism $G \to \mathcal{F}$ factors uniquely via $h_!h^!\mathcal{F}$. In particular, the second map in (13) is an isomorphism.

The first map in (13) is an isomorphism because $h_!$ is an equivalence of categories $\mathrm{Sh}_{\mathcal{R}_W} \to \mathrm{Sh}^W_{\mathcal{R}_X}$. □

As $f_!$ is in general not right exact, it cannot have right adjoint on the abelian level. However, there exists

$$f^!\colon \mathcal{D}^+(\mathrm{Sh}_{\mathcal{R}_Y}) \to \mathcal{D}^+(\mathrm{Sh}_{\mathcal{R}_X})$$

such that

$$\mathrm{Hom}(Rf_!\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, f^!\mathcal{G}).$$

Functor $f^!$ is the *inverse image with compact support.* For $f^!\mathcal{G}$ we should have

$$R\Gamma(U, f^!\mathcal{G}) = R\,\mathrm{Hom}(\mathcal{R}_U, f^!\mathcal{G}) = R\,\mathrm{Hom}(f_!\mathcal{R}_U, \mathcal{G}).$$

Unfortunately, $U \mapsto R\,\mathrm{Hom}(f_!\mathcal{R}_U, \mathcal{G})$ is not a sheaf on X. We need to replace the constant sheaf $\mathcal{R}_X$ by its flat resolution:

$$0 \to \mathcal{L}_n \to \ldots \to \mathcal{L}_0 \to 0.$$

Then, for an injective complex $\mathcal{G}^\bullet$, $f^!(\mathcal{G}^\bullet)$ is the total complex of $\mathcal{H}om(f_!\mathcal{L}, \mathcal{G}^\bullet)$.

For a finite-dimensional, locally compact topological space X with a constant sheaf $\mathbb{Z}_X$ (or k_X for some field k) let

$$D^\bullet_X = f^!(\mathbb{Z}), \tag{14}$$

where $f\colon X \to \mathrm{pt}$. Object $D^\bullet_X$ is the *dualizing complex* on X. By adjunction, we have

$$\mathrm{Hom}(\Gamma_c(X, \mathcal{F}), \mathbb{Z}) \simeq \mathrm{Hom}(\mathcal{F}, D^\bullet_X).$$

Exercise 5.9. If X is a smooth manifold with the constant sheaf k_X then $D^\bullet_X = k_X[\dim X]$. Show that the $f_! \dashv f^!$ adjunction implies Poincare duality for X, $H^i_c(X) \simeq H^{n-i}(X)^\vee$.

For $\mathcal{F} \in \mathcal{D}^b(\mathrm{Sh}_{\mathbb{Z}_X})$ we put

$$\mathscr{D}_X(\mathcal{F}) := R\,\mathcal{H}\mathrm{om}(\mathcal{F}, D_X^\bullet).$$

Object $\mathscr{D}_X(\mathcal{F})$ is the *Verdier dual* of $\mathcal{F}$.

As we always have $\mathcal{F} \mapsto \mathrm{Hom}(\mathrm{Hom}(\mathcal{F}, \mathcal{G}))$, we have a natural transformation

$$\alpha_X\colon \mathrm{Id}_{\mathcal{D}^b(\mathrm{Sh}_{\mathbb{Z}_X})} \to \mathscr{D}_X\mathscr{D}_X.$$

α is generally not an isomorphism.

However, let us assume that X is a stratified space $X = \bigsqcup_{s\in\mathcal{S}} S$ whose strata are non-singular topological spaces. A sheaf $\mathcal{F}$ on X is *constructible* with respect to $\mathcal{S}$ if for any $S \in \mathcal{S}$ and its embedding $i_S\colon S \to X$, $i_S^\bullet(\mathcal{F})$ is locally constant on S. (Recall that a constant sheaf is a sheafification of a constant presheaf whose all restriction morphisms are the identity morphisms. A sheaf is locally constant if every point has an open neighbourhood such that restriction to this neighbourhood is a constant sheaf). Let $\mathcal{D}^b_{\mathcal{S}}(\mathrm{Sh}_{\mathbb{Z}_X}) \subset \mathcal{D}^b(\mathrm{Sh}_{\mathbb{Z}_X})$ be the full subcategory consisting of complexes with constructible cohomology. Then $D_X^\bullet \in \mathcal{D}^b_{\mathcal{S}}(\mathrm{Sh}_{\mathbb{Z}_X})$ and α is an isomorphism when restricted to $\mathcal{D}^b_{\mathcal{S}}(\mathrm{Sh}_{\mathbb{Z}_X})$. $\mathscr{D}_X|_{\mathcal{D}^b_{\mathcal{S}}(\mathrm{Sh}_{\mathbb{Z}_X})}$ is the *Verdier duality.*

Let $f\colon X \to Y$ be a continuous morphism of locally compact and finite-dimensional spaces. For $\mathcal{F} \in \mathcal{D}(\mathrm{Sh}_{\mathcal{R}_X})$ and $\mathcal{G} \in \mathcal{D}(\mathrm{Sh}_{\mathcal{R}_Y})$ we have

$$\begin{aligned}
&f^\bullet(\mathcal{G}_1 \otimes^L \mathcal{G}_2) \simeq f^\bullet(\mathcal{G}_1) \otimes^L f^\bullet(\mathcal{G}_2),\\
&R\,\mathcal{H}\mathrm{om}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_3) \simeq R\,\mathcal{H}\mathrm{om}(\mathcal{F}_1, R\,\mathcal{H}\mathrm{om}(\mathcal{F}_2, \mathcal{F}_3)),\\
&Rf_\bullet R\,\mathcal{H}\mathrm{om}(f^\bullet\mathcal{G}, \mathcal{F}) \simeq R\,\mathcal{H}\mathrm{om}(\mathcal{G}, Rf_\bullet\mathcal{F}),\\
&Rf_!(\mathcal{F} \otimes^L f^\bullet\mathcal{G}) \simeq Rf_!\mathcal{F} \otimes^L \mathcal{G},\\
&f^! R\,\mathcal{H}\mathrm{om}(\mathcal{G}_1, \mathcal{G}_2) \simeq R\,\mathcal{H}\mathrm{om}(f^\bullet\mathcal{G}_1, f^!\mathcal{G}_2),\\
&f^!\mathscr{D}_Y(\mathcal{G}) = f^! R\,\mathcal{H}\mathrm{om}(\mathcal{G}, D_Y^\bullet) \simeq R\,\mathcal{H}\mathrm{om}(f^\bullet\mathcal{G}, D_X^\bullet) \simeq \mathscr{D}_X(f^\bullet\mathcal{G}),
\end{aligned}$$

The proofs consist of checking equalities for sheaves and then considering appropriate resolutions.

5.3. **Recollement for a closed subset.**

Let X be a locally compact finite-dimensional space and $i\colon F \to X$ an embedding of a closed subset. Then $i_\bullet = i_!$, so the functor

$$i_\bullet\colon \mathrm{Sh}_{\mathbb{Z}_F} \to \mathrm{Sh}_{\mathbb{Z}_X}$$

has left and right adjoints

$$i^\bullet, i^! \colon \mathrm{Sh}_{\mathbb{Z}_X} \to \mathrm{Sh}_{\mathbb{Z}_F}\,.$$

Let now $j\colon U \to X$ be an embedding of an open subset. Then, for any sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathbb{Z}_X}$ $\mathcal{F}^U$ is isomorphic to $j_\bullet j^\bullet \mathcal{F}$, hence $j^\bullet = j^!$. It follows that functor

$$j^\bullet \colon \mathrm{Sh}_{\mathbb{Z}_X} \to \mathrm{Sh}_{\mathbb{Z}_U}$$

has left and right adjoints

$$j_!, j_\bullet \colon \mathrm{Sh}_{\mathbb{Z}_U} \to \mathrm{Sh}_{\mathbb{Z}_X}\,.$$

Functors $i_\bullet$, $i^\bullet$, $j^\bullet$ and $j_!$ are exact while $i^!$ and $j_\bullet$ are left exact.

The sheaf $j_!\mathcal{E}$ is a sheafification of the presheaf

$$V \mapsto \begin{cases} \mathcal{E}(V) & \text{if } V \subset U \\ 0 & \text{otherwise.} \end{cases}$$

Let us now assume that $U = X \setminus F$ is the complementary open set to F. Then $j^\bullet i_\bullet = 0$, hence also $i^\bullet j_! = 0$ and $i^! j_\bullet = 0$.

Given a sheaf $\mathcal{F}$ on X sequence

$$0 \to j_! j^\bullet \mathcal{F} \to \mathcal{F} \to i_\bullet i^\bullet \mathcal{F} \to 0$$

is exact, as the stalks of $j_! j^\bullet \mathcal{F}$ are either zero or equal to the stalks of $\mathcal{F}$. Precisely where the stalks are zero, the stalks of $i_\bullet i^\bullet \mathcal{F}$ are equal to the stalks of $\mathcal{F}$.

On the other hand, $j_\bullet j^\bullet \mathcal{F}(V) = \mathcal{F}(U \cap V)$, for any $V \subset U$ closed. Hence, the kernel of the morphism of (pre)sheaves $\mathcal{F} \to j_\bullet j^\bullet \mathcal{F}$ on V consists of $s \in \mathcal{F}(V)$ such that $\operatorname{supp}\, s \subset F$. It follows that sequence

$$0 \to i_\bullet i^! \mathcal{F} \to \mathcal{F} \to j_\bullet j^\bullet \mathcal{F}$$

is exact for any $\mathcal{F} \in \mathrm{Sh}_{\mathbb{Z}_X}$.

An injective sheaf $\mathcal{I}$ is in particular *flabby*, i.e. the map $\mathcal{I} \to j_\bullet j^\bullet \mathcal{I}$ is epi and we have

$$0 \to i_\bullet i^! \mathcal{I} \to \mathcal{I} \to j_\bullet j^\bullet \mathcal{I} \to 0.$$

Considering bounded below complexes of injective objects we get distinguished triangles

$$\begin{aligned} & j_! j^\bullet \mathcal{I}^\cdot \to \mathcal{I}^\cdot \to i_\bullet i^\bullet \mathcal{I}^\cdot \to j_! j^\bullet \mathcal{I}^\cdot[1], \\ & i_\bullet i^! \mathcal{I}^\cdot \to \mathcal{I}^\cdot \to j_\bullet j^\bullet \mathcal{I}^\cdot \to i_\bullet i^! \mathcal{I}^\cdot[1]. \end{aligned} \tag{15}$$

The fact that the triangles are distinguished follows from the fact that they are term-wise exact sequences, see Proposition 3.27.

Let now $\mathcal{F}$ be any object of $\mathcal{D}^+(\mathrm{Sh}_{\mathbb{Z}_X})$ and $\mathcal{I}^\bullet$ its injective resolution. Then triangles (15) can be read as

$$j_!j^\bullet\mathcal{F} \to \mathcal{F} \to i_\bullet i^\bullet\mathcal{F} \to j_!j^\bullet\mathcal{F}[1],$$
$$i_\bullet Ri^!\mathcal{F} \to \mathcal{F} \to Rj_\bullet j^\bullet\mathcal{F} \to i_\bullet Ri^!\mathcal{F}[1].$$

In the first triangle we all four functors are exact, so we can just apply them to any complex termwise and get derived functor. In the second triangle $i^!$ should be applied to an injective resolution (as it is) while $j_\bullet$ to a flabby one (which again is the case).

Functors $i_\bullet$, $Rj_\bullet$ and $j_!$ considered as functors between derived categories are fully faithful, because the compositions with the adjoints $i^\bullet i_\bullet$, $j^\bullet Rj_\bullet$, $j^\bullet j_!$ are isomorphic to identity, which can be checked term-wise (on complexes of injective objects, if necessary) *cf.* [Huy06, Remark 1.23].

Now, for any $A \in \mathcal{D}^+(\mathrm{Sh}_{\mathbb{Z}_U})$ and $B \in \mathcal{D}^+(\mathrm{Sh}_{\mathbb{Z}_F})$

$$\mathrm{Hom}(j_!A, i_\bullet B) \simeq 0 \simeq \mathrm{Hom}(i_\bullet B, Rj_\bullet A).$$

Indeed, both $i_\bullet$ and $j_\bullet$ map injective objects to injective objects hence the above Hom-spaces can be computed term-wise using injective resolution of B, resp. A.

Exercise 5.10. A functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories with exact left adjoint maps injective objects to injective objects.

Thus, we have proved

Proposition 5.11. *Let $F \subset X$ be a closed subset of a topological space with the complementary open set U. Then category $\mathcal{D}^+(Sh_{\mathbb{Z}_X})$ admits a recollement*

$$\mathcal{D}^+(Sh_{\mathbb{Z}_F}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathcal{D}^+(Sh_{\mathbb{Z}_X}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathcal{D}^+(Sh_{\mathbb{Z}_U})$$

5.4. **Perverse sheaves.**

Let now X be a topological space with a fixed sheaf of rings $\mathcal{O}_X$. We suppose that X is partitioned by a finite family $\mathscr{S} = \{S\}$ of non-empty locally closed subspaces. We assume that

- The closure $\overline{S}$ of any stratum is a union of strata.
- $\mathcal{O}_X$ is the constant sheaf R on X, where R is a left Noetherian ring.
- The components of a given strata are topological manifolds, all of the same dimension; if S is contained in the closure $\overline{T}$ of a stratum T, then $\dim S < \dim T$.

- Let $i_S\colon S \to X$ be the inclusion map of a stratum S into X. The direct image functor j_* from $\mathcal{O}_S$-modules to $\mathcal{O}_X$-modules has finite homological dimension (there exists N_S such that $R^l i_{S*}\mathcal{F} = 0$ for $l > N_S$ and any $\mathcal{F} \in \mathrm{Sh}_{\mathcal{O}_S}$).

An example of such stratification is Whitney stratification: if S is of dimension i and T of dimension j and sequences of points $s_i \in S$, $t_i \in T$ converging to $t \in T$ are such that secant lines between s_i and t_i converge to a line L and the sequence of tangent i-planes to $s_i \in S$ converge to an i-plane then L is contained in T.

In general, we shall consider X to be a complex analytic or algebraic variety stratified by (real or complex) subvarieties.

We consider

$$X_i = \bigcup_{\dim S \leq i} S.$$

Then $X_i \subset X$ is a closed variety. The assumptions on the stratification imply that the recollement given by Proposition 5.11 restricts to the bounded and bounded below derived category of complexes of sheaves with $\mathscr{S}$-constructible cohomology, which we shall denote by $\mathcal{D}^b_{\mathcal{S}}(X)$ and $\mathcal{D}^+_{\mathcal{S}}(X)$. It follows that we have recollements

$$\mathcal{D}^+_{\mathcal{S}}(X_{i-1}) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}^+_{\mathcal{S}}(X_i) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}^+_c(S_i) \tag{16}$$

where S_i is the union of strata of dimension i. Analogously for $\mathcal{D}^b_{\mathcal{S}}(X)$.

Category $\mathcal{D}^+_c(S_i)$ of complexes with locally constant cohomology has the standard t-structure and its shifts $(\mathcal{D}^+_c(S_i)^{\leq p}, \mathcal{D}^+_c(S_i)^{\geq p+1})$ for $p \in \mathbb{Z}$.

We fix *perversity function*

$$p\colon \mathscr{S} \to \mathbb{Z}$$

and define the category of *perverse sheaves* $\mathcal{M}_{\mathcal{S}}(p, X)$ as the t-structure on $\mathcal{D}^+_{\mathcal{S}}(X)$ glued along recollements (16) from the shifts by $p(S)$ of the standard t-structure on $\mathcal{D}^+_c(S)$.

For $p = 0$ we get the standard t-structure

$$\mathrm{Sh}_{\mathcal{O}_X} \simeq \mathcal{M}_{\mathcal{S}}(0, X).$$

Proposition 5.12. *The aisles of the perverse t-structure are:*

$$\begin{aligned} {}^p\mathcal{D}^+_{\mathcal{S}}(X)^{\leq 0} &= \{K \in \mathcal{D}^+_{\mathcal{S}}(X) \mid i_S^* K \in \mathcal{D}^+_c(S)^{\leq p(S)} \forall S \in \mathscr{S}\}, \\ {}^p\mathcal{D}^+_{\mathcal{S}}(X)^{\geq 0} &= \{K \in \mathcal{D}^+_{\mathcal{S}}(X) \mid i_S^! K \in \mathcal{D}^+_c(S)^{\geq p(S)} \forall S \in \mathscr{S}\}. \end{aligned}$$

Proof. We proceed by induction on the dimension of X that The case $\dim X = 0$ is clear. Let us now assume that the above description holds for X of dimension $< n$ and let X be a stratified topological space of dimension n.

The embedding $i_{n-1}\colon X_{n-1} \to X$ is closed and $S_n \subset X$ is open. Hence, for any S of dimension n, $i_S^* \simeq i_S^!$. For $T \in \mathscr{S}$ with $\dim T < n$ let $k_T\colon T \to X_{n-1}$ denote the embedding. Then, the definition of the glued t-structure and the inductive hypothesis imply:

$$\begin{aligned}
{}^p\mathcal{D}^+_{\mathcal{S}}(X)^{\leq 0} =& \{K \in \mathcal{D}^+_{\mathcal{S}}(X) \mid i^*_S(K) \in \mathcal{D}^+_c(S)^{\leq p(S)}\, \forall S \ \dim S = n, i^*_{n-1}K \in {}^p\mathcal{D}^+_{\mathcal{S}}(X_{n-1})^{\leq 0}\} = \\
=& \{K \in \mathcal{D}^+_{\mathcal{S}}(X) \mid i^*_S(K) \in \mathcal{D}^+_c(S)^{\leq p(S)}\, \forall S \ \dim S = n, \\
& k^*_T i^*_{n-1}K \in \mathcal{D}^+_c(T)^{\leq p(S)}\, \forall T \ \dim T < n\}, \\
{}^p\mathcal{D}^+_{\mathcal{S}}(X)^{\geq 0} =& \{K \in \mathcal{D}^+_{\mathcal{S}}(X) \mid i^*_S(K) \in \mathcal{D}^+_c(S)^{\geq p(S)}\, \forall S \ \dim S = n, i^!_{n-1}K \in {}^p\mathcal{D}^+_{\mathcal{S}}(X_{n-1})^{\geq 0}\} = \\
=& \{K \in \mathcal{D}^+_{\mathcal{S}}(X) \mid i^!_S(K) \in \mathcal{D}^+_c(S)^{\leq p(S)}\, \forall S \ \dim S = n, \\
& k^!_T i^!_{n-1}K \in \mathcal{D}^+_c(T)^{\geq p(T)}\, \forall T \ \dim T < n\}.
\end{aligned}$$

For $S \in \mathscr{S}$ with $\dim S < n$ the composition of the embedding $k_S\colon S \to X_{n-1}$ with i_{n-1} is the embedding i_S. Hence $i^*_S = k^*_S \circ i^*_{n-1}$ and $i^!_S = k^!_S \circ i^!_{n-1}$. The statement follows. □

Let us now assume that $\mathcal{O}_X = k_X$ is the constant sheaf with value k, for some field k.

Recall that the dualising complex (14) yields Verdier duality $\mathscr{D}_X\colon \mathcal{D}^+_{\mathcal{S}}(X)^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{D}^+_{\mathcal{S}}(X)$. For a smooth manifold S we have $D^\bullet_S = k_S[\dim_{\mathbb{R}} S]$.

Exercise 5.13. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of triangulated categories. Show that a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1})$ on the category $\mathcal{C}$ induces a t-structure on $\mathcal{D}$.

Exercise 5.14. Show that a t-structure on category $\mathcal{C}$ induces a t-structure on the category $\mathcal{C}^{\mathrm{op}}$.

Proposition 5.15. *Verdier duality $\mathscr{D}_S$ on $\mathcal{D}^+_c(S)$, for a smooth manifold S with $\dim_{\mathbb{R}}(S) = n$, maps the t-structure $(\mathcal{D}^+_c(S)^{\leq p}, \mathcal{D}^+_c(S)^{\geq p+1})$ to the t-structure $(\mathcal{D}^+_c(S)^{\leq -p-n}, \mathcal{D}^+_c(S)^{\geq -p-n})$.*

Proof. We know that $(\mathscr{D}_S(\mathcal{D}^+_c(S)^{\geq p+1}), \mathscr{D}_S(\mathcal{D}^+_c(S)^{\leq p})$ is a t-structure on $\mathcal{D}^+_c(S)$. We check that $\mathscr{D}_S(\mathcal{D}^+_c(S)^{\leq p}) \in \mathcal{D}^+_c(S)^{\geq -p-n}$ and $\mathscr{D}_S(\mathcal{D}^+_c(S)^{\geq q}) \in \mathcal{D}^+_c(S)^{\leq -q-n}$.

Let $A \in \mathcal{D}^+_c(S)$. Spectral sequence with $E^{p,q}_2 = \mathrm{Ext}^q(\mathcal{H}^-p(A), k_S[n])$ converges to $H^i(\mathscr{D}_S(A)) \simeq \mathrm{Ext}^i(A, k_S[n])$. As $E^{p,q}_2$ are Ext-groups of vector spaces they are non-zero only when they are Hom's. It follows that the spectral sequence degenerates and

$$H^i(\mathscr{D}_S(A)) \simeq \mathrm{Ext}^i(A, k_S[n]) \simeq \mathrm{Ext}^{i+n}(A, k_S) \simeq \mathrm{Hom}(H^{-i-n}(A), S).$$

Then, if $H^i(A) = 0$ for $i > p$ then $H^j(\mathscr{D}_S(A)) = 0$ for $j < -p-n$. Similarly, if $H^i(A) = 0$ for $i < q$ then $H^j(\mathscr{D}_S(A)) = 0$ for $j > -q-n$. □

Proposition 5.16. *For perversity $p\colon \mathscr{S} \to \mathbb{Z}$ define $p^*(S) = -p(S) - \dim_R(S)$. Then the image under Verdier duality of the t-structure for perversity p is the t-structure for perversity p^*.*

Proof. We know that $(\mathscr{D}_X({}^p\mathcal{D}^+_{\mathcal{S}}(X)^{\geq 0}), \mathscr{D}_X({}^p\mathcal{D}^+_{\mathcal{S}}(X)^{\leq -1})$ is a t-structure on $\mathcal{D}^+_{\mathcal{S}}(X)$. It thus suffices to check that $\mathcal{F} \in {}^p\mathcal{D}^+_{\mathcal{S}}(X)^{\leq 0}$ if and only if $\mathscr{D}_X(\mathcal{F}) \in {}^{p^*}\mathcal{D}^+_{\mathcal{S}}(X)^{\geq 0}$.

By Proposition 5.12 and the fact that Verdier duality exchanges i^*_S with $i^!_S$, we have

$$\begin{aligned}
&\mathcal{F} \in {}^p\mathcal{D}^+_{\mathcal{S}}(X)^{\leq 0} \Leftrightarrow \\
&\Leftrightarrow i^*_S(F) \in \mathcal{D}^+_c(S)^{\leq p(S)}\ \forall S \in \mathscr{S} \Leftrightarrow \\
&\Leftrightarrow \mathscr{D}_S(i^*_S(F)) \in \mathcal{D}^+_c(S)^{\geq -p(S)-\dim_{\mathbb{R}}(S)}\ \forall S \in \mathscr{S} \Leftrightarrow \\
&\Leftrightarrow i^!_S\mathscr{D}_X\mathcal{F} \in \mathcal{D}^+_c(S)^{\geq -p(S)-\dim_{\mathbb{R}}(S)}\ \forall S \in \mathscr{S} \Leftrightarrow \\
&\Leftrightarrow \mathscr{D}_X(\mathcal{F}) \in {}^{p^*}\mathcal{D}^+_{\mathcal{S}}(S)^{\geq 0}.
\end{aligned}$$

□

Corollary 5.17. *Assume that X is a topological space with a stratification $\mathscr{S}$ whose all strata have even dimension over $\mathbb{R}$. Let*

$$p_{1/2}(S) = -\frac{1}{2}\dim_{\mathbb{R}}(S).$$

Then Verdier duality restricts to a duality on the category $\mathcal{M}_{\mathcal{S}}(p_{1/2}, X)$.

Proof. It suffices to check that $p^*(S) = -p(S) - \dim_{\mathbb{R}}(S) = \frac{1}{2}\dim_{\mathbb{R}}(S) - \dim_{\mathbb{R}}(S) = -\frac{1}{2}\dim_{\mathbb{R}}(S) = p(S)$. □

The perversity $p_{1/2}$ is often called the *middle perversity*. It exists whenever X is a complex manifold stratified by complex submanifolds.

An open set $U \subset X$ has stratification induced from $\mathscr{S}$ Let $U_n \subset U$ be the intersection of U with the open strata. The *intersection complex $IC(U)$* is defined as the intermediate extension $j_{!*}$ of $k_{U_n}[-1/2\dim_{\mathbb{R}}(U_n)]$. These were first described as explicit complexes by Goresky and MacPherson [GM80].

5.4.1. *Subdivision of stratification.* Let a stratification $\mathcal{T}$ be a subdivision of a stratification $\mathcal{S}$ (each stratum of $\mathcal{S}$ is the union of several strata of $\mathcal{T}$). Then we have an embedding of categories $\mathcal{D}^+_{\mathcal{S}}(X) \to \mathcal{D}^+_{\mathcal{T}}(X)$. Let $p\colon \mathcal{S} \to \mathbb{Z}$, $q\colon \mathcal{T} \to \mathbb{Z}$ be perversities such that

$$p(S) \leq q(T) \leq p(S) + \dim S - \dim T$$

whenever $T \subset S$.

Proposition 5.18. *Under the above assumptions the t-structure of perversity q on $\mathcal{D}^+_{\mathcal{T}}(X)$ induces the t-structure of perversity p on $\mathcal{D}^+_{\mathcal{S}}(X)$ (under the embedding $\mathcal{D}^+_{\mathcal{S}}(X) \subset \mathcal{D}^+_{\mathcal{T}}(X)$). In particular, any p-perverse sheaf is also q-perverse.*

Consider now the case when X is a complex variety, $\mathcal{S}$ is a stratification of X by non-singular subvarieties and the perversity p depend only on the dimension of $S \in \mathcal{S}$. We also assume that

$$0 \leq p(n) - p(m) \leq m - n \quad \text{for } n \leq m. \tag{17}$$

Under these assumptions the subdivision of stratification is compatible with t-structures and we can define a t-structure on the triangulated category $\mathcal{D}^b_c(X)$ of complexes with cohomology constructible with respect to some stratification. The following proposition describes perverse sheaves.

Proposition 5.19. *For $\mathcal{F} \in \mathcal{D}^b_c(X)$ the following conditions are equivalent:*

(1) $\mathcal{F}$ is a perverse sheaf,
(2) Any irreducible submanifold $S \subset X$ contains a Zariski open $U \subset S$ such that, for $j\colon U \to X$, $H^i(j^\bullet \mathcal{F}) = 0$ for $i > p(\dim_{\mathbb{R}}(S))$, $H^i(Rj^! \mathcal{F}) = 0$ for $i < p(\dim_{\mathbb{R}}(S))$.

5.4.2. *Simple objects.* In case the perversity depends only on the dimension of strata and satisfies (17) the category $\mathcal{M}_{\mathcal{S}}(p, X)$ admits the following description

Proposition 5.20. *The category $\mathcal{M}_{\mathcal{S}}(p, X)$ is Artinian. Its simple objects are of the form $L(S, \mathcal{E}) = {}^p(i_S)_{!*}\mathcal{E}[p(S)]$ where $S \in \mathcal{S}$ and $\mathcal{E}$ is an irreducible locally constant sheaf of vector spaces on $\mathcal{S}$. In particular, if all strata are simply connected, simple objects on $\mathcal{M}_{\mathcal{S}}(p, X)$ are in one-to-one correspondence with strata $S \in \mathcal{S}$.*

If we do not fix a stratification then the category of perverse shaves is Noetherian. However, if $p = p_{1/2}$ the Verdier duality implies that the category is also Artinian. In particular, if X is a complex variety we obtain the following proposition

Proposition 5.21. *The category $\mathcal{M}(p_{1/2}, X)$ of perverse sheaves with the cohomology that are constructible with respect to some stratification with non-singular varieties is Artinian. Its simple objects are of the form $L(S, \mathcal{E})$ where $S \subset X$ is a non-singular irreducible variety and $\mathcal{E}$ is an irreducible locally constant sheaf on $\mathcal{S}$. $L(S, \mathcal{E}) \simeq L(S', \mathcal{E}')$ if and only if $S \cap S'$ is dense in S and S' and $\mathcal{E}|_{S\cap S'} = \mathcal{E}'|_{S \cap S'}$.*

5.5. **Gluing of perverse sheaves.**

Finally, we consider a smooth complex variety X, its closed subvariety Z defined by $f = 0$ and the complement $U = X \setminus Z$. We consider the self-dual perversity $p_{1/2}$ and discuss how to 'glue' a perverse sheaf on Z and on U to get a perverse sheaf on X.

We denote $B = \mathbb{C}^1$, $B^* = \mathbb{C}^1 \setminus \{0\}$ and $\widetilde{B}^*$ the universal covering of B^*, $p\colon \widetilde{B}^* \to B^*$ the covering map. We have the following commutative diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{i} & X & \xleftarrow{j} & U \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \{0\} & \longrightarrow & B & \longleftarrow & B^* \end{array}$$

Denote by $\widetilde{X}^*$ the fiber product

$$\begin{array}{ccc} \widetilde{X}^* & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{B}^* & \longrightarrow & B \end{array}$$

For $\mathcal{F} \in \mathcal{D}_c^+(X)$ define

$$\psi_f(\mathcal{F}) = i^\bullet R\pi_\bullet \pi^\bullet \mathcal{F}.$$

Proposition 5.22 (SGA VII). *ψ_f extends to the* nearby cycles functor $\psi_f\colon \mathcal{D}_c^+(X) \to \mathcal{D}_c^+(Z)$. *If $\mathcal{F} \in \mathcal{M}(p_{1/2}, X)$ then $\psi_f(\mathcal{F}) \in \mathcal{M}(p_{1/2}, Z)$.*

Since $\pi(\widetilde{X}^*) \subset U$, $\psi_f\mathcal{F}$ depends only on $j^\bullet\mathcal{F}$. Hence

$$\psi_f\colon \mathcal{D}_c(U) \to \mathcal{D}_c(Z).$$

The unit $\mathrm{Id} \to R\pi_\bullet\pi^\bullet$ gives a morphism

$$\theta\colon i^\bullet \to \psi_f.$$

The *vanishing cycles functor* is a functor $\varphi_f\colon \mathcal{D}_c(X) \to \mathcal{D}_c(Z)$ such that for any $\mathcal{F} \in \mathcal{D}_c(X)$

$$i^\bullet\mathcal{F} \xrightarrow{\theta_{\mathcal{F}}} \psi_f\mathcal{F} \xrightarrow{\mathrm{can}_{\mathcal{F}}} \varphi_f\mathcal{F} \to i^\bullet\mathcal{F}[1]$$

is distinguished.

Proposition 5.23. *If $\mathcal{F} \in \mathcal{M}(p_{1/2}, X)$ then $\varphi_f(\mathcal{F}) \in \mathcal{M}(p_{1/2}, Z)$.*

We have the complete turn $t\colon \widetilde{B}^* \to \widetilde{B}^*$ and $p \circ t = p$. It determines $\tau\colon \widetilde{X}^* \to \widetilde{X}^*$ with $\pi \circ \tau = \tau$. The unit $\mathrm{Id} \to R\tau_\bullet\tau^\bullet$ gives a morphism $\lambda\colon R\pi_\bullet\pi^\bullet \to R\pi_\bullet\pi^\bullet$ which induces

$$T\colon \psi_f \to \psi_f.$$

Morphism T, called *monodromy* satisfies $T \circ \theta = \theta$, hence it induces the monodromy action on vanishing cycles

$$T\colon \varphi_f \to \varphi_f.$$

Since t is an isomorphism, T is an automorphism of functors.

The definition of φ_f gave us morphism can: $\psi_f \to \varphi_f$. We also have var: $\varphi_f \to \psi_f$ defined as a lift to φ_f of $T - \mathrm{Id}\colon \psi_f \to \psi_f$ (its composition with θ is zero, as $T \circ \theta = \theta$).

Then, we have

$$\mathrm{var} \circ \mathrm{can} = T - \mathrm{Id}.$$

We define the category Glue(T, U). Objects are quadruples $(\mathcal{G}, \mathcal{H}, a, b)$ where $\mathcal{G} \in \mathcal{M}(U, p_{1/2})$, $\mathcal{H} \in \mathcal{M}(Z, p_{1/2})$, $a\colon \psi_f(\mathcal{G}) \to \mathcal{H}$, $b\colon \mathcal{H} \to \psi_f(\mathcal{G})$ are such that $b \circ a = \mathcal{T}_{\mathcal{G}} - \mathrm{Id}$.

Theorem 5.24 (Beilinson)**.** *Functor $\mathcal{M}(X, p_{1/2}) \to$ Glue(T, U), $\mathcal{F} \mapsto (j^{\bullet}\mathcal{F}, \varphi_f\mathcal{F}, can_{\mathcal{F}}, var_{\mathcal{F}})$ is an equivalence of categories.*

Consider the simplest example $X = B$, $Z = \{0\}$, $U = B^*$, $f(z) = z$ and the category of perverse sheaves with respect to the fixed stratification by Y and U. Then $\mathcal{M}(U)$ is the category of locally constant sheaves on U, i.e. its is equivalent to the category of pairs (V, A) where V is a finite dimensional vector space and $A\colon V \to V$ is an invertible linear operator (monodromy). The category $\mathcal{M}(Z)$ is the category of vector spaces. Functor ψ_f maps (V, A) to V. The monodromy operator is $A - \mathrm{Id}$. It follows that the category of perverse sheaves is equivalent to the category whose objects are diagrams of vector spaces

$$\Phi \underset{E}{\overset{F}{\rightleftarrows}} \Psi \tag{18}$$

such that $\mathrm{Id} + FE$ is invertible.

Another example is $X = \mathbb{C}^n$ stratified by $X_I = \{(x_1, \ldots, x_n) \,|\, x_i = 0 \text{ for } i \in I, x_i \neq 0 \text{ for } i \notin I\}$, where $I \subset \{1, \ldots, n\}$ is any subset. The category of perverse sheaves is equivalent to the category whose objects are finite-dimensional vector spaces V_I and linear maps:

$$E_{I,i}\colon V_I \to V_{I \cup \{i\}}, \qquad F_{I,i}\colon V_{I \cup \{i\}} \to V_I,$$

satisfying

$$\begin{aligned}
&E_{I\cup\{j\},i}E_{I,j} = E_{I\cup\{i\},j}E_{I,i},\\
&F_{I,j}F_{I\cup\{j\},i} = F_{I,i}F_{I\cup\{i\},j},\\
&E_{I\setminus\{j\},i}F_{I\setminus\{j\},j} = F_{I\cup\{i\}\setminus\{j\},j}E_{I,i},\\
&F_{I,i}E_{I,i} + \mathrm{Id} \ \text{ is invertible.}
\end{aligned}$$

5.6. Perverse sheaves on hyperplane arrangements.

M. Kapranov and V. Schechtman in [KS16] considered the case $X = \mathbb{C}^n$. The stratification they considered came from an arrangement $\mathcal{H}$ of linear hyperplanes in $\mathbb{R}^n$. Let $H = \{f_H = 0\}$, for all $H \in \mathcal{H}$. $\mathcal{L} = \mathcal{L}_{\mathcal{H}}$ is the poset of *flats* of $\mathcal{H}$, i.e. linear subspaces $\bigcap_{H\in\mathcal{I}} H$ for various subsets $\mathcal{I} \subset \mathcal{H}$. We assume that $\mathcal{L}$ contains $\{0\}$.

The stratification of $\mathbb{C}^n$ is given by

$$L^o_{\mathbb{C}} = L_{\mathbb{C}} \setminus_{H \not\supseteq L} H_{\mathbb{C}}.$$

We consider the category $\mathcal{M}(X, \mathcal{H})$ of perverse sheaves with middle perversity.

Each $x \in \mathbb{R}^n$ gives a *sign vector* $(\mathrm{sgn} f_H(x))_{H\in\mathcal{H}}$. Level sets of this vector subdivide $\mathbb{R}^n$ into locally closed subsets, *faces*. We denote by $\mathcal{C} = \mathcal{C}_{\mathcal{H}}$ the poset of faces with $C' \preceq C$ if $C' \subset \overline{C}$. An ordered triple (A, B, C) of faces is *collinear* if there are point $a \in A$, $b \in B$, $c \in C$ such that b lies in the straight line segment $[a, c]$.

A perverse sheaf gives a vector space E_C for every $C \in \mathcal{C}$. If $A \preceq B$ then we have the restriction map $\gamma_{A,B}\colon E_A \to E_B$, as every open set which contains a point of A has an open subset which is contained in B. Then, γ yield a representation of a quiver of $\mathcal{C}$ with arrows $A \to B$ if $A \preceq B$.

On the other hand, we have the Verdier duality which preserves $\mathcal{M}(X, \mathcal{H})$. The restriction morphisms for the Verdier dual give maps $\delta_{B,A}\colon E_B \to E_A$ which yield a representation of the opposite quiver.

Theorem 5.25. *The category $\mathcal{M}(X, \mathcal{H})$ is equivalent to the category of representations of the double quiver of the poset $\mathcal{C}$ satisfying the following conditions*

- *For any $A \preceq B$ we have $\gamma_{A,B}\delta_{B,A} = Id_{E_B}$. It allows us to define for any $A, B \in \mathcal{C}$ the transition map $\varphi_{A,B} = \gamma_{C,B}\delta_{A,C}\colon E_A \to E_B$ where C is any cell less than A and B.*
- *If (A, B, C) is a collinear triple of faces then $\varphi_{A,C} = \varphi_{B,C} f_{A,B}$.*
- *If C_1, C_2 are faces of the same dimension d lying in the same d-dimensional subspace, on the opposite sides of a $(d-1)$-dimensional face D then φ_{C_1,C_2} is an isomorphism.*

Let us again consider $X = \mathbb{C}$. Then we have 3 faces in $\mathcal{C}$: $-$, 0 and $+$. The category $\mathcal{M}(X, \mathcal{H})$ is equivalent to the category whose objects are diagrams of vector spaces

$$E_- \underset{\gamma_-}{\overset{\delta_-}{\rightleftarrows}} E_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} E_+ \tag{19}$$

such that

$$\gamma_-\delta_- = \mathrm{Id}_{E_-}, \qquad \gamma_+\delta_+ = \mathrm{Id}_{E_+},$$
$$\gamma_+\delta_- \text{ is an isomorphism}, \qquad \gamma_-\delta_+ \text{ is an isomorphism}.$$

Exercise 5.26. Show that the two descriptions of the category of perverse sheaves on $\mathbb{C}^1$ stratified by a point and its complement are equivalent.

(1) Show that the category with objects (19) is equivalent to the category $\mathcal{B}$ whose objects are (E_0, P_+, P_-) where $P_+, P_-\colon E_0 \to E_0$ are idempotents such that $P_-\colon \mathrm{Im}(P_+) \to \mathrm{Im}(P_-)$ and $P_+\colon \mathrm{Im}(P_-) \to \mathrm{Im}(P_+)$ are isomorphisms.
(2) For $(E_0, P_+, P_-) \in \mathcal{B}$ put $\Phi = \ker(P_-)$, $\Psi = \mathrm{Im}(P_+)$, $F = P_+$, $E = P_- - \mathrm{Id}$. Show that it defines a functor Υ from the category $\mathcal{B}$ to the category with objects (18).
(3) Given an object (18) let $E_0 = \Phi \oplus \Psi$, $P_+ = \begin{pmatrix} 0 & 0 \\ F & 1 \end{pmatrix}$ and $P_- = \begin{pmatrix} 0 & E \\ 0 & 1 \end{pmatrix}$. Show that this way we get a functor quasi-inverse to Υ.

6. Derived categories of coherent sheaves

This section is mainly based on [Huy06].

6.1. Crash course on spectral sequences.

A *spectral sequence* in an abelian category $\mathcal{A}$ is a collection of objects $(E_r^{p,q}, E^n), n, p, q, r \in \mathbb{Z}$, $r \geq 1$ and morphisms $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that

(i) $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$,
(ii) $E_{p+1}^{r,q}$ is isomorphic to $H^0(E_r^{p+\bullet r,q-\bullet r+1})$, the isomorphisms are part of the data
(iii) For any (p,q) there exists r_0 such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for $r \geq r_0$. In particular, $E_r^{p,q} \simeq E_{r_0}^{p,q}$ for all $r \geq r_0$. This object is called $E_\infty^{p,q}$.
(iv) There is a decreasing filtration $\ldots \subset F^{p+1}E^n \subset F^pE^n \subset \ldots \subset E^n$ such that $\bigcap F^pE^n = 0$, $\bigcup F^pE^n = E^n$ and $E_\infty^{p,q} \simeq F^pE^{p+q}/F^{p+1}E^{p+q}$.

We write it as

$$E_r^{p,q} \Rightarrow E^{p+q}.$$

We shall mostly encounter spectral sequences given by *double complexes* $K^{\bullet,\bullet}$. They have $d_I^{i,j}\colon K^{i,j} \to K^{i+1,j}$ and $d_{II}^{i,j}\colon K^{i,j} \to K^{i,j+1}$ satisfying

$$d_I^2 = 0, \qquad d_{II}^2 = 0, \qquad d_I d_{II} + d_{II} d_I = 0.$$

The total complex $K^\bullet$ is $K^n = \bigoplus_{i+j=n} K^{i,j}$ with $d = d_I + d_{II}$.

The total complex has a filtration $F^l K^n = \bigoplus_{j \geq l} K^{n-j,j}$. If $K^{n-l,l} = 0$ for $|l| >> 0$ then there is a spectral sequence

$$E_2^{p,q} = H_{II}^p H_I^q(K^{\bullet,\bullet}) \Rightarrow H^{p+q}(K^\bullet).$$

Proposition 6.1. *Let $F_1 \colon K^+(\mathcal{A}) \to K^+(\mathcal{B})$ and $F_2 \colon K^+(\mathcal{B}) \to K^+(\mathcal{C})$ be two exact functors. Suppose that $\mathcal{A}$ and $\mathcal{B}$ contain enough injectives and that the image under F_1 of a complex $I^\bullet$ of injective objects in $\mathcal{A}$ is cntained in an F_2-adapted triangulated subcategory K_{F_2}. Then for any $A^\bullet \in \mathcal{D}^+(\mathcal{A})$ there exists a spectral sequence*

$$E_2^{p,q} = R^p F_2(R^q F_1(A^\bullet)) \Rightarrow R^{p+q}(F_2 \circ F_1)(A^\bullet).$$

Proof. We take the Cartan-Eilenberg resolution of $A^\bullet$, i.e. we resolve by injective objects the cohomology and every term of $A^\bullet$. Our double complex is F_1 of this resolution. Then $H_{II}^p H_I^q(K^{\bullet,\bullet}) \simeq R^p F(H^q(A^\bullet))$. □

Let $A^\bullet$, $B^\bullet$ be complexes in $\mathcal{D}(\mathcal{A})$ with $B^\bullet$ bounded below. If $\mathcal{A}$ has enough injectives then there exists a spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(A^\bullet, H^q(B^\bullet)[p]) \Rightarrow \mathrm{Hom}(A^\bullet, B^\bullet[p+q]).$$

The first functor we use is the identity, the second $\mathrm{Hom}(A^\bullet, -)$.

Looking directly at a double complex induced by an injective resolution of $B^\bullet$ we also get

$$E_2^{p,q} = \mathrm{Hom}(H^{-q}(A^\bullet), B^\bullet[p]) \Rightarrow \mathrm{Hom}(A^\bullet, B^\bullet[p+q]).$$

6.2. **Preliminaries.**

A scheme X is quasi-separated if the diagonal map $\Delta \colon X \to X \times X$ is quasi-compact, i.e. the inverse image of any quasi-compact open set is quasi-compact.

By a sheaf on X we mean a sheaf of $\mathcal{O}_X$-modules. We denote by $\mathrm{Sh}(X)$ the category of sheaves of $\mathcal{O}_X$ modules and by $\mathrm{QCoh}(X)$ the category of quasi-coherent sheaves. Let $\mathcal{D}(\mathrm{Sh}(X))$ denote the unbounded derived category and $\mathcal{D}_{\mathrm{QCoh}(X)}(\mathrm{Sh}(X))$ its full subcategory of objects with quasi-coherent cohomology sheaves. By $\mathrm{Perf}(X)$ we denote the full subcategory of $\mathcal{D}(\mathrm{Sh}(X))$ whose objects are *perfect*, i.e. locally isomorphic to a bounded complex of vector bundles. If X is quasi-compact and quasi-separated then $\mathrm{Perf}(X)$ consists precisely of compact objects of $\mathcal{D}_{\mathrm{QCoh}(X)}(\mathrm{Sh}(X))$, i.e. objects such that morphisms from them commute with arbitrary direct sums.

If X is quasi-compact and separated then the functor $\mathcal{D}(\mathrm{QCoh}(X)) \to \mathcal{D}(\mathrm{Sh}(X))$ induces an equivalence $\mathcal{D}(\mathrm{QCoh}(X)) \xrightarrow{\simeq} \mathcal{D}_{\mathrm{QCoh}(X)}(\mathrm{Sh}(X))$.

Lemma 6.2. *If $G \twoheadrightarrow F$ is an $\mathcal{O}_X$-module homomorphism from a quasi-coherent sheaf $\mathcal{G}$ onto a coherent sheaf $\mathcal{F}$ on a noetherian scheme X, then there exists a coherent subsheaf $\mathcal{G}' \subset \mathcal{G}$ such that the composition $\mathcal{G}' \to \mathcal{G} \to \mathcal{F}$ is still surjective.*

Proof. The statement is clear for modules. We cover X by finitely many affine open, find a coherent subsheaf $\mathcal{H}_U$ for any U and take as $\mathcal{G}'$ the coherent subsheaf of $\mathcal{G}$ such that $\mathcal{H}_U \subset \mathcal{G}'|_U$. □

Proposition 6.3. *If X is noetherian we have an equivalence $\mathcal{D}^-(Coh(X)) \xrightarrow{\simeq} \mathcal{D}^-_{Coh(X)}(QCoh(X))$. This remains true if we replace '$-$' by 'b'.*

Proof. Let $\dots \mathcal{G}^n \to \dots \to \mathcal{G}^m \to 0$ be a bounded above complex of quasi-coherent sheaves. Assume that $\mathcal{G}^i$ are coherent for $i > j$. We have surjective morphisms $\mathcal{G}^j \to \mathrm{Im} d^j$ and $\ker(d^j) \to \mathcal{H}^j(\mathcal{G}^\bullet)$. $\mathrm{Im} d^j$ is a subsheaf of $\mathcal{G}^{j+1}$, hence it is coherent and, by assumption, so is $\mathcal{H}^j(\mathcal{G}^\bullet)$. Hence, there exist coherent subsheaves $\mathcal{G}^j_1 \subset \mathcal{G}^j$ and $\mathcal{G}^j_2 \subset \ker d^j \subset \mathcal{G}^j$. Let $\mathcal{G}'^j$ be the coherent subsheaf of $\mathcal{G}^j$ generated by them. Let also $\mathcal{G}'^{j-1}$ be the fiber product $\mathcal{G}^{j-1} \times_{\mathcal{G}^j} \mathcal{G}'^j$. Then complex with $\mathcal{G}^j$ and $\mathcal{G}^{j-1}$ exchanged by $\mathcal{G}'^j$ and $\mathcal{G}'^{j-1}$ is quasi-isomorphic to the original complex and has coherent terms for $i \geq j$. □

Proposition 6.4. *Let X be a Noeatherian separated scheme. Then $Perf(X) = \mathcal{D}^b(Coh(X))$ implies that X is regular. If X is of finite dimension the converse is also true.*

We shall be mostly interested in

$$\mathcal{D}^b(X) = \mathcal{D}^b(\mathrm{QCoh}(X)).$$

Category $\mathrm{Coh}(X)$ does not contain enough injective objects so in order to compute derived functors we pass to the category $\mathrm{QCoh}(X)$.

If X is projective over a field k then cohomology $H^i(X, F)$ of any coherent sheaf F is finite dimensional. This can be used to show that $\mathrm{Ext}^i(E, F)$ is finite dimensional for any two coherent sheaves E, F. Then the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(E^\bullet, \mathcal{H}^q(F^\bullet)) \Rightarrow \mathrm{Ext}^{p+q}(E^\bullet, F^\bullet)$$

implies that $\mathrm{Ext}^i(E^\bullet, F^\bullet)$ are finite dimensional for any $E^\bullet, F^\bullet \in \mathcal{D}^b(X)$.

Definition 6.5. The support of a complex $F^\bullet$ in $\mathcal{D}^b(X)$ is the union of the support of all its cohomology sheaves.

Lemma 6.6. *Suppose $F^\bullet \in \mathcal{D}^b(X)$ and $\operatorname{supp} F^\bullet = Z_1 \sqcup Z_2$, where $Z_1, Z_2 \subset X$ are disjoint closed subsets. Then $F^\bullet \simeq F_1^\bullet \oplus F_2^\bullet$ with $\operatorname{supp} F_i \subset Z_i$, for $i = 1, 2$.*

Proof. We proceed by induction on the length n of the complex. If $n = 1$ the case is clear. Let $F^\bullet$ be a complex of length at least two. Suppose m is minimal such that $\mathcal{H}^m(F^\bullet) \neq 0$. Then $H = \mathcal{H}^m(F^\bullet)$ can be decomposed as $H = H_1 \oplus H_2$ with $\operatorname{supp} H_i \subset Z_i$.

Consider distinguished triangle

$$H[-m] \to F^\bullet \to G^\bullet \to H[-m+1]$$

given by the truncation in the standard t-structure. By inductive hypothesis $G^\bullet = G_1^\bullet \oplus G_2^\bullet$ with $\operatorname{supp} G_i^\bullet \subset Z_i$.

We use spectral sequence

$$E_2^{p,q} = \operatorname{Hom}(\mathcal{H}^{-q}(G_1^\bullet), H_2[p]) \Rightarrow \operatorname{Hom}(G_1^\bullet, H_2[p+q])$$

to conclude that $\operatorname{Ext}^i(G_1^\bullet, H_2) = 0$, for all $i \in \mathbb{Z}$. Indeed, $\mathcal{H}^{-q}(G_1^\bullet)$ and H_2 are coherent sheaves with disjoint support so there are no morphisms between them (we can take injective resolution of push-forward of H_2 to $X\backslash Z_2$. As the embedding j of the complement of Z_2 to X is open, functor j_* maps injective objects to injective objects. This way we get an injective resolution of H_2 whose support is disjoint with the support of cohomology of $G_2^\bullet$, so all Ext-groups vanish.)

Similarly we show that there are no Ext-groups between $G_2^\bullet$ and H_1. So the map $G_1^\bullet \oplus G_2^\bullet \simeq G^\bullet \to H[-m+1] \simeq H_1[-m+1] \oplus H_2[-m+1]$ is diagonal. It follows that $F^\bullet$ is isomorphic to $F_1^\bullet \oplus F_2^\bullet$, for F_i defined as the fiber of $G_i^\bullet \to H_i[1-m]$:

$$H_i[-m] \to F_i^\bullet \to G_i^\bullet \to H_i[-m+1].$$

□

A triangulated category $\mathcal{D}$ is *decomposed into triangulated subcategories* $\mathcal{D}_1, \mathcal{D}_2$ if $\mathcal{D}_1$ and $\mathcal{D}_2$ are both non-trivial and $\mathcal{D}$ admits semi-orthogonal decompositions $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle = \langle \mathcal{D}_2, \mathcal{D}_1 \rangle$. Category $\mathcal{D}$ is *indecomposable* if it cannot be decomposed.

Proposition 6.7. *Let X be a noetherian scheme. Then $\mathcal{D}^b(X)$ is an indecomposable triangulated category if and only if X is connected.*

Proof. If $X = X_1 \sqcup X_2$ we take $\mathcal{D}_1 = \mathcal{D}^b(X_1)$ and $\mathcal{D}_2 = \mathcal{D}^b(X_2)$.

Assume now that X is connected and assume that $\mathcal{D}^b(X)$ is decomposed into $\mathcal{D}_1$ and $\mathcal{D}_2$. Let $\mathcal{O}_X = F_1 \oplus F_2$ be a decomposition of the structure sheaf. Both F_1 and F_2 are concentrated in zero'th cohomology, hence we can assume they are pure sheaves. As F_i is a subsheaf of $\mathcal{O}_X$ it is a coherent ideal sheaf of some closed subscheme $X_i \subset X$. Moreover, $\mathcal{O}_X = I_{X_1} + I_{X_2} \subset I_{X_1 \cap X_2}$ and $I_{X_1 \cup X_2} = I_{X_1} \cap I_{X_2} = 0$. Therefore, $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. As X is connected, we can assume that $X = X_1$. Therefore $\mathcal{O}_X \in \mathcal{D}_1$.

If $x \in X$ is a closed point then sheaf $\mathcal{O}_x$ is indecomposable. Therefore, $\mathcal{O}_x \in \mathcal{D}_1$ or $\mathcal{O}_x \in \mathcal{D}_2$. As $\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_x) \neq 0$, we know that $\mathcal{O}_x \in \mathcal{D}_1$.

Let now $F^\bullet$ be an object of $\mathcal{D}_2$. Let m be maximal such that $\mathcal{H}^m(F^\bullet) \neq 0$. Pick a closed point x in the support of $H = \mathcal{H}^m(F^\bullet)$. There exists surjection $H \to \mathcal{O}_x$. Applying $\operatorname{Hom}(-, \mathcal{O}_x[-m])$ to the distinguished triangle

$$\tau_{\leq m-1} F^\bullet \to F^\bullet \to H[-m] \to \tau_{\leq m-1} F^\bullet[1]$$

yields an isomorphism

$$\operatorname{Hom}(H[-m], \mathcal{O}_x[-m]) \simeq \operatorname{Hom}(F^\bullet, \mathcal{O}_x[-m])$$

as $\operatorname{Hom}(\tau_{\leq m-1} F^\bullet[1], \mathcal{O}_x[-m]) \simeq 0 \simeq \operatorname{Hom}(\tau_{\leq m-1} F^\bullet, \mathcal{O}_x[-m])$ for degree reasons ($\tau_{\leq m-1} F^\bullet \in \mathcal{D}^b(X)^{\leq m-1}$, $\tau_{\leq m-1} F^\bullet[1] \in \mathcal{D}^b(X)^{\leq m-2}$ and $\mathcal{O}_x[m] \in \mathcal{D}^b(X)^{\geq m}$). This contradicts the assumption that $\operatorname{Hom}(\mathcal{D}_1, \mathcal{D}_2) = 0$. $\square$

6.3. **Hom and $\mathcal{H}$om.**

Let $\mathcal{A}$ be an abelian category and $A^\bullet$ a complex of objects of $\mathcal{A}$. We have

$$\operatorname{Hom}^\bullet(A^\bullet, -) \colon K^+(\mathcal{A}) \to K(\mathrm{Ab})$$

which to a complex $B^\bullet$ assigns the complex $\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)$ with $\operatorname{Hom}^i(A^\bullet, B^\bullet) = \bigoplus \operatorname{Hom}(A^k, B^{i+k})$ and $d(f) = d_B \circ f - (-1)^i f \circ d_A$.

Assume $\mathcal{A}$ has enough injective objects. The category of complexes of injective objects is adapted to this functor and we may define

$$R\operatorname{Hom}^\bullet(A^\bullet, -) \colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}(\mathrm{Ab}).$$

Then

$$\operatorname{Ext}^i(A^\bullet, B^\bullet) := H^i(R\mathcal{H}om^\bullet(A^\bullet, B^\bullet)).$$

One checks that $\operatorname{Ext}^i(A^\bullet, B^\bullet) \simeq \operatorname{Hom}_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet[i])$.

The Ext-groups depend only on the quasi-isomorphism class of $A^\bullet$, so in fact we have

$$R\operatorname{Hom}^\bullet(=, -) \colon \mathcal{D}(\mathcal{A})^{\mathrm{op}} \otimes \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}(\mathrm{Ab}).$$

If $\mathcal{A}$ has enough projectives, we get

$$R\operatorname{Hom}^\bullet(=, -) \colon \mathcal{D}^-(\mathcal{A})^{\mathrm{op}} \otimes \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathrm{Ab}).$$

If $\mathcal{A} = \mathrm{QCoh}(X)$ a sheaf $F \in \mathrm{QCoh}(X)$ defines a left exact functor

$$\mathcal{H}om(F, -) \colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(X).$$

As $\mathrm{QCoh}(X)$ has enough injective objects, we have the derived functor

$$R\,\mathcal{H}\mathrm{om}(F,-)\colon \mathcal{D}^+(\mathrm{QCoh}(X)) \to \mathcal{D}^+(\mathrm{QCoh}(X)).$$

By definition

$$\mathcal{E}xt^i(F,E) = R^i\,\mathcal{H}\mathrm{om}(F,E).$$

If F is coherent, the definition is local:

$$\mathcal{E}xt^i(F,E)_x \simeq \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(F_x,E_x).$$

Restricting to coherent sheaves yields the functor

$$R\,\mathcal{H}\mathrm{om}(F,-)\colon \mathcal{D}^+(X) \to \mathcal{D}^+(X).$$

If X is regular, it can be further restricted to

$$R\,\mathcal{H}\mathrm{om}(F,-)\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X).$$

Let now $F^\bullet \in \mathcal{D}^-(X)$. We have

$$\begin{aligned}
&\mathcal{H}\mathrm{om}^\bullet(F^\bullet,-)\colon K^+(\mathrm{QCoh}(X)) \to K^+(\mathrm{QCoh}(X)),\\
&\mathcal{H}\mathrm{om}^i(F^\bullet,E^\bullet) := \prod \mathcal{H}\mathrm{om}(F^p,E^{i+p}), \quad d = d_E - (-1)^i d_F.
\end{aligned}$$

The class of complexes of injective sheaves is adapted to the functor $\mathcal{H}\mathrm{om}^\bullet(F^\bullet,-)$. As in the global case we get

$$R\,\mathcal{H}\mathrm{om}(=,-)\colon \mathcal{D}^-(\mathrm{QCoh}(X))^{\mathrm{op}} \otimes \mathcal{D}^+(\mathrm{QCoh}(X)) \to \mathcal{D}^+(\mathrm{QCoh}(X)).$$

If $F^\bullet$ is a complex of locally free sheaves then $R\,\mathcal{H}\mathrm{om}(F^\bullet,-)$ can be computed as $\mathcal{H}\mathrm{om}(F^\bullet,-)$. This is the consequence of the fact that we can calculate cohomology of $\mathcal{H}\mathrm{om}$ locally and free modules are projective.

The theory of spectral sequences work here and we have:

$$\begin{aligned}
E_2^{p,q} &= \mathcal{E}xt^p(F^\bullet,\mathcal{H}^q(E^\bullet)) \Rightarrow \mathcal{E}xt^{p+q}(F^\bullet,E^\bullet),\\
E_2^{p,q} &= \mathcal{E}xt^p(\mathcal{H}^{-q}(F^\bullet),E^\bullet) \Rightarrow \mathcal{E}xt^{p+q}(F^\bullet,E^\bullet).
\end{aligned}$$

We also have the trace map

$$R\,\mathcal{H}\mathrm{om}(E^\bullet,E^\bullet) \to \mathcal{O}_X.$$

We assume that X is regular and replace $E^\bullet$ with a complex of locally free sheaves. Then $R\,\mathcal{H}\mathrm{om}(E^\bullet,E^\bullet) \simeq \mathcal{H}\mathrm{om}(E^\bullet,E^\bullet)$ and $\mathcal{H}\mathrm{om}^0(E^\bullet,E^\bullet) \simeq \bigoplus \mathcal{H}\mathrm{om}(E^i,E^i)$ On each of the component we have the trace map $\mathcal{H}\mathrm{om}(E^i,E^i) \to \mathcal{O}_X$ and

$$\mathrm{tr}_{E^\bullet} = \bigoplus (-1)^i \mathrm{tr}_{E^i}.$$

The dual of $F^\bullet \in \mathcal{D}^-(\mathrm{QCoh}(X))$ is

$$F^{\bullet\vee} := R\mathcal{H}\mathrm{om}(F^\bullet, \mathcal{O}_X) \in \mathcal{D}^+(\mathrm{QCoh}(X)).$$

If $F^\bullet$ is a complex of locally free sheaves, $F^{\bullet\vee}$ is the complex with terms $\mathcal{H}\mathrm{om}(F^i, \mathcal{O}_X)$.

6.4. **Serre duality.**

Definition 6.8. Let X be a smooth projective variety of dimension n. Then one defined the *Serre functor* $\mathbb{S}_X$ as the composition:

$$\mathcal{D}^*(X) \xrightarrow{\omega_X \otimes (-)} \mathcal{D}^*(X) \xrightarrow{[n]} \mathcal{D}^*(X)$$

where $* = b, +, -$.

Theorem 6.9. *Let X be a smooth projective variety. Then for any $E^\bullet, F^\bullet$ there exists functorial isomorphism*

$$Hom(E^\bullet, F^\bullet) \xrightarrow{\simeq} Hom(F^\bullet, \mathbb{S}_X(E^\bullet))^\vee.$$

Exercise 6.10. Suppose that E and F are coherent sheaves on a smooth projective variety X of dimension n. Prove that $\mathrm{Ext}^i(E, F) = 0$, for $i > n$.

Exercise 6.11. Let C be a smooth projective curve. Prove that any object of $\mathcal{D}^b(C)$ is isomorphic to a direct sum $\bigoplus E_i[i]$, where $E_i \in \mathrm{Coh}(C)$.

6.5. **Derived functors in algebraic geometry.**

Cohomology Let X be a noetherian k-scheme. Global section functor $\Gamma\colon \mathrm{QCoh}(X) \to$ mod-k is left exact. As $\mathrm{QCoh}(X)$ has enough injective objects, we can form the derived functor

$$R\Gamma\colon \mathcal{D}^+(\mathrm{QCoh}(X)) \to \mathcal{D}^+(\text{mod-}k).$$

Theorem 6.12 (Grothendieck)**.** *For any quasi-coherent sheaf F on a noetherian scheme X, $H^i(X, F) = 0$ for $i > \dim(X)$.*

Therefore, we can restrict $R\Gamma$ to $\mathcal{D}^b(\mathrm{QCoh}(X))$, the functor will take values in $\mathcal{D}^b(\text{mod-}k)$.

Theorem 6.13 (Serre)**.** *If F is a coherent sheaf on a projective scheme X over a field, then all cohomology groups $H^i(X, F)$ are of finite dimension.*

Direct image Let $f\colon X \to Y$ be a morphism of noetherian schemes. The direct image is a left exact functor

$$f_*\colon \text{QCoh}(X) \to \text{QCoh}(Y).$$

As QCoh(X) has enough injective objects, we can derive to get

$$Rf_*\colon \mathcal{D}^+(\text{QCoh}(X)) \to \mathcal{D}^+(\text{QCoh}(Y)).$$

The cohomology sheaves of $Rf_*(F^\bullet)$ are called *higher direct images.*

Theorem 6.14. *For a quasi-coherent sheaf F on X and a morphism $f\colon X \to Y$ of noetherian schemes the higher direct images R^if_*F are trivial for $i > \dim X$.*

Therefore, we can restrict to bounded derived categories of quasi-coherent sheaves. Finally

Theorem 6.15. *If $f\colon X \to Y$ is a projective (or proper) morphism of noetherian schemes then the higher direct images $R^if_*(F)$ of a coherent sheaf on X are again coherent.*

Flabby sheaves, i.e. sheaves such that for any $U \subset X$ open the restriction map $F(X) \to F(U)$ is surjective, form an adapted class for the direct image functor.

Lemma 6.16. *Any injective $\mathcal{O}_X$-sheaf is flabby. Any flabby sheaf F on X is f_*-acyclic for any morphism $f\colon X \to Y$ and, moreover, f_*F is again flabby.*

For a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ we can conclude that $R(g\circ f)_* = Rg_* \circ Rf_*$. Moreover, we get *Leray* spectral sequence:

$$E_2^{p,q} = R^pg_*(R^qf_*(F^\bullet)) \Rightarrow R^{p+q}(g\circ f)_*(F^\bullet)$$

for any $F^\bullet \in \mathcal{D}^b(\text{QCoh}(X))$.

Tensor product Let F be a coherent sheaf on a projective k-scheme X. It defines right exact functor

$$F \otimes (-)\colon \text{Coh}(X) \to \text{Coh}(X).$$

Any coherent sheaf admits a resolution by locally free sheaves (we use the fact that X is projective here. More generally, the *resolution property* holds for quasi-projective schemes over affine schemes but there are some normal 3-dimensional toric examples for which the resolution property is not known). As locally free sheaves form an adapted class for tensor product (at every point they are projective, hence flat), we get

$$F \otimes (-)\colon \mathcal{D}^-(X) \to \mathcal{D}^-(X).$$

By definition

$$\mathcal{T}\text{or}_i(F, E) = \mathcal{H}^{-i}(F \otimes^L E).$$

If X is smooth of dimension n any coherent sheaf admits a locally free resolution of length n. Hence, then the functor $F \otimes^L (-)$ restricts to $\mathcal{D}^b(X)$.

In more general situation let $F^\bullet$ be a bounded above complex of coherent sheaves on X. Define

$$F^\bullet \otimes (-)\colon \mathcal{K}^-(\mathrm{Coh}(X)) \to \mathcal{K}^-(\mathrm{Coh}(X)),$$
$$(F^\bullet \otimes E^\bullet)^i = \bigoplus_{p+q=i} F^p \otimes E^q, \qquad d = d_F \otimes 1 + (-1)^i 1 \otimes d_E.$$

The subcategory of complexes of locally free sheaves is adapted to $F^\bullet \otimes (-)$ (we use two spectral sequences of the double complex and the fact that locally free sheaves are adapted to tensor product with a sheaf to show that if $E^\bullet$ is an acyclic complex of locally free sheaves then $F^\bullet \otimes E^\bullet$ is also acyclic). So we can consider the derived functor

$$F^\bullet \otimes^L (-)\colon \mathcal{D}^-(X) \to \mathcal{D}^-(X).$$

For a complex of locally free sheaves $E^\bullet$ and an acyclic complex $F^\bullet$ the complex $F^\bullet \otimes E^\bullet$ is acyclic, therefore the functor

$$(-) \otimes^L (-)\colon \mathcal{K}^-(\mathrm{Coh}(X)) \times D^-(X) \to D^-(X)$$

need not be derived in the first factor and descends to the bifunctor $(-) \otimes^L (-)$ for the derived categories.

If X is smooth, the functor restricts to the bounded derived categories.

Computing the derived tensor product as the ordinary tensor product of complexes of locally free sheaves yields functorial isomorphisms:

$$F^\bullet \otimes^L E^\bullet \simeq E^\bullet \otimes^L F^\bullet,$$
$$F^\bullet \otimes^L (E^\bullet \otimes^L G^\bullet) \simeq (E^\bullet \otimes^L F^\bullet) \otimes^L G^\bullet.$$

We also have spectral sequence

$$E_2^{p,q} = \mathcal{T}\mathrm{or}_{-p}(\mathcal{H}^q(F^\bullet), E^\bullet) \Rightarrow \mathcal{T}\mathrm{or}_{-p-q}(F^\bullet, E^\bullet).$$

We may assume that $E^\bullet$ is a complex of locally free sheaves. Then $\mathcal{T}\mathrm{or}_{-p}(\mathcal{H}^q(F^\bullet), E^\bullet)$ can be computed via the p-th cohomology of the complex $\mathcal{H}^q(F^\bullet) \otimes E^\bullet$. Similarly, $\mathcal{T}\mathrm{or}_{-p-q}(F^\bullet, E^\bullet)$ is the $p+q$-th cohomology of the complex $F^\bullet \otimes E^\bullet$ which is the double complex of a double complex. The spectral sequence is then the standard spectral sequence of a double complex.

Inverse image Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then

$$f^*\colon \mathrm{Sh}_{\mathcal{O}_Y}(Y) \to \mathrm{Sh}_{\mathcal{O}_X}(X)$$

is a right exact functor, the composition of an exact functor f^{-1} with $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (-)$. Then

$$Lf^* := (\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} (-)) \circ f^{-1} \colon \mathcal{D}^-(Y) \to \mathcal{D}^-(X)$$

and we have a spectral sequence

$$E_2^{p,q} = L^p f^*(\mathcal{H}^q(E^\bullet)) \Rightarrow L^{p+q} f^*(E^\bullet).$$

If $f\colon X \to Y$ is flat, functor f^* is exact and need not to be derived.

Consider a morphism $S \to X$. For a closed point $x \in X$ denote $i_x \colon S_x \to S$ the closed embedding of the fibre over x.

Lemma 6.17. *Suppose $Q \in \mathcal{D}^b(S)$ and assume that for all closed points $x \in X$ the derived pull-back $Li_x^*Q \in \mathcal{D}^b(S_x)$ is a complex concentrated in degree zero, i.e. a sheaf. Then Q is isomorphic to a sheaf which is flat over X.*

Proof. We look at the spectral sequence

$$E_2^{p,q} = \mathcal{H}^p(Li_x^*\mathcal{H}^q(Q)) \Rightarrow \mathcal{H}^{p+q}(Li_x^*Q).$$

The right hand side is trivial except possibly for $p+q=0$. Choose m maximal with $\mathcal{H}^m(Q) \neq 0$. Then there exists a closed point $x \in X$ with $E_2^{0,m} = \mathcal{H}^0(Li_x^*\mathcal{H}^m(Q)) \neq 0$. This non-triviality survives passing to the limit of the spectral sequence, so $m = 0$.

Also $E_2^{0,-1} = \mathcal{H}^{-1}(Li_x^*\mathcal{H}^0(Q))$ with $x \in X$ arbitrary also survives and must, therefore, be trivial. This shows that $\mathcal{H}^0(Q)$ is flat over X. Hence, $E_2^{p,0} = \mathcal{H}^p(Li_x^*\mathcal{H}^0(Q)) = 0$.

It remains to show that Q has no non-trivial negative cohomology. Again, choose m maximal among all non-trivial negative cohomology and a point x in the support of $\mathcal{H}^m(Q)$. Since all $E_2^{-p,q} = \mathcal{H}^{-p}(Li_x^*\mathcal{H}^q(Q)) = 0$ for $q > m$ and $p < 0$, this would again yield the contradiction $E^m = \mathcal{H}^m(Li_x^*Q) = \mathcal{H}^0(Li_x^*\mathcal{H}^m(Q)) \neq 0$ in the limit. $\square$

As higher derived functors of f_* can be calculated by integrating along the fibers of f we know that $R^i f_*(\mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on X and $i \geq \max \dim X_f$. The following proposition tells us when higher derived functors of f^* vanish.

Proposition 6.18. *For $f\colon X \to Y$, a projective morphism between smooth projective varieties $L^k f^*(\mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on Y and $k > dimY - dimX + max\ dimX_f$.*

Proof. As f is projective we have a factorization

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X \times Y \\ & \searrow^{f} & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where i is given by the graph of f and p is flat. Let $Z \subset Y$ be a closed subscheme of Y. Then

$$L^\bullet f^*(\mathcal{O}_Z) = L^\bullet i^* p^*(\mathcal{O}_Z) = L^\bullet i^* \mathcal{O}_{p^{-1}(Z)}.$$

The functor i_* is exact and hence understanding cohomology sheaves of the complex $L^\bullet i^*(\mathcal{O}_{p^{-1}(Z)})$ is the same as understanding the cohomology sheaves of $i_* L^\bullet i^*(\mathcal{O}_{p^{-1}(Z)}) = \mathcal{O}_{p^{-1}(Z)} \otimes^L i_* \mathcal{O}_X$. The image of X in $X \times Y$ is locally a complete intersection and hence there exists a vector bundle E of rank $r = \dim Y$ and $s \in H^0(X \times Y, E^*)$ such that $X = Z(s)$ and we have a Koszul complex

$$\ldots \to \Lambda^2 E \to E \to \mathcal{O}_{X \times Y}$$

which we can also restrict to $p^{-1}(Z)$. The set of zeroes of s restricted to $p^{-1}(Z)$ is $Z(s|_{p^{-1}(Z)}) = i(f^{-1}(Z))$. We can estimate $\dim f^{-1}(Z) \leq \dim Z + \max \dim X_f$.

We need one fact about Koszul complex of a not necessarily regular section. Namely, let E be a vector bundle on a smooth scheme W and $s \in H^0(W, E^*)$ let be a section. Put $t = \mathrm{codim}_W(\{s = 0\})$. Then

$$\max\{n \mid \mathcal{H}^{-n}(\lambda^\bullet E) \neq 0\} = \mathrm{rk}(E) - t.$$

Indeed, locally $E = \mathcal{O}^{\oplus \mathrm{rk}(E)}$ and s is given by $\mathrm{rk}(E)$ functions. If the functions do not form a regular sequence then after permutation we can assume that the first t functions do. The Koszul resolution for s is then a tensor product of the exact Koszul complex for the first t functions and the Koszul sequence for the remaining $\mathrm{rk}(E) - t$ functions which is not exact. When $\mathcal{E}^\bullet$ is an exact complex of flat modules such that $H^i(\mathcal{E}^\bullet) = 0$ for $i \neq 0$ and $\mathcal{F}^\bullet$ is any complex then the second page of a spectral sequence

$$E_2^{pq} = H^p(\mathcal{E}^\bullet \otimes \mathcal{F}^q) \Rightarrow H^{p+q}(\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet)$$

degenerates and hence $\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet$ has nonzero cohomology groups in the same places as $\mathcal{F}^\bullet$ does.

Now, let our W be $p^{-1}(Z)$. Then $\dim W = \dim Z + \dim X$. We also know that $\dim Z(s|_{p^{-1}(Z)}) \leq \dim Z + \max \dim X_f$. Hence,

$$\mathrm{codim}_W Z(s|_{p^{-1}(Z)}) = \dim W - \dim Z(s|_{p^{-1}(Z)}) \geq \dim X - \max \dim X_f.$$

Then, the bound for the nonzero cohomology of the Koszul complex of $i_* \mathcal{O}_X$ when restricted to $p^{-1}(Z)$ is

$$k = \mathrm{rk}(E) - \mathrm{codim}_W Z(s|_{p^{-1}(Z)}) \leq \dim Y - \dim X + \max \dim X_f.$$

We have hence proved that for structure sheaves of closed subschemes $L^k f^*(\mathcal{O}_Z) = 0$ for $k > \dim Y - \dim X + \max \dim X_f$. The question about vanishing of higher derived functors of f^* is local on Y and so we can assume that $Y = \mathrm{Spec} R$ for some noetherian ring R and that we have a finitely generated R-module M. Then (by theorem 7.E from Matsumura, for example) M admits a finite filtration with quotients of the form R/P for some prime ideal $P \subset R$. Let $M' \hookrightarrow M \to R/P$ be a short exact sequence. Then M' has a shorter filtration than M and we can assume by induction that the higher Tor-groups vanish both for M' and R/P. The long exact sequence then proves that higher Tor-functors vanish also for M. □

Compatibilities

(1) Let $f\colon X \to Y$ be a proper morphism of projective schemes over a field k. For any $F^\bullet \in \mathcal{D}^b(X)$, $E^\bullet \in \mathcal{D}^b(Y)$ there exists natural isomorphism, *projection formula*

$$Rf_*(F^\bullet) \otimes^l E^\bullet \simeq Rf_*(F^\bullet \otimes Lf^* E^\bullet).$$

To prove it we present $E^\bullet$ as a complex of locally free sheaves.

(2) Let $f\colon X \to Y$ be a morphism of projective schemes and let $E^\bullet, F^\bullet \in \mathcal{D}^b(Y)$. Then

$$Lf^*(F^\bullet) \otimes^L Lf^*(E^\bullet) \simeq Lf^*(F^\bullet \otimes E^\bullet).$$

(3) Let $f\colon X \to Y$ be a projective morphism. Then $Lf^* \dashv Rf_*$ because we can present the complex on Y as a complex of locally free sheaves and the complex on X as a complex of injective sheaves.

(4) Suppose X is smooth and projective and take $E^\bullet, F^\bullet, G^\bullet \in \mathcal{D}^b(X)$. Then

$$R\mathcal{H}\mathrm{om}(F^\bullet, E^\bullet) \otimes^L G^\bullet = R\mathcal{H}\mathrm{om}(F^\bullet, E^\bullet \otimes^L G^\bullet),$$
$$R\mathcal{H}\mathrm{om}(F^\bullet, R\mathcal{H}\mathrm{om}(E^\bullet, G^\bullet)) \simeq R\mathcal{H}\mathrm{om}(F^\bullet \otimes^L E^\bullet, G^\bullet),$$
$$R\mathcal{H}\mathrm{om}(F^\bullet, E^\bullet \otimes^L G^\bullet) \simeq R\mathcal{H}\mathrm{om}(R\mathcal{H}\mathrm{om}(E^\bullet, F^\bullet), G^\bullet).$$

because we can assume that all three complexes are complexes of locally free sheaves.

(5) Let $F^\bullet \in \mathcal{D}^-(X)$. Then $\Gamma \circ \mathcal{H}\mathrm{om}^\bullet(F^\bullet, -) = \mathrm{Hom}^\bullet(F^\bullet, -)$. Hence

$$R\Gamma \circ R\mathcal{H}\mathrm{om}(F^\bullet, -) = R\mathcal{H}\mathrm{om}(F^\bullet, -)$$

and we get a spectral sequence

$$E_2^{p.q} = H^p(X, \mathcal{E}\mathrm{xt}^q(F^\bullet, E^\bullet)) \Rightarrow \mathrm{Ext}^{p+q}(F^\bullet, E^\bullet).$$

(6) Let $f\colon X \to Y$ be a morphism of projective schemes, $F^\bullet \in \mathcal{D}^-(Y)$, $E^\bullet \in \mathcal{D}^b(Y)$. Then

$$Lf^* R\mathcal{H}\mathrm{om}_Y(F^\bullet, E^\bullet) \simeq R\mathcal{H}\mathrm{om}_X(Lf^*F^\bullet, Lf^*E^\bullet).$$

(7) Consider a fibre product diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{v} & Y \\ \downarrow g & & \downarrow f \\ X & \xrightarrow{u} & Z \end{array}$$

with $u\colon X \to Z$ flat and $f\colon Y \to Z$ proper. Then *flat base change* asserts, for any $F^{\bullet} \in \mathcal{D}(\mathrm{QCoh}(Y))$,

$$u^* Rf_* F^{\bullet} \xrightarrow{\simeq} Rg_* v^* F^{\bullet}.$$

Let us consider the special case of the product $X \times Y$, $q\colon X \times Y \to X$, $p\colon X \times Y \to Y$. For $F^{\bullet} \in \mathcal{D}^b(Y)$ the flat base change yields

$$q_* p^* F^{\bullet} \simeq R\Gamma(Y, F^{\bullet}) \otimes \mathcal{O}_X.$$

As a consequence we get *Künneth formula*, for $F \in \mathcal{D}^b(Y)$, $E \in \mathcal{D}^b(X)$,

$$R\Gamma(X \times Y, q^* E^{\bullet} \otimes^L p^* F^{\bullet}) \simeq R\Gamma(X, Rq_*(q^* E^{\bullet} \otimes^L p^* F^{\bullet})) \simeq$$
$$\simeq R\Gamma(X, E^{\bullet} \otimes^L Rq_* p^* F^{\bullet}) \simeq R\Gamma(X, E^{\bullet} \otimes R\Gamma(Y, F^{\bullet}) \otimes \mathcal{O}_X) \simeq$$
$$\simeq R\Gamma(X, E^{\bullet}) \otimes R\Gamma(Y, F^{\bullet}).$$

6.6. **Grothendieck-Verdier duality.**

This part is based on [Nee10] and [Sta13, Section 46.3].

Let $f\colon X \to Y$ be a morphism of quasi-separated, quasi-compact schemes. The pushforward functor $Rf_*\colon \mathcal{D}(\mathrm{QCoh}(X)) \to \mathcal{D}(\mathrm{QCoh}(Y))$ (defined via h-injective complexes) has right adjoint $f^!$. The proof is 'abstract nonsense', functor Rf_* maps direct sums to direct sums, hence it has a right adjoint.

In order to formulate some properties of the functor $f^!$ we need to introduce the following

Conjecture 6.19. *Let X be a quasi-compact separated scheme and let $U \subset X$ be a quasi-compact open subset. Let $j\colon U \to X$ be the inclusion. Then there exists a compact object $E \in \mathcal{D}(QCoh(X))$ and an integer $n \geq 1$ such that $Rj_* \mathcal{O}_U \in \langle E \rangle_n$, the smallest subcategory obtained from $\mathcal{E}$, its shifts and direct sums of copies of those, by taking n cones.*

The conjecture holds under the assumptions that

(1) X is noetherian, finite dimensional and smooth over a finite dimensional noetherian ring,
(2) X is a locally closed subscheme of Y and Conjecture holds for Y,
(3) There is an affine morphism $X \to Y$ and the Conjecture holds for Y.

Let $f\colon X \to Y$ be such that Rf_* maps perfect complexes to perfect complexes and X satisfies the Conjecture. Then $f^!$ restricts to a functor $\mathcal{D}^b(Y) \to \mathcal{D}^b(X)$.

Grothendieck-Verdier duality states that for $F^\bullet \in \mathcal{D}_{\mathrm{QCoh}(X)}(\mathcal{O}_X)$ and $E^\bullet \in \mathcal{D}_{\mathrm{QCoh}(Y)}(\mathcal{O}_Y)$ functors Rf_* and $f^!$ are locally adjoint:

$$Rf_* R\mathcal{H}\mathrm{om}_X(F^\bullet, f^!(E^\bullet)) \simeq R\mathcal{H}\mathrm{om}_Y(Rf_*(F^\bullet), E^\bullet).$$

Let now X be a noetherian separated scheme. There exists *dualizing complex* $\omega_X^\bullet \in \mathcal{D}^b(X)$ so that the natural functor

$$R\mathcal{H}\mathrm{om}(-, \omega_X^\bullet)\colon \mathcal{D}^b(X)^{\mathrm{op}} \to \mathcal{D}^b(X)$$

is an equivalence.

The dualizing complex is unique up to shift and twist by a line bundle. The variety X is Gorenstein if and only if $\omega_X^\bullet$ is a line bundle. X is Cohen-Macaulay if and only if $\omega_X^\bullet$ is a sheaf (up to shift, of course), [Har66].

If $f\colon X \to Y$ is such that Rf_* maps perfect complexes to perfect complexes and X satisfies the Conjecture, then $f^!\omega_Y^\bullet \simeq \omega_X^\bullet$.

As a result, we can define the dualizing complex for a proper noetherian scheme X over a field k for which the Conjecture holds. Let $p\colon X \to \operatorname{Spec} k$ be the structure morphism. Then

$$\omega_X^\bullet \simeq p^!(\mathcal{O}_{\operatorname{Spec} k}).$$

6.7. **Spanning classes in the derived category.**

Definition 6.20. A collection Ω of objects in a triangulated category $\mathcal{D}$ is a *spanning class* of $\mathcal{D}$ if for any $D \in \mathcal{D}$

(i) if $\mathrm{Hom}(A, D[i]) = 0$ for all $A \in \Omega$, all $i \in \mathbb{Z}$, then $D \simeq 0$.
(ii) If $\mathrm{Hom}(D[i], A) = 0$ for all $A \in \Omega$, all $i \in \mathbb{Z}$, then $D \simeq 0$.

Exercise 6.21. If $\mathcal{D}$ is endowed with a Serre functor conditions (i) and (ii) of the above definition are equivalent.

Proposition 6.22. *Let X be a smooth projective variety. Then the objects $\mathcal{O}_x$, for $x \in X$ closed, span $\mathcal{D}^b(X)$.*

Proof. As $\mathcal{D}^b(X)$ has Serre functor, it suffices to check that for any non-zero $F^\bullet \in \mathcal{D}^b(X)$ there exists $x \in X$ such that $\mathrm{Hom}(F^\bullet, \mathcal{O}_x[i]) \neq 0$, for some i.

Let m be maximal such that $\mathcal{H}^m(F^\bullet)$ is non-trivial. Let x be a closed point in the support of $\mathcal{H}^m(F^\bullet)$. Then $\mathrm{Hom}(\mathcal{H}^m(F^\bullet), \mathcal{O}_x) \neq 0$. Distinguished triangle

$$\tau_{\leq m-1}F^\bullet \to F^\bullet \to \mathcal{H}^m(F^\bullet)[-m]$$

implies an isomorphism $\mathrm{Hom}(F^\bullet, \mathcal{O}_x[-m]) \simeq \mathrm{Hom}(\mathcal{H}^m(F^\bullet)[-m], \mathcal{O}_X[-m])$ which finishes the proof. □

Spanning classes prove useful when proving fully faithfulness:

Exercise 6.23. Let $F\colon \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories with left and right adjoints $G \dashv F \dashv H$. Suppose that Ω is a spanning class for $\mathcal{D}$ and that for all $A, B \in \Omega$ the natural homomorphisms

$$F\colon\ \mathrm{Hom}(A, B[i]) \to \mathrm{Hom}(F(A), F(B)[i])$$

are bijective for all i. Show that F is fully faithful.

Definition 6.24. A sequence of objects $L_i \in \mathcal{A}$, $i \in \mathbb{Z}$ in a k-linear abelian category $\mathcal{A}$ is *ample* if for any $A \in \mathcal{A}$ there exists an integer $i_0(A)$ such that for $i < i_0(A)$ the following conditions are satisfied:

(i) The natural morphism $\mathrm{Hom}(L_i, A) \otimes L_i \to A$ is surjective.
(ii) If $j \neq 0$ then $\mathrm{Hom}(L_i, A[j]) = 0$.
(iii) $Hom(A, L_i) = 0$.

An abelian category $\mathcal{A}$ has *finite cohomological dimension* if there exists N such that for $i > N$ and any $A, B \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(A, B[i]) = 0$.

Spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(H^{-q}(A^\bullet), B[p]) \Rightarrow \mathrm{Hom}(A^\bullet, B[p+q])$$

implies that for any $A^\bullet \in \mathcal{D}^b(\mathcal{A})$ and $B \in \mathcal{A}$ there exists N such that $\mathrm{Hom}(A^\bullet, B[i]) = 0$ for $i > N$.

Proposition 6.25. *Let L_i, $i \in \mathbb{Z}$ be an ample sequence in a k-linear abelian category $\mathcal{A}$ of finite cohomological dimension. Then L_i span the derived category $\mathcal{D}^b(\mathcal{A})$.*

Proof. Let $A^\bullet \in \mathcal{D}^b(\mathcal{A})$ be such that $\mathrm{Hom}(L_i, A^\bullet[j]) = 0$ for all i, j. Let n be minimal such that $\mathcal{H}^n(A^\bullet) \neq 0$. Then, as $\mathrm{Hom}(L_i[-n], \tau_{>n}A^\bullet[-1]) = 0$, $\mathrm{Hom}(L_i[-n], \mathcal{H}^n(A^\bullet)[-n]) \subset \mathrm{Hom}(L_i[-n], A^\bullet)$. By assumption, the latter space is zero, hence $\mathrm{Hom}(L_i, \mathcal{H}^n(A^\bullet)) = 0$ for all i. On the other hand, there exists i_0 such that $\mathrm{Hom}(L_i, \mathcal{H}^n(A^\bullet)) \otimes L_i \to H^n(A^\bullet)$ is surjective. The contradiction implies that $A^\bullet \simeq 0$.

Let us now assume that $\mathrm{Hom}(A^\bullet, L_i[j]) = 0$ for all i, j. Let n be maximal such that $H^n(A^\bullet) \neq 0$. Let i be such that $\mathrm{Hom}(L_i, H^n(A^\bullet)) \otimes L_i \to H^n(A^\bullet)$ is surjective and let B_1 denote its kernel. As $\mathrm{Hom}(A^\bullet, L_i[j]) = 0$ for all j, $0 \neq \mathrm{Hom}(A^\bullet, H^n(A^\bullet)[-n]) \subset \mathrm{Hom}(A^\bullet, B_1[1-n])$.

We continue exchanging $H^n(A^\bullet)$ with B_1; there exists $i_1 < i$ such that $\mathrm{Hom}(L_{i_1}, B_1) \otimes L_{i_1} \to B_1$ is surjective. If B_2 is its kernel then $0 \neq \mathrm{Hom}(A^\bullet, B_1[1-n]) \subset \mathrm{Hom}(A^\bullet, B_2[2-n])$. This way we obtain a sequence of objects $B_j \in \mathcal{A}$ such that $\mathrm{Hom}(A^\bullet, B_j[j-n]) \neq 0$ which contradicts the assumption that $\mathcal{A}$ is of finite homological dimension. Hence $\mathcal{H}^n(A^\bullet) = 0$ for all n. □

Proposition 6.26. *Let X be a projective variety over a field. If L is an ample line bundle on X then powers of L, L^i, $i \in \mathbb{Z}$ form an ample sequence in the abelian category $Coh(X)$.*

Proof. By definition, an ample line bundle L has the property that for any coherent sheaf F on X there exists n_0 such that for any $n \geq n_0$ the sheaf $F \otimes L^n$ is globally generated. This means that

$$H^0(X, F \otimes L^n) \otimes O_X \to F \otimes L^n$$

is surjective. Tensoring with L^{-n} we get a surjective map

$$L^n \otimes \mathrm{Hom}(L^{-n}, F) \to F.$$

Condition (ii) follows from the fundamental theorem of Serre that $H^i(X, F \otimes L^n) = 0$ for any $i > 0$ and any $n > n_0$.

It remains to check that $\mathrm{Hom}(F, L^i) = 0$. We can assume taht L is very ample (by passing to some power L^k and proving the vanishing for $F \otimes L^i$, $i = 1, \ldots, k$).

As L is very ample for any $x \in X$ there exists a section $0 \neq s_x \in H^0(X, L)$ with $0 = s_x(x) \in L(x)$. If $0 \neq \varphi\colon F \to L^i$ then there exists a closed point x such that $\varphi(x)\colon F(x) \to L^i(x)$ is non-trivial. Hence, φ is not in the image of the inclusion $\mathrm{Hom}(F, L^{i-1}) \xrightarrow{s_x} \mathrm{Hom}(F, L^i)$. It follows that $\dim \mathrm{Hom}(F, L^{i-1}) < \dim \mathrm{Hom}(F, L^i)$. As $\mathrm{Hom}(F, L)$ is finitely dimensional, it can happen only finitely many times before $\mathrm{Hom}(F, L^i)$ become trivial. □

Theorem 6.27. *Let $F\colon \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A})$ be an exact autoequivalence. Suppose $f\colon Id_{\{L_i\}} \xrightarrow{\simeq} F|_{\{L_i\}}$ is an isomorphism of functors on the full subcategory $\{L_i\}$ given by an ample sequence L_i in $\mathcal{A}$. Then there exists a unique extension to an isomorphism $\widetilde{f}\colon Id_{\mathcal{D}^b(\mathcal{A})} \to F$.*

Proof. **Step 1** We characterize objects in $\mathcal{A}$ in terms of the ample sequence: An object $A^\bullet \in \mathcal{D}^b(\mathcal{A})$ is isomorphic to an object in $\mathcal{A}$ if and only if

$$\mathrm{Hom}(L_i, A^\bullet[j]) = 0$$

for all $j \neq 0$ and $i << 0$.

One direction is immediate from the definition of ample sequence and the other can be verified using the spectral sequence:

$$E_2^{p,q} = \mathrm{Hom}_{\mathcal{A}}(L_i, H^q(A^{\bullet})[p]) \Rightarrow \mathrm{Hom}(L_i, A^{\bullet}[p+q]).$$

(We assume that $\mathcal{A}$ has enough injectives, for $\mathrm{Coh}(X)$ we consider the embedding into $\mathrm{QCoh}(X)$.) Since $A^{\bullet}$ is bounded, its cohomology is concentrated in degree $[-k, k]$. Hence, $E_2^{p,q} = 0$ for $|q| > k$. We can find i_0 such that $\mathrm{Hom}(L_i, H^q(A)[p]) = 0$ for $i \leq i_0$ and $p \neq 0$. Then the spectral sequence is supported in vertical axis, hence $\mathrm{Hom}(L^i, H^q(A^{\bullet})) = \mathrm{Hom}(L^i, A^{\bullet}[q])$. By definition, $\mathrm{Hom}(L^i, H^q(A^{\bullet})) \neq 0$ for $i << 0$, as soon as $H^q(A) \neq 0$. As $\mathrm{Hom}(L_i, A^{\bullet}[q]) = 0$ for $q \neq 0$, we conclude that $H^q(A) = 0$, for $q \neq 0$.

Step 2 We show that for any $A \in \mathcal{A}$ also $F(A) \in \mathcal{A}$. It follows from the isomorphisms

$$\mathrm{Hom}(L_i, F(A)[j]) \simeq \mathrm{Hom}(F(L_i), F(A)[j]) \simeq \mathrm{Hom}(L_i, A[j])$$

(we assumed that F is an equivalence, hence it is fully faithful) and the previous step.

Step 3 We construct for any $A \in \mathcal{A}$ an isomorphism $\widetilde{f}_A \colon A \to F(A)$ which is functorial in A and extends f.

We have short exact sequence:

$$0 \to B \to L_i^{\oplus k} \to A \to 0$$

with $i << 0$. Its image under F is again a short exact sequence in $\mathcal{A}$ (by the previous step):

(20)
$$\begin{array}{ccccc} F(B) & \longrightarrow & F(L_i^{\oplus}k) & \longrightarrow & F(A) \\ & & {\scriptstyle f}\big\uparrow & & \dashuparrow \\ B & \longrightarrow & L_i^{\oplus k} & \longrightarrow & A \end{array}$$

We want to show that the dashed arrow exists and is unique. To show its existence we shall check that $g \colon B \to L_i^{\oplus k} \to F(L_i^{\oplus k}) \to F(A)$ is zero. If the map exists, then it is unique as $\mathrm{Hom}(A, F(A)) \subset \mathrm{Hom}(L_i^{\oplus k}, F(A))$.

We choose a surjective map $L_j^{\oplus l} \to B$ and get a commutative diagram

$$\begin{array}{ccc} F(L_j^{\oplus l}) & \longrightarrow & F(L_i^{\oplus k}) \\ {\scriptstyle f}\big\uparrow & & \big\uparrow \\ L_j^{\oplus l} & \longrightarrow & L_i^{\oplus k} \end{array}$$

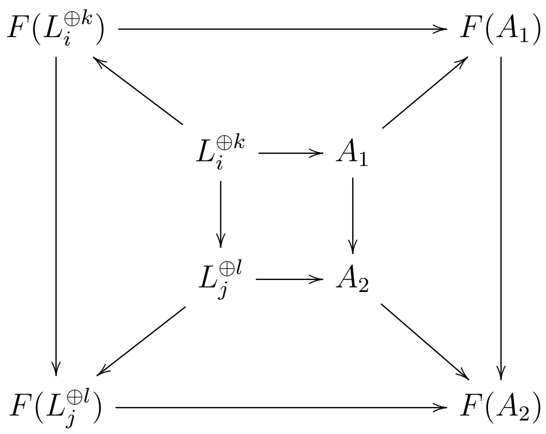
as $f|_{\{L_i\}}$ is a natural transformation. As $L_j^{\oplus l} \to L_i^{\oplus k} \to A$ is zero, so is $F(L_j^{\oplus l}) \to F(L_i^{\oplus k}) \to F(A)$. So, the map $L_j^{\oplus l} \to B \xrightarrow{g} F(A)$ is zero. As the first map is a surjection, it implies that $g = 0$.

We have constructed a map $A \to F(A)$. Now we check that it does not depend on the choice of L_i. Any two surjections can be dominated by a third one, hence it is enough to consider

$$\begin{array}{ccccc} F(L_j^{\oplus l}) & \longrightarrow & F(L_i^{\oplus k}) & \longrightarrow & F(A) \\ {\scriptstyle f}\uparrow & & {\scriptstyle f}\uparrow & & {\scriptstyle \widetilde{f}}\uparrow \\ L_j^{\oplus l} & \longrightarrow & L_i^{\oplus k} & \longrightarrow & A \end{array}$$

If th smaller square commutes then the bigger one does too. We have seen that there is unique morphism $A \to F(A)$ with this property, hence $\widetilde{f}$ is equal to the morphism constructed via the surjection $L_j^{\oplus l} \to A$.

We check that $A \to F(A)$ is functorial. Morphism $\varphi\colon A_1 \to A_2$ can be lifted to the morphisms of kernels, because $\mathrm{Ext}^1(L_i^{\oplus k}, B) = 0$ for $i << 0$. We get



and we want to show commutativity of the right part. As everything else commutes, $L_i^{\oplus k} \to A_1$ composed with the upper path $A_1 -> F(A_2)$ equals $L_i^{\oplus k} \to A_1$ composed with the lower path. As $L_i^{\oplus k} \to A_1$ is surjective, we get that $\widetilde{f}$ is functorial.

It remains to check that it's an isomorphism. In (20) we have not only map $A \to F(A)$ but also $B -> F(B)$. Snake Lemma implies that $A \to F(A)$ is surjective, for any A. Hence, also $B \to F(B)$ is surjective and Snake Lemma again implies that $A \to F(A)$ is injective.

Step 4 We define $\widetilde{f}_{A^\bullet}$ for any $A \in \mathcal{D}^b(\mathcal{A})$ recursively on the length of the complex. We will assume that we have constructed an isomorphism $\widetilde{f}_{A^\bullet} \colon A^\bullet \to F(A^\bullet)$ for any complex $A^\bullet$ with

$$\mathrm{length}(A^\bullet) := \max\{q_1 - q_2 \,|\, H^{q_1}(A^\bullet) \neq = \neq H^{q_2}(A^\bullet)\} + 1 < N$$

functorial in $A^\bullet$.

Suppose length of $A^\bullet$ is N. We can assume that $A^\bullet$ looks like $\dots A^{m-1} \to A^m \to 0$ and $H^m(A^\bullet) \neq 0$. There exists i such that $\operatorname{Hom}(H^m(A^\bullet), L_i) = 0$ and there exists surjection $L_i^{\oplus k} \to A^m$. This surjection gives us a distinguished triangle

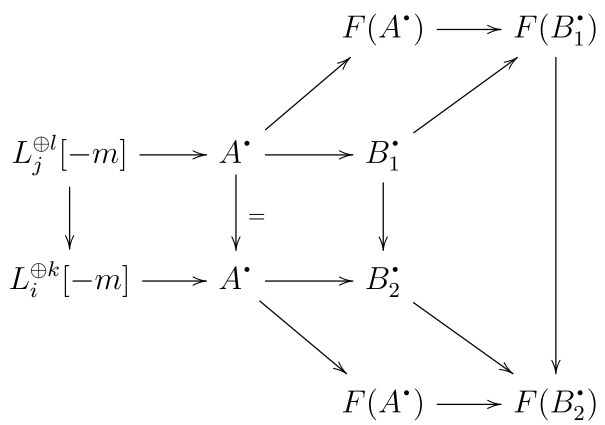
$$L_i^{\oplus k}[-m] \to A^\bullet \to B^\bullet \to L_i^{\oplus k}[-m+1].$$

The map $L^{\oplus k} \to A^m \to H^m(A^\bullet)$ is surjective, hence $H^m(B^\bullet) = 0$. It follows from the long exact sequence of cohomology that $H^i(A^\bullet) \simeq H^i(B^\bullet)$ for $i < m-1$. Hence, $B^\bullet$ is of length less than N. We have

$$\begin{array}{ccccccc}
F(L_i^{\oplus k})[-m] & \longrightarrow & F(A^\bullet) & \longrightarrow & F(B) & \longrightarrow & F(L_i^{\oplus k})[-m+1] \\
\uparrow{\scriptstyle \widetilde{f}} & & \uparrow{\scriptstyle \widetilde{f}} & & \uparrow{\scriptstyle \widetilde{f}} & & \uparrow{\scriptstyle \widetilde{f}} \\
L_i^{\oplus k}[-m] & \longrightarrow & A^\bullet & \longrightarrow & B & \longrightarrow & L_i^{\oplus k}[-m+1]
\end{array}$$

Morphism $\widetilde{f}\colon A^\bullet \to F(A^\bullet)$ exists because of TR3. It is unique as $\operatorname{Hom}(A^\bullet, F(L_i^{\oplus k})[-m]) \simeq \operatorname{Hom}(A^\bullet, L_i^{\oplus k}[-m]) \simeq \operatorname{Hom}(H^m(A^\bullet), L_i^{\oplus k}) \simeq 0$. Also it is an isomorphism as the remaining two morphisms are.

We need to check that $\widetilde{f}_{A^\bullet}$ is independent of the choices and functorial. For independence of the choices we again consider $L_j^{\oplus l} \to L_i^{\oplus k} \to A^m$. We have a diagram:



which implies that two ways of getting from $A^\bullet$ to $F(B_2^\bullet)$ are identical. In the diagram, existence of $B_1^\bullet \to B_2^\bullet$ is ensured by TR3 and the right part commutes because of inductive hypothesis. We have already seen that the map $\operatorname{Hom}(A^\bullet, F(A^\bullet)) \to \operatorname{Hom}(A^\bullet, F(B_2^\bullet))$ is injective (the proof of uniqueness of $\widetilde{f}$), so also the two ways of getting $A^\bullet \to F(A^\bullet)$ are equal.

We are left with functoriality. Let $\varphi\colon A^\bullet \to C^\bullet$ be a morphism of complexes of length less than or equal to N. Assume $A^\bullet$ is quasi-isomorphic to $\dots \to A^{n-1} \to A^n \to 0$ and $C^\bullet$ is quasi-isomorphic to $\dots \to C^{m-1} \to C^m \to 0$.

First assume that $m < n$. Let $L_i^{\oplus k} \to A^n$ be a surjection and $B^\bullet$ the cone of $L_i^{\oplus k}[-n] \to A^\bullet \to B^\bullet$. Then $\mathrm{Hom}(B^\bullet, C^\bullet) \to \mathrm{Hom}(A^\bullet, C^\bullet)$ is surjective. Indeed, we can assume that there are no higher Ext-groups from L_i^k to the cohomology of $C^\bullet$. Then the usual spectral sequence implies that $\mathrm{Hom}(L_i^k[-n], C^\bullet)$ is zero. Then φ admits a lift to $\varphi_1\colon B^\bullet \to C^\bullet$. By inductive hypothesis $\widetilde{f}$ is functorial with respect to $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$, as the length of $B^\bullet$ is less than the length of $A^\bullet$:

$$\begin{array}{ccccc} & & \overset{\varphi}{\longrightarrow} & & \\ A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle \widetilde{f}} \\ F(A^\bullet) & \longrightarrow & F(B^\bullet) & \longrightarrow & F(C^\bullet) \\ & & \underset{F(\varphi)}{\longrightarrow} & & \end{array}$$

If $n \le m$ we choose $L_j^{\oplus l} \to C^m$ and consider the triangle $L_j^{\oplus l}[-m] \to C^\bullet \to D^\bullet$. As the length of $D^\bullet$ is less than the length of $C^\bullet$, morphism $\widetilde{f}$ is functorial with respect to $\varphi_1\colon A^\bullet \to C^\bullet \to D^\bullet$ and $\psi\colon C^\bullet \to D^\bullet$. It follows the two maps $A^\bullet \to F(C^\bullet)$ are equal when composed with $F(\psi)$. We just need to check that $\mathrm{Hom}(A^\bullet, F(C^\bullet)) \to \mathrm{Hom}(A^\bullet, F(D^\bullet))$ is injective. Its kernel is a quotient of $\mathrm{Hom}(A^\bullet, L_j^{\oplus l}[-m])$. For degree reasons, $\mathrm{Hom}(A^\bullet, L_j^{\oplus l}[-m]) \simeq \mathrm{Hom}(H^m(A^\bullet), L_j^{\oplus l})$. By definition of an ample sequence, the latter space vanishes for $j << 0$.

□

7. Full exceptional collections

We shall discuss non-standard t-structures on $\mathcal{D}^b(X)$ induced by (strong) full exceptional collections.

A *full exceptional collection* in a k-linear triangulated category $\mathcal{D}$ is a semi-orthogonal decomposition of the form

$$\mathcal{D} = \langle \mathcal{D}^b(k), \dots, \mathcal{D}^b(k) \rangle.$$

Equivalently, it is a collection $\langle E_1, \dots, E_n \rangle$ of objects of $\mathcal{D}$ such that

(1) $R\,\mathrm{Hom}^\bullet(E_i, E_i) \simeq k$,
(2) $R\,\mathrm{Hom}^\bullet(E_i, E_j) \simeq 0$ for $i \ge j$,
(3) The smallest triangulated subcategory of $\mathcal{D}$ containing $E_1, \dots, E_n$ is equivalent to $\mathcal{D}$.

Moreover, an exceptional collection is *strong* if

$$\mathrm{Hom}(E_i, E_j[l]) = 0, \text{ for } l \neq 0.$$

7.1. **Beilinson's result.**

The first example of a full strong exceptional collection (though it was not called this way) was constructed by A. Beilinson on projective space [Beĭ78].

The structure sheaf of the diagonal on $\mathbb{P}^n \times \mathbb{P}^n$ admits resolution

$$0 \to p_1^*(\Omega^n(n)) \otimes p_2^*(\mathcal{O}(-n)) \to \ldots \to p_1^*(\Omega^1(1)) \otimes p_2^*(\mathcal{O}(-1)) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0.$$

Let now $E^\bullet$ be an object of $\mathcal{D}^b(\mathbb{P}^n)$. Then $Rp_{2*}(p_1^* E^\bullet \otimes^L \mathcal{O}_\Delta) \simeq E^\bullet$.

Object $p_1^* E^\bullet \otimes^L \mathcal{O}_D$ is quasi-isomorphic to

$$p_1^*(E^\bullet \otimes \Omega^n(n)) \otimes p_2^*(\mathcal{O}(-n)) \to \ldots \to p_1^*(E^\bullet \otimes \Omega^1(1)) \otimes p_2^*(\mathcal{O}(-1)) \to p_1^*(E^\bullet).$$

We get a spectral sequence

$$E_1^{r,s} = H^s(\mathbb{P}^n, \Omega^{-r}(-r) \otimes E^\bullet) \otimes \mathcal{O}(r) \Rightarrow \begin{cases} E^\bullet & \text{if } r+s=0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $E^\bullet$ belongs to the smallest triangulated subcategory of $\mathcal{D}^b(\mathbb{P}^n)$ containing $R\Gamma(E^\bullet \otimes \Omega^i(i)) \otimes \mathcal{O}(-i)$. In other words, the smallest triangulated subcategory of $\mathcal{D}^b(\mathbb{P}^n)$ containing $\mathcal{O}(-n), \ldots, \mathcal{O}$ is equivalent to $\mathcal{D}^b(\mathbb{P}^n)$. Similarly, for $\Omega^n(n), \ldots, \Omega_1(1), \mathcal{O}$.

Standard computation shows that $\langle \mathcal{O}(-n), \ldots, \mathcal{O} \rangle$, $\langle \Omega^n(n), \ldots, \Omega^1(1), \mathcal{O} \rangle$ are strong exceptional collections in $\mathcal{D}^b(\mathbb{P}^n)$.

7.2. **Equivalence of categories.**

Theorem 7.1. [Bon89] *Let $\langle E_1, \ldots, E_n \rangle$ be a full exceptional collection in a k-linear Hom-finite triangulated category $\mathcal{D}$. Then functor*

$$R\,Hom^\bullet(\bigoplus E_i, -) \colon \mathcal{D} \to \mathcal{D}^b(End(\bigoplus E_i))$$

is an equivalence of $\mathcal{D}$ with the bounded derived category of the category of finite dimensional right modules over $End(\bigoplus E_i)$.

Let us come back to the example of $\mathbb{P}^n \simeq \mathbb{P}(V^\vee)$. Then

$$\mathrm{Hom}(\mathcal{O}(-i), \mathcal{O}(-j)) \simeq S^{i-j}(V)$$

and the composition in $\mathrm{End}(\bigoplus_{i=-n}^0 \mathcal{O}(-i))$ agrees with the composition in $S^\bullet(V)$. In other words, Theorem 7.1 implies

$$\mathcal{D}^b(\mathbb{P}(V^\vee)) \simeq \mathcal{D}^b(S^\bullet(V)/S^{n-1}V).$$

Exterior powers of the Euler sequence $0 \to \Omega^1 \to \mathcal{O}(-1) \otimes V \to \mathcal{O} \to 0$ yield short exact sequences

$$0 \to \Omega^i(i) \to \Lambda^i(V) \otimes \mathcal{O} \to \Omega^{i-1}(i) \to 0.$$

By induction one proves that

$$\operatorname{Hom}(\Omega^i(i), \Omega^j(j)) \simeq \Lambda^{i-j}(V^\vee).$$

The multiplication in the exterior algebra again coincides with the composition of morphisms in the collection $\Omega(i)$, hence

$$\mathcal{D}^b(\mathbb{P}(V^\vee)) \simeq \mathcal{D}^b(\Lambda^\bullet(V^\vee)).$$

7.3. **Braid group action.**

The subcategory $\langle E \rangle$ generated by an exceptional object E is always admissible, functor left adjoint to the embedding $i\colon \mathcal{D}^b(k) \to \mathcal{D}$, $i(k) = E$ is $i^*(-) = R\operatorname{Hom}^\bullet(-, E)^\vee$, the right adjoint is $i^!(-) = R\operatorname{Hom}^\bullet(E, -)$.

It follows that an exceptional object induces two semi-orthogonal decompositions

$$\mathcal{D} = \langle \mathcal{D}_0, \mathcal{D}^b(k) \rangle = \langle \mathcal{D}^b(k), \mathcal{D}_1 \rangle$$

Functors

$$i_0^* i_1 \colon \mathcal{D}_1 \to \mathcal{D}_0, \qquad\qquad i_1^! i_0 \colon \mathcal{D}_0 \to \mathcal{D}_1$$

are quasi-inverse equivalences called *mutation functors*.

In the case of an exceptional collection $\langle E_1, \dots, E_n \rangle$ we can consider the subcategory $\langle E_i, E_{i+1} \rangle$ and perform the mutation there. The *left mutation* $L_{E_i} E_{i+1}$ of E_{i+1} over E_i fits into a distinguished triangle

$$E_i \otimes R\operatorname{Hom}^\bullet(E_i, E_{i+1}) \to E_{i+1} \to L_{E_i} E_{i+1} \to E_i \otimes R\operatorname{Hom}^\bullet(E_i, E_{i+1})[1]$$

(we use the triangle associated to the semi-orthogonal decomposition $\langle L_{E_i} E_{i+1}, E_i \rangle$). Note that then

$$R\operatorname{Hom}^\bullet(E_{i+1}, L_{E_i} E_{i+1}) \simeq k, \qquad R\operatorname{Hom}^\bullet(E_i, L_{E_i} E_{i+1}) \simeq 0,$$
$$R\operatorname{Hom}^\bullet(E_{i+1}, E_i) \simeq 0, \qquad R\operatorname{Hom}^\bullet(E_i, E_i) \simeq k.$$

The *right mutation* $R_{E_{i+1}} E_i$ of E_i over E_{i+1} fits into a distinguished triangle

$$R_{E_{i+1}} E_i \to E_i \to E_{i+1} \otimes R\operatorname{Hom}^\bullet(E_i, E_{i+1})^\vee \to R_{E_{i+1}} E_i[1].$$

Again, we have

$$R\operatorname{Hom}(R_{E_{i+1}}E_i, E_i) \simeq k, \qquad R\operatorname{Hom}(E_{i+1}, E_i) \simeq 0,$$
$$R\operatorname{Hom}(R_{E_{i+1}}E_i, E_{i+1}) \simeq 0, \qquad R\operatorname{Hom}(E_{i+1}, E_{i+1}) \simeq k.$$

Mutations L_{E_i} and $R_{E_{i+1}}$ define the action of the braind group with n strands on the set of exceptional collections in a triangulated category $\mathcal{D}$.

In particular, the half twist gives the collection $\langle F_n, \ldots, F_1\rangle$ *left dual* to $\langle E_1, \ldots, E_n\rangle$:

$$F_i = L_{E_1} \ldots L_{E_{i-1}} E_i.$$

One checks that

$$R\operatorname{Hom}^\bullet(E_i, F_j) \simeq \begin{cases} k & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

If the collection $\langle E_1, \ldots, E_n\rangle$ is full, the left dual collection $\langle F_n, \ldots, F_1\rangle$ is determined by the Kronecker-type condition on $R\operatorname{Hom}^\bullet$.

One checks that on $\mathbb{P}^n$ collection $\langle \Omega^n(n)[n], \ldots, \Omega^1(1)[1], \mathcal{O}\rangle$ is left dual to $\langle \mathcal{O}, \ldots, \mathcal{O}(n)\rangle$.

7.4. **Glued t-structure.**

A full exceptional collection $\langle E_1, \ldots, E_n\rangle$ yields two admissible filtrations

$$\langle E_1\rangle \subset \langle E_1, E_2\rangle \subset \ldots \langle E_1 \ldots E_n\rangle$$
$$\langle E_n\rangle \subset \langle E_{n-1}, E_n\rangle \subset \ldots \langle E_1, \ldots E_n\rangle.$$

In both cases the 'graded components' are equivalent to $\mathcal{D}^b(k)$. We can use recollements

$$\langle E_1, \ldots E_{i-1}\rangle \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \langle E_1, \ldots E_i\rangle \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \mathcal{D}^b(k)$$

and the standard t-structure on $\mathcal{D}^b(k)$ to glue t-structure on $\mathcal{D}$.

Proposition 7.2. *Let $\langle E_1, \ldots, E_n\rangle$ be a full exceptional collection in a triangulated category $\mathcal{D}$. Denote by $\langle F_n, \ldots, F_1\rangle$ the left dual exceptional collection.*

The t-structure glued on $\mathcal{D}$ along the filtration

$$\langle E_1\rangle \subset \langle E_1, E_2\rangle \subset \ldots \langle E_1 \ldots E_n\rangle$$

from the standard t-structures on $\mathcal{D}^b(k)$ is

$$\mathcal{D}^{\leq 0} = \{D \in \mathcal{D} \mid R\,Hom^\bullet(D, F_i) \in \mathcal{D}^b(k)^{\geq 0}, \forall i\},$$
$$\mathcal{D}^{\geq 0} = \{D \in \mathcal{D} \mid R\,Hom^\bullet(E_i, D) \in \mathcal{D}^b(k)^{\geq 0}, \forall i\}.$$

7.5. **Full exceptional collections on $\mathbb{P}^2$ and Markov numbers.**

A. Rudakov studied full exceptional collection on $\mathbb{P}^2$ obtained by mutations from the collection $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$.

He found that these collections contain (up to shift) line bundles. If a, b, c are ranks of the bundles then (a, b, c) is a solution to the *Markov equation*:

$$a^2 + b^2 + c^2 = 3abc.$$

All solutions of the Markov equation are obtained by 'mutations' from the solution $(1, 1, 1)$.

If (a, b, c) is a solution to Markov equation then putting $b' = 3ac - b$ we get a new solution (a, b', c). If we do this procedure again, we recover the original solution (a, b, c). However, we can change a or c. Solutions to the Markov equation can be organised in a tree with vertices (a, b, c) and arrows joining two solutions related by a mutation. Generally, from every solution there are three arrows:

$$(a, b, c) \to (a, 3ac - b, c), \quad (a, b, c) \to (a, 3ab - c, b), \quad (a, b, c) \to (b, 3bc - a, c).$$

The only singular solutions are $(1, 1, 1)$ and $(1, 2, 1)$ where some of the triples coincide.

$$(1,1,1) \to (1,2,1) \to (1,5,2)$$

$$(1,5,2) \to (1,13,5), \qquad (1,5,2) \to (5,29,2)$$

7.6. **Full exceptional collections on homogeneous spaces and toric varieties.**

Further examples of full exceptional collections were constructed by M. Kapranov on Grassmannians and flag varieties. They were also constructed via a 'nice' resolution of the structure sheaf of the diagonal.

Consider $G(k, V)$, the Grassmannian of k-dimensional subspaces in V. We denote by S the tautological k-dimensional bundle on $G(k, V)$ and by U the tautological $n-k$-dimensional quotient bundle.

The resolution of $\mathcal{O}_\Delta$ implied that $\Sigma^\alpha U$ generate $\mathcal{D}^b(G(k, V))$ where Σ^α is the Schur functor assigned to a Young tableau α and α fits into a $k \times (n-k)$ rectangle.

(Recall, that given a vector space W and a Young tableau α with l entries $\Sigma^\alpha W$ is the projection of the subspace of $W^{\otimes l}$ consisting of tensors which are unchanged by any permutation of boxes in the same row to the subspace of vectors that change sign under any permutation that changes boxes in the same column. This operation is functorial, so it can be applied to vector bundles.)

Full exceptional collections have been also constructed by Yu. Kawamata on projective toric varieties. (Quotient singularities are allowed but then one needs to consider derived category of a relevant Deligne-Mumford stack.)

7.7. **Derived category under blow-up and projective bundles.**

Beilinson's result was generalised in a different direction by D. Orlov [Orl92]. He constructed semi-orthogonal decompositions for projective bundles and blow-ups.

Let X be a smooth variety and let E be a vector bundle of rank r on X. Let $p\colon \mathbb{P}(E) \to X$ be the projective bundle determined by E. Let $\mathcal{O}_E(1)$ be the relative ample line bundle.

Morphism p is flat, moreover the derived functor $p^*\colon \mathcal{D}^b(X) \to \mathcal{D}^b(\mathbb{P}(E))$ is fully faithful:

$$\mathrm{Hom}(p^*E^\bullet, p^*F^\bullet) \simeq \mathrm{Hom}(E^\bullet, Rp_*(p^*F^\bullet)) \simeq \mathrm{Hom}(E^\bullet, F^\bullet \otimes Rp_*(\mathcal{O})) \simeq \mathrm{Hom}(E^\bullet, F^\bullet).$$

We denote by $\mathcal{D}(X)_0$ the full-subcategory $p^*\mathcal{D}^b(X) \subset \mathcal{D}^b(\mathbb{P}(E))$. By $\mathcal{D}(X)_l$ we denote the full subcategory with objects $p^*(F^\bullet) \otimes \mathcal{O}_E(k)$, for $F^\bullet \in \mathcal{D}(X)$.

Lemma 7.3. *For $0 \leq l \leq r-1$, $Hom(\mathcal{D}(X)_l, \mathcal{D}(X)_0) = 0$.*

Proof.

$$\mathrm{Hom}(p^*(F^\bullet) \otimes \mathcal{O}_l, p^*(G^\bullet)) \simeq \mathrm{Hom}(F^\bullet, Rp_*(p^*(G^\bullet) \otimes \mathcal{O}(-l)) \simeq$$
$$\simeq \mathrm{Hom}(F^\bullet, G^\bullet \otimes Rp_*\mathcal{O}(-l)) \simeq 0.$$

□

Beilinson's resolution of the diagonal works also in this case and we get

Theorem 7.4. [Orl92] *Let E be a rank r vector bundle on a smooth projective variety X. Then $\mathcal{D}^b(\mathbb{P}(E))$ admits a semi-orthogonal decomposition*

$$\mathcal{D}^b(\mathbb{P}(E)) = \langle \mathcal{D}(X)_0, \ldots, \mathcal{D}(X)_{r-1} \rangle.$$

Let Y be a smooth variety, $Z \subset Y$ a closed subvariety (which is locally a complete intersection) and $f\colon X \to Y$ the blow-up of Y along Z. Let $E \subset X$ be the exceptional divisor of f. Then E is a Cartier divisor [Har77, Proposition II.7.3] and we have a diagram

$$\begin{array}{ccc} E & \xrightarrow{i} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{j} & Y \end{array}$$

Moreover, since Z is a locally complete intersection, $\mathcal{O}_E(E) \simeq \mathcal{O}_\pi(-1)$.

Lemma 7.5. *For $C \in \mathcal{D}^b(E)$, we have $i_*i^!i_*C \simeq (i_*C \oplus i_*C \otimes \mathcal{O}_X(E)[-1])$.*

Proof. Local duality $i_*R\mathcal{H}\mathrm{om}_E(-, i^!(=)) \simeq R\mathcal{H}\mathrm{om}_X(i_*(-), =)$ implies isomorphisms:

$$\begin{aligned} i_*i^!i_*C &\simeq i_*R\mathcal{H}\mathrm{om}_E(\mathcal{O}_E, i^!i_*C) \simeq R\mathcal{H}\mathrm{om}_X(i_*\mathcal{O}_E, i_*C) \\ &\simeq R\mathcal{H}\mathrm{om}_X([\mathcal{O}_X(-E) \to \mathcal{O}_X], i_*C) \simeq i_*C \oplus i_*C(E)[-1]. \end{aligned}$$

The direct sum decomposition follows from the fact that the adjunction unit $i_*C \to i_*i^!i_*C$ splits the distinguished triangle obtained by applying $R\mathcal{H}\mathrm{om}(-, i_*C)$ to the exact sequence

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0.$$

□

Lemma 7.6. *Functor $Ri_* \circ L\pi^* \colon \mathcal{D}^b(Z) \to \mathcal{D}^b(X)$ is fully faithful.*

Proof. For $A_1, A_2 \in \mathcal{D}^b(Z)$, we have

$$\mathrm{Hom}_X(Ri_*L\pi^*A_1, Ri_*L\pi^*A_2) \simeq \mathrm{Hom}_Z(A_1, R\pi_*i^!Ri_*L\pi^*A_2).$$

We shall show that $R\pi_*$ of the adjunction unit $L\pi^*A_2 \to i^!Ri_*L\pi^*A_2$ is an isomorphism, i.e. that for a distinguished triangle

$$L\pi^*A_2 \to i^!Ri_*L\pi^*A_2 \to B \to L\pi^*A_2[1] \tag{21}$$

we have $R\pi_*B = 0$. Since j is a closed embedding, it suffices to show that $Rj_*R\pi_*B = Rf_*Ri_*B = 0$. In view of Lemma 7.5, we have

$$Ri_*B \simeq Ri_*L\pi^*A_2(E)[-1] \simeq Ri_*(L\pi^*A_2 \otimes Li^*\mathcal{O}_X(E))[-1] \simeq Ri_*(L\pi^*A_2 \otimes \mathcal{O}_\pi(-1))[-1].$$

Since Ri_* is conservative, $B \simeq L\pi^*A_2 \otimes \mathcal{O}_\pi(-1)[1]$. Then,

$$Rf_*Ri_*B \simeq Rj_*R\pi_*B \simeq Rj_*R\pi_*(L\pi^*A_2 \otimes \mathcal{O}_\pi(-1))[-1] \simeq Rj_*(A_2 \otimes R\pi_*\mathcal{O}_\pi(-1))[-1] \simeq 0.$$ □

For $k \in \mathbb{Z}$, we put

$$\widetilde{\mathcal{D}}(Z)_k := Ri_*L\pi^*\mathcal{D}^b(Z) \otimes \mathcal{O}_X(kE) \subset \mathcal{D}^b(X).$$

Let $d = \mathrm{codim}_Y Z$.

Lemma 7.7. *For any $k \in \{1, \dots, d\}$ and any $A \in \mathcal{D}^b(Z)$, object $Ri_* L\pi^* A \otimes \mathcal{O}_X(kE)$ lies in the kernel of $Rf_* \colon \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$.*

Proof. It follows immediately from the projection formula:

$$\begin{aligned} Rf_*(Ri_* L\pi^* A \otimes \mathcal{O}_X(kE)) &\simeq Rf_* Ri_*(L\pi^* A \otimes Li^* \mathcal{O}_X(kE)) \\ &\simeq Rj_* R\pi_*(L\pi^* A \otimes \mathcal{O}_\pi(-k)) \simeq Rj_*(A \otimes R\pi_*(\mathcal{O}_\pi(-k))) \simeq 0, \end{aligned}$$

□

Lemma 7.8. *For any $k \in \{1, \dots, d-1\}$ and $A \in \mathcal{D}^b(Z)$, we have*

$$Rf_* Ri_* i^!(Ri_* L\pi^* A \otimes \mathcal{O}_X(kE)) = 0.$$

Proof. Local duality gives

$$\begin{aligned} &Rf_* Ri_* i^!(Ri_* L\pi^* A \otimes \mathcal{O}_X(kE)) \simeq Rf_* Ri_* R\mathcal{H}\mathrm{om}_E(\mathcal{O}_E, i^!(Ri_* L\pi^* A \otimes \mathcal{O}_X(kE))) \\ &\simeq Rf_* R\mathcal{H}\mathrm{om}_X(\mathcal{O}_E, Ri_* L\pi^* A \otimes \mathcal{O}_X(kE)) \simeq Rf_*(R\mathcal{H}\mathrm{om}_X(\mathcal{O}_E, Ri_* L\pi^* A) \otimes \mathcal{O}_X(kE)) \\ &\simeq Rf_*(Ri_* i^! Ri_* L\pi^* A \otimes \mathcal{O}_X(kE)). \end{aligned}$$

In view of Lemma 7.5, we have

$$\begin{aligned} &Rf_*(Ri_* i^! Ri_* L\pi^* A \otimes \mathcal{O}_X(kE)) \simeq Rf_*((Ri_* L\pi^* A \oplus Ri_* L\pi^* A \otimes \mathcal{O}_X(E)[-1]) \otimes \mathcal{O}_X(kE)) \\ &\simeq Rf_*(Ri_* L\pi^* A \otimes \mathcal{O}_X(kE)) \oplus Rf_*(Ri_* L\pi^* A \otimes \mathcal{O}_X((k+1)E)[-1]). \end{aligned}$$

The statement follows from Lemma 7.7. □

Lemma 7.9. *For $0 \le k < l \le d-1$, we have $Hom_X(\widetilde{\mathcal{D}}(Z)_k, \widetilde{\mathcal{D}}(Z)_l) = 0$.*

Proof. For $A_1, A_2 \in \mathcal{D}^b(Z)$, we have

$$\begin{aligned} &\mathrm{Hom}_X(Ri_* L\pi^* A_1 \otimes \mathcal{O}_X(kE), Ri_* L\pi^* A_2 \otimes \mathcal{O}_X(lE)) \\ &\simeq \mathrm{Hom}_Z(A_1 \otimes R\pi_* i^!(Ri_* L\pi^! A_2 \otimes \mathcal{O}_X((l-k)E)). \end{aligned}$$

In view of Lemma 7.7, object $Rj_* R\pi_* i^!(Ri_* L\pi^* A_2 \otimes \mathcal{O}_X((l-k)E)) \simeq Rf_* Ri_* i^!(Ri_* L\pi^* A_2 \otimes \mathcal{O}_X((l-k)E))$ is zero. Since j is a closed embedding, it follows that $R\pi_* i^!(Ri_* L\pi^* A_2 \otimes \mathcal{O}_X((l-k)E)) = 0$ which finishes the proof. □

Proposition 7.10. *Let $\mathcal{F} \in \mathcal{D}^b(X)$ be such that $Rf_* \mathcal{F} = 0$. Then $\mathcal{F}$ lies in the subcategory of $\mathcal{D}^b(X)$ generated by $i_* \mathcal{D}^b(E)$.*

Proof. We shall show that all cohomology sheaves of $\mathcal{F}$ are (set-theoretically) supported on E. Then, we check that any $\widetilde{\mathcal{F}} \in \mathcal{D}^b(X)$ whose cohomology are set-theoretically supported on E is an iterated extension of shifts of sheaves supported on E.

Since $Rf_*\mathcal{F} = 0$, spectral sequence with the $E_2^{p,q} = R^p f_* \mathcal{H}^q \mathcal{F}$ converges to zero. We note that $R^i f_* \mathcal{H}^l(\mathcal{F})$ together with their subobjects and quotients are supported on Z, for $i \geq 1$ and arbitrary l. It follows that $E_w^{p,q}$ is a sheaf whose support is contained in Z, as soon as $p \geq 1$ and $w \geq 2$. On the other hand, if $\mathcal{H}^k(\mathcal{F})$ is not supported on E, then, since f is an isomorphism outside of E, $f_*\mathcal{H}^k(\mathcal{F})$ is non-zero and its support is not contained in Z. It follows that $E_\infty^{0,k}$ is non-zero which contradicts the fact that $E^{p,q} \Rightarrow 0$.

For a sheaf $\mathcal{G} \in \mathrm{Coh}(X)$ set-theoretically supported on E, let $l(\mathcal{G}) \in \mathbb{N}$ be minimal such that $\mathcal{G} \in \mathcal{D}^b(l(\mathcal{G})E)$. Then, for the kernel $\mathcal{G}'$ of the restriction $\mathcal{G} \to \mathcal{G}|_E$, we have $l(\mathcal{G}') = l(\mathcal{G}) - 1$. For a complex $\mathcal{F} \in \mathcal{D}^b(X)$ with all cohomology sheaves supported on E, we put $l(\mathcal{F}) := \sum l(\mathcal{H}^i(\mathcal{F}))$.

We prove by descending induction on $l(\mathcal{F})$ that $\mathcal{F}$ lies in triangulated subcategory of $\mathcal{D}^b(X)$ generated by $\mathcal{D}^b(E)$. Let k be maximal such that $\mathcal{H}^k(\mathcal{F}) \neq 0$ and let $\mathcal{F}' \in \mathcal{D}^b(X)$ be a complex defied by mattress:

$$\begin{array}{ccccc}
\tau_{\leq k-1}\mathcal{F} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{H}^k(\mathcal{F})|_E[-k] \\
\simeq\uparrow & & \uparrow & & \uparrow \\
\tau_{\leq k-1}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{H}^k(\mathcal{F})[-k] \\
 & & \uparrow & & \uparrow \\
 & & \mathcal{G}'[-k] & \xrightarrow{\simeq} & \mathcal{G}'[-k]
\end{array}$$

Then $l(\mathcal{F}') = l(\mathcal{F}) - 1$, hence, by inductive hypothesis, $l(\mathcal{F}) \in \langle i_*\mathcal{D}^b(E)\rangle$. The statement follows from the fact that $\mathcal{H}^k(\mathcal{F})|_E$ is isomorphic to $i_*\mathcal{G}$, for some $\mathcal{G} \in \mathrm{Coh}(E)$. □

Lemma 7.11. *Let $\mathcal{E}$ be a locally free sheaf of rank d on Y and $Z \subset Y$ be given by the zeros of a section of $\mathcal{E}$. Then, for $i = 1, \dots, d$, the sheaf $L^i f^* \mathcal{O}_Z = \Omega_\pi^i(i)$ lies in the subcategory $\langle \widetilde{\mathcal{D}}(Z)_{d-1}, \dots, \widetilde{\mathcal{D}}(Z)_1 \rangle$.*

Proof. We use Koszul resolution

$$0 \to \Lambda^d \mathcal{E}^* \to \ldots \to \mathcal{E}^* \to \mathcal{O}_X \to 0$$

of $\mathcal{O}_Z$ to get $L^i f^* \mathcal{O}_Z = \Omega_\pi^i(i)$.

To prove that $\Omega_\pi^i(i) \in \langle \widetilde{\mathcal{D}}(Z)_{d-1}, \dots, \widetilde{D}(Z)_1 \rangle \subset \mathcal{D}^b(X)$ we use relative Euler sequence

$$0 \to \Omega^p(p) \to f^*\Lambda^p \mathcal{E}^* \to \Omega^{p-1}(p) \to 0$$

and its twists by $\mathcal{O}_X(kE)$. We prove by decreasing induction on p that $\Omega^p(k) \in \langle \widetilde{\mathcal{D}}(Z)_{d-1}, \dots, \widetilde{\mathcal{D}}(Z)_1 \rangle$, for $k = 1, \dots, p$ and $p = 1, \dots, d-1$.

Since $\Omega_\pi^{d-1} = \mathcal{O}_E(dE)$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_\pi(-1)$, we have $\Omega_\pi^{d-1}(k) = \mathcal{O}_E((d-k)E) \in \widetilde{\mathcal{D}}(Z)_{d-k}$ and the case $p = d-1$ is clear. Let us now assume that the statement holds for

p. The sheaf $\Omega^{p-1}(p-k)$ belongs to any triangulated subcategory of $\mathcal{D}^b(X)$ containing $\Omega^p(p-k)$ and $f^*\Lambda^p\mathcal{E}^* \otimes \mathcal{O}_X(kE)$. The Lemma follows from the fact that the first sheaf belongs to $\langle \widetilde{\mathcal{D}}(Z)_{d-1}, \ldots, \widetilde{\mathcal{D}}(Z)_1 \rangle$ for $k = 0, \ldots, p-1$, the second for $k = 1, \ldots, d-1$. $\square$

Theorem 7.12. *Category $\mathcal{D}^b(X)$ admits a semi-orthogonal decomposition*

$$\mathcal{D}^b(X) \simeq \langle \widetilde{\mathcal{D}}(Z)_{d-1}, \ldots, \widetilde{\mathcal{D}}(Z)_1, Lf^*\mathcal{D}^b(Y) \rangle \tag{22}$$

Proof. The fact that categories are semi-orthogonal follows from Lemmas 7.7 and 7.9. It remains to show that any object $B \in \mathcal{D}^b(X)$ admits a "filtration" with subquotients in $\widetilde{\mathcal{D}}(Z)_k$ and $Lf^*\mathcal{D}^b(Y)$.

Let $\mathcal{F} \in \mathcal{D}^b(X)$. Then

$$Lf^*Rf_*\mathcal{F} \to \mathcal{F} \to \mathcal{F}' \to Lf^*Rf_*\mathcal{F}[1]$$

is a decomposition of $\mathcal{F}$ into $Lf^*Rf_*\mathcal{F} \in Lf^*\mathcal{D}^b(Y)$ and $\mathcal{F}'$ such that $Rf_*\mathcal{F}' = 0$, hence $\mathrm{Hom}(Lf^*\mathcal{D}^b(Y), \mathcal{F}') = 0$.

For any $\mathcal{F} \in \mathcal{D}^b(X)$, the $L\pi^* \dashv R\pi_*$ and $i_* \dashv i^!$ adjunction counits give a morphism of functors $Ri_*L\pi^*R\pi_*i^!((-) \otimes \mathcal{O}_X(-kE)) \to (-) \otimes \mathcal{O}_X(-kE)$, hence

$$\kappa_k \colon Ri_*L\pi^*R\pi_*i^!((-) \otimes \mathcal{O}_X(-kE)) \otimes \mathcal{O}_X(kE) \to (-).$$

The cone $\overline{\mathcal{F}}$ of κ_k applied to $\mathcal{F} \in \mathcal{D}^b(X)$ is such that $\mathrm{Hom}_X(\widetilde{\mathcal{D}}(Z)_k, \overline{\mathcal{F}}) = 0$.

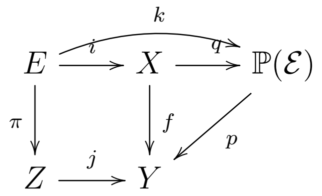
Let now $\mathcal{F}$ be an arbitrary element of $\mathcal{D}^b(X)$. The cone $\mathcal{F}_0$ of the $Lf^* \dashv Rf_*$ adjunction unit is an element of $\mathcal{D}^b(X)$ orthogonal to $Lf^*\mathcal{D}^b(Y)$. Let $\mathcal{F}_1$ be the cone of κ_1 applied to $\mathcal{F}_0$. Since both $\mathcal{F}_0$ and $Ri_*L\pi^*R\pi_*i^!(\mathcal{F}_0 \otimes \mathcal{O}_X(-E)) \otimes \mathcal{O}_X(E)$ are orthogonal to $Lf^*\mathcal{D}^b(Y)$, Lemma 7.8, so is $\mathcal{F}_1$. Moreover, $\mathcal{F}_1$ is orthogonal to $\widetilde{\mathcal{D}}(Z)_1$. Continuing, we define $\mathcal{F}_k$ as the cone of κ_k applied to $\mathcal{F}_{k-1}$. Then, Lemmas 7.8 and 7.9 imply that $\mathcal{F}_k$ is orthogonal to $\langle \widetilde{\mathcal{D}}(Z)_k, \ldots, \widetilde{\mathcal{D}}(Z)_1, Lf^*\mathcal{D}^b(Y) \rangle$. It follows that (22) is a semi-orthogonal decomposition if and only if $\mathcal{F}_{d-1}$ is zero, i.e. κ_{d-1} applied to $\mathcal{F}_{d-2}$ is an isomorphism.

The $Lf^* \dashv Rf_*$ adjunction counit and morphisms κ_k are defined locally over Y. Therefore, to prove that (22) is a semi-orthogonal decomposition, it suffices to choose an open cover of Y and prove that (22) holds for an arbitrary open set $U \subset Y$ in the cover and the blow-up f_U of U along $Z \cap U$. Therefore, one can assume that $Z \subset Y$ is the zero set of a section of a locally free sheaf of rank d.

Let $\mathcal{F}$ be an arbitrary element of $\mathcal{D}^b(X)$ and let $\mathcal{F}'$ be the cone of the $Lf^* \dashv Rf_*$ adjunction counit applied to $\mathcal{F}$. Then $Rf_*\mathcal{F}' = 0$, hence by Proposition 7.10, $\mathcal{F}'$ belongs to the triangulated subcategory of $\mathcal{D}^b(X)$ generated by $i_*\mathcal{D}^b(E)$. Thus, in order to prove that $\mathcal{F}'$ belongs to $\langle \widetilde{\mathcal{D}}(Z)_{d-1}, \ldots, \widetilde{\mathcal{D}}(Z)_1, Lf^*\mathcal{D}^b(Y) \rangle \subset \mathcal{D}^b(X)$ it suffices to check that any element of $i_*\mathcal{D}^b(E)$ belongs to this subcategory of $\mathcal{D}^b(X)$.

By Theorem 7.4 we have $\mathcal{D}^b(E) = \langle \pi^*\mathcal{D}^b(Z) \otimes \mathcal{O}_E((d-1)E), \dots, \pi^*\mathcal{D}^b(Z) \otimes \mathcal{O}_E(E), \pi^*\mathcal{D}^b(Z)\rangle$. Thus, to show that $i_*\mathcal{D}^b(E)$ is contained in $\langle \widetilde{\mathcal{D}}(Z)_{d-1}, \dots, \widetilde{\mathcal{D}}(Z)_1, Lf^*\mathcal{D}^b(Y)\rangle$, it suffices to check that $i_*\pi^*\mathcal{D}^b(Z)$ is contained there. Since morphism π is flat and i is a closed embedding, functor $i_*\pi^*$ is t-exact. Hence, to show that $i_*\pi^*\mathcal{F}$ belongs to $\langle \widetilde{\mathcal{D}}(Z)_{d-1}, \dots, \widetilde{\mathcal{D}}(Z)_1, Lf^*\mathcal{D}^b(Y)\rangle$, for any $\mathcal{F} \in \mathcal{D}^b(Z)$, it suffices to check it for $\mathcal{F} \in \mathrm{Coh}(Z)$.

Since $Z \subset Y$ is given by a zero-section of a locally free sheaf $\mathcal{E}$, morphism f fits into a commuting diagram



Since morphism p is flat and E is the fiber product of $\mathbb{P}(\mathcal{E})$ and Z over Y, we have, $k_*\pi^*\mathcal{F} \simeq p^*j_*\mathcal{F}$, for any $\mathcal{F} \in \mathrm{Coh}(Y)$. Further,

$$q_*Lf^*j_*\mathcal{F} \simeq q_*Lq^*p^*j_*\mathcal{F} \simeq q_*Lq^*q_*i_*\pi^*\mathcal{F} \simeq \bigoplus q_*(\Lambda^k\mathcal{N}^*[k] \otimes i_*\pi^*\mathcal{F}) \tag{23}$$

where $\mathcal{N}$ is the normal bundle to the embedding q. The last isomorphism follows from [Huy06, Proposition 11.1] and the fact that $X \subset \mathbb{P}(\mathcal{E})$ is the set of zeros of a section of $p^*\Lambda^2\mathcal{E}^* \otimes \mathcal{O}_g(1)$.

In particular, it follows that $i_*\pi^*\mathcal{F} = f^*j_*\mathcal{F}$. In view of distinguished triangle

$$\tau_{\le -1}Lf^*j_*\mathcal{F} \to Lf^*j_*\mathcal{F} \to f^*j_*\mathcal{F} \to \tau_{\le -1}Lf^*j_*\mathcal{F}[1], \tag{24}$$

in order to show that $i_*\pi^*\mathcal{F}$ lies in $\langle \widetilde{\mathcal{D}}(Z)_{d-1}, \dots, \widetilde{\mathcal{D}}(Z)_1, Lf^*\mathcal{D}^b(Y)\rangle$, it suffices to check that $\tau_{\le -1}Lf^*j_*\mathcal{F}$ is an element of this category. Since q_* is t-exact, (23) implies that $L^kf^*j_*\mathcal{F} \simeq \Lambda^k\mathcal{N}^* \otimes i_*\pi^*\mathcal{F}$. It follows from Lemma 7.11 that $\Lambda^k\mathcal{N}^*|_E \simeq \Omega^k_\pi(k)$. Finally, as $i_*\pi^*\mathcal{F}$ is scheme-theoretically supported on E, we have $L^kf^*j_*\mathcal{F} \simeq \Omega^k_\pi(k) \otimes i_*\pi^*\mathcal{F} \simeq i_*(\Omega^k_\pi(k) \otimes \pi^*\mathcal{F})$.

As in the proof of Lemma 7.11 we use the relative Euler sequence

$$0 \to \Omega^p_\pi(p) \to f^*\Lambda^p\mathcal{E} \to \Omega^{p-1}(p) \to 0$$

and its twist by $\mathcal{O}_pi(k)$ and $\pi^*\mathcal{F}$ (note that all sheaves in the sequence are locally free, hence it remains exact after twist with an arbitrary sheaf on E) to prove by decreasing induction on p that $i_*(\Omega^p_\pi(k) \otimes \pi^*\mathcal{F})$ is an element of $\langle \widetilde{\mathcal{D}}(Z)_{d-1}, \dots, \widetilde{\mathcal{D}}(Z)_1\rangle$, for $k = 1, \dots, p$. It implies that $\tau_{\le -1}Lf^*j_*\mathcal{F}$ is an element of this category, hence, in view of (24), $f^*j_*\mathcal{F} \simeq i_*\pi^*\mathcal{F}$ is an element of $\langle \widetilde{\mathcal{D}}(Z)_{d-1}, \dots, \widetilde{\mathcal{D}}(Z)_1, Lf^*\mathcal{D}^b(Y)\rangle$, which finishes the proof. □

8. Modern approach

8.1. The stable category of spectra.

The first example of a 'distinguished triangle' is the Puppe sequence. Given a continuous morphism $f\colon X \to Y$ of pointed topological spaces, we have the mapping cone

$$Mf = \{(x, \omega) \in X \times Y^I \,|\, \omega(0) = y_0, \omega(1) = f(x)\}$$

which fits into a sequence

$$\Omega Y \to Mf \to X \to Y.$$

Remark 8.1. Note that we calculated Mf as a homotopy fiber product

$$\begin{array}{ccc} Mf & \longrightarrow & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y \end{array}$$

which means that we replaced the map $* \to Y$ with a fibration $Y^I \to Y$ and calculated the usual fiber product:

$$\begin{array}{ccc} Mf & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y^I & \longrightarrow & Y. \end{array}$$

The sequence can be extended to the left and for any topological space Z the sequence

$$[Z, \Omega Y] \to [Z, Mf] \to [Z, X] \to [Z, Y]$$

is exact, where $[Z, X]$ denotes the (set of) homotopy classes of pointed continuous morphisms $Z \to X$.

Puppe sequence almost makes Top_* into a triangulated category; the problem is that not every space is a loop space, so Ω is not an equivalence.

We denote by HoTop_* the homotopy category of pointed compactly generated weak Hausdorff spaces with homotopy classes of pointed continuous morphisms. There is a functor

$$\Sigma^\infty\colon \mathrm{HoTop}_* \to \mathrm{HoSpectra}_*$$

to the *stable homotopy category.* It is a category with suspension functor $\Sigma\colon \mathrm{HoSpectra}_* \to \mathrm{HoSpectra}_*$ which agrees with the suspension on HoTop_* and is an equivalence.

The functor Σ^∞ has a right adjoint

$$\Omega^\infty\colon \mathrm{HoSpectra}_* \to \mathrm{HoTop}_*.$$

Moreover, as all objects of $\mathrm{HoSpectra}_*$ are suspensions, $[X, Y]$ is an abelian group for any pair of objects $X, Y \in \mathrm{HoSpectra}_*$.

The *sphere spectrum*

$$\mathbb{S} := \Sigma^\infty S^0$$

allows us to define stable homotopy groups

$$\pi_n(X) = [\mathbb{S}, X]_n = [\Sigma^n \mathbb{S}, X].$$

A spectrum is *connective* if $\pi_n(X) = 0$ for $n < 0$.

The category $\mathrm{HoSpectra}_*$ is a symmetric monoidal category with respect to $X \wedge Y$. The unit object is $\mathbb{S}$.

A *prespectrum* E is a sequence of based spaces $E_0, E_1, \dots$ along with structure maps $\Sigma E_n \to E_{n+1}$. A map of prespectra is a sequence $f_n\colon E_n \to F_n$ which commute with the structure maps.

A prespectrum is a *CW prespectrum* if all E_i are CW complexes and all structure maps $\Sigma E_n \to E_{n+1}$ are inclusions of subcomplexes. A cell of a CW prespectrum is a cell of one of the E_i's together with all of its suspensions. A k-cell of E_n is a stable $(k-n)$-cell of $E_\bullet$.

CW perspectra are objects of $\mathrm{HoSpectra}_*$. Morphisms are homotopy classes of 'eventually defined' $f\colon E_\bullet \to F_\bullet$. f is a map on each m-cell of $E_\bullet$ which is defined on $(m-n)$-cell of E_n for $n >> 0$.

This is the original definition due to Adams. A modern point of view on $\mathrm{HoSpectra}_*$ is as the homotopy category of a stable $(\infty, 1)$-category.

8.2. **DG categories and DG enhancements.**

The first solution to the problem that cones are not functorial and that a triangulated category is a structure not a property was proposed by Bondal and Kapranov in [BK90]. They proposed, in some sense, not to make the last step and, in particular, view derived categories as the homotopy categories of complexes of injective objects.

More precisely, a *DG category* is a (pre)additive category $\mathcal{D}$ together with a grading and a differential ∂ on $\mathrm{Hom}_{\mathcal{D}}(D_1, D_2)$ for any pair D_1, D_2 of objects of $\mathcal{D}$. We assume that the graded Leibniz rule is satisfied

$$\partial(fg) = \partial(f)g + (-1)^{|f|} f \partial(g)$$

and that for any $D \in \mathcal{D}$, Id_D is closed of degree zero.

Examples of DG categories are categories of complexes over an additive category $\mathcal{A}$ with homogeneous morphisms; $f^\bullet\colon A^\bullet \to B^\bullet$ is of degree n if f^i is a morphism $A^i \to B^{i+n}$.

The differential is

$$(\partial f)^i = f^{i+1} d_A + (-1)^{n+1} d_B f^i.$$

A *homotopy category* $\underline{\mathcal{D}}$ of a DG category $\mathcal{D}$ has the same objects as $\mathcal{D}$ and morphisms are the zero cohomology groups of morphisms in $\mathcal{D}$.

The DG category $\mathcal{D}$ is *pretriangulated* if for any $D \in \mathcal{D}$ and $n \in \mathbb{Z}$ functor $\mathrm{Hom}(-, D)[n]$ is representable and if for any closed degree zero morphism $f\colon D_1 \to D_2$ the functor $\mathrm{Cone}(\mathrm{Hom}(-, D_1) \to \mathrm{Hom}(-, D_2))$ is representable.

If a DG category $\mathcal{D}$ is pretriangulated then $\underline{\mathcal{D}}$ is triangulated.

Let $\mathcal{T}$ be a triangulated category. A *DG enhancement* of $\mathcal{T}$ is a pretriangulated DG category $\mathcal{D}$ together with an equivalence $\underline{\mathcal{D}} \xrightarrow{\simeq} \mathcal{T}$.

Let $\mathcal{A}$ be an abelian category. If $\mathcal{A}$ has enough projectives then the DG category of bounded above complexes of projective objects in $\mathcal{A}$ is a DG enhancement for $\mathcal{D}^-(\mathcal{A})$. If $\mathcal{A}$ has enough injectives then the DG category of bounded below complexes of injectives in $\mathcal{A}$ is a DG enhancement for $\mathcal{D}^+(\mathcal{A})$. Note that the category of complexes of objects of $\mathcal{A}$ is not a DG enhancement for the derived category.

V. Lunts and D. Orlov proved that the DG enhancement for $\mathcal{D}^b(X)$, for a reasonable scheme X is unique, [LO10]. The result was further generalised by A. Canonaco and P. Stellari [CS18].

8.3. **Infinity categories.**

The following is based mostly on [Lur09].

Nowadays, the language of infinity categories is believed to be the most useful. Naively speaking, an infinity category is a category with objects, morphisms, morphisms between morphisms and so on. These have to satisfy axioms subject to higher coherence conditions.

The first approach to infinity categories was suggested by Dwyer and Kan. For them an infinity category was a category enriched over simplicial sets. Their construction turned out to be not enough to construct limits of infinity categories.

The remedy is to define infinity categories as Kan complexes. Before we do it, we give THE example of an infinity category. Let X be a topological space. We can form an infinity category πX whose objects are points of X. Morphisms in πX are given by paths $I \to X$, 2-morphisms by homotopies between paths, 3-morphisms as homotopies between homotopies and so on. The category πX is often called an *infinity groupoid* as all morphisms are invertible. In fact, every infinity groupoid is πX, for some topological space X. These are often referred to as *homotopy types*.

We shall be mostly interested in $(\infty, 1)$ categories, i.e. infinity categories in which all k-morphisms, for $k > 1$ are invertible. Therefore, it is a collection of objects and $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is an infinity groupoid, i.e. a homotopy type. One way to define an $(\infty, 1)$ category is to say that it is a category enriched over the category of compactly generated and weakly Hausfdorff topological spaces. This is a 'strict' version of an $(\infty, 1)$-category, therefore it is not very convenient to work with. However, the 'strict' version is equivalent to a weak version which is defined via simplicial sets.

Following Lurie, we shall refer to $(\infty, 1)$-categories as infinity categories.

If X is a topological space then the simplicial set $\mathrm{Sing}X$ is a Kan complex, i.e. it satisfies the property that for any $0 \le i \le n$ any diagram of solid arrows can be completed to a diagram of dotted arrows:

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{Sing}X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Here, Λ^n_i denotes the i'th horn obtained from Δ^n by deleting the i'th face and the interior. According to Quillen the functor Sing and geometric realisation provide mutually inverse equivalences of the category of topological spaces and Kan complexes.

We can also assign a simplicial set to a category $\mathcal{A}$, the *nerve* of a category. $N(\mathcal{A})_n = \mathrm{Map}(\Delta^n, \mathcal{A})$ is the set of n composable arrows. In fact, a simplicial set K is equivalent to $N(\mathcal{A})$, for some category $\mathcal{A}$ if and only if any $0 < i < n$ any diagram of solid arrows can be completed to a diagram of dotted arrows:

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

We can view an arbitrary simplicial set K is a generalised category whose objects are vertices of K, morphisms are the edges of K and a 2-simplex $\Delta^2 \to K$ should be thought of as

$$\begin{array}{ccccc} & & Y & & \\ & \overset{\phi}{\nearrow} & & \overset{\psi}{\searrow} & \\ X & & \xrightarrow{\quad\theta\quad} & & Z \end{array}$$

together with a 'homotopy' between θ and $\psi\phi$. However, this way we cannot always compose morphisms in K unless it satisfies the additional property:

Definition 8.2. An $(\infty,1)$-category is a simplicial set K which has the following property, for any $0 < i < n$ any map $f_0\colon \Lambda^n_i \to K$ admits an extension to $f\colon \Delta^n \to K$. The simplicial set K is often referred to as a *weak Kan complex.*

This approach to infinity categories was developed by Joyal.

If $\mathcal{C}$ is a topological category it is easy to define its homotopy category $\underline{\mathcal{C}}$. Its objects are objects of $\mathcal{C}$ and morphisms are $\pi_0\mathrm{Map}_{\mathcal{C}}(X,Y)$.

As every $(\infty,1)$-category $\mathcal{C}$ is equivalent to a topological category, we can consider a homotopy category of an infinity category. There is also a (nontrivial) way to write down explicitly the simplicial set $\mathrm{Map}_{\mathcal{C}}(X,Y)$.

The role of vector spaces in the world of infinity categories is played by the category $\mathcal{S}$ of *spaces*. This is the (appropriate ∞-enhancement of the) subcategory of the category of simplicial sets consisting of Kan complexes.

An important source of ∞-categories come from localisation. For a category $\mathcal{C}$ and a set W of morphisms there is an infinity category $L(C,W)$ which is universal with respect to functors to infinite categories that map W to equivalences. Its homotopy category is equivalent to the usual localisation.

The localisation procedure can be applied to the category of ∞-categories and strict ∞-functors for W being equivalences (F is an equivalence when it is fully faithful, $\mathrm{Map}(X,Y) \to \mathrm{Map}(F(X),F(Y))$ are weak homotopy equivalences, and essentially surjective, which can be checked on homotopy categories). This way we get an infinity category of infinity categories which allows us, in particular, to consider homotopy limits of infinity categories.

8.4. **Stable infinity categories.**

We would like to enhance triangulated categories to infnity categories. Therefore, we need to know when the homotopy category of an infnity category is triangulated. This is precisely the condition of stability.

Definition 8.3. Let $\mathcal{C}$ be an infinity category. A *zero object* of $\mathcal{C}$ is an object which is both initial and final. We say that $\mathcal{C}$ is *pointed* if it contains a zero object.

An object $0 \in \mathcal{C}$ is zero if for any object X spaces $\mathrm{Map}(0,X)$, $\mathrm{Map}(X,0)$ are contractible.

Definition 8.4. Let $\mathcal{C}$ be a pointed infinity category. A *triangle* in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ depicted as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

A triangle is a *fiber sequence* if it is a pullback square and a *cofiber sequence* if it is a pushout square.

A triangle is a pair of morphisms f, g, a 2-simplex corresponding to the diagram

$$\begin{array}{ccccc} & & Y & & \\ & {}^{f}\nearrow & & \searrow^{g} & \\ X & & \xrightarrow{\quad h \quad} & & Z \end{array}$$

and a 2-simplex

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & & \searrow & \\ X & & \xrightarrow{\quad h \quad} & & Z \end{array}$$

Definition 8.5. Let $\mathcal{C}$ be a pointed category and $g \colon X \to Y$ a morphism. A *fiber* of g is a fiber sequence

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Y \end{array}$$

A *cofiber* of g is a cofiber sequence

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

Definition 8.6. An infinity category $\mathcal{C}$ is *stable* if it satisfies the following

(1) There exists a zero object $0 \in \mathcal{C}$,
(2) Every morphism in $\mathcal{C}$ admits a fiber and a cofiber,
(3) A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Let $\mathcal{C}$ be a stable category and $X \in \mathcal{C}$ an object of $\mathcal{C}$. The shift $X[1] \in \mathcal{C}$ is defined via the pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X[1] \end{array}$$

Definition 8.7. Let $\mathcal{C}$ be a pointed category which admits cofibers. Suppose given a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in the homotopy category $\underline{\mathcal{C}}$. We will say that it is a *distinguished triangle* if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$

$$\begin{array}{ccccc} X & \xrightarrow{\widetilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \widetilde{g}} & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow{\widetilde{h}} & W \end{array}$$

satisfying

(1) Objects 0, $0'$ are zero,
(2) Both square are pushout diagrams,
(3) Morphisms $\widetilde{f}$ and $\widetilde{g}$ represent f and g, respectively,
(4) The map $h\colon Z \to X[1]$ is the composition of the homotopy class of $\widetilde{h}$ with the equivalence $W \simeq X[1]$ determined by the outer rectangle.

Theorem 8.8. *Let $\mathcal{C}$ be a stable infinity category. Then the homotopy category $\underline{\mathcal{C}}$ with the above distinguished triangles is a triangulated category.*

The useful property of stable categories is that the categories of functors are also stable

Proposition 8.9. *Let $\mathcal{C}$ be a stable category and K a simplicial set. Then the ∞-category Fun$(K, \mathcal{C})$ is stable.*

8.5. **Examples of stable infinity categories.**

The first example of a triangulated category is the derived category of an abelian category $\mathcal{A}$. We shall briefly discuss how to construct its $(\infty, 1)$-enhancement.

For simplicity, let us assume that has enough injective objects. In this case we already know the DG enhancement for $\mathcal{D}^+(\mathcal{A})$. Therefore, it suffices to describe an $(\infty, 1)$-enhancement of a DG category $\mathcal{D}$.

Definition 8.10. Let $\mathcal{D}$ be a DG category. The *differential graded nerve of* $\mathcal{D}$ is a simplicial set $N_{dg}(\mathcal{D})$ with $\mathrm{Hom}(\Delta^n, N_{dg}(\mathcal{D})) = (\{X_i\}_{0\leq i\leq n}, \{f_I\})$ where

(1) for $0 \leq i \leq n$ X_i is an object of $\mathcal{D}$,
(2) For every subset $I = \{i_- < i_m < i_{m-1} < \ldots < i_1 < i_+\} \subset [n]$ with $m \geq 0$ f_I is an element of $\mathrm{Hom}^m(X_{i_-}, X_{i_+})$ satisfying

$$\partial f_I = \sum_{1\leq j\leq m} (-1)^j (f_{I\setminus\{i_j\}} - f_{i_j<\ldots<i_+} \circ f_{i_-<\ldots<i_j}).$$

If $\alpha\colon [m] \to [n]$ is a nondecreasing function then the induced map $N_{dg}(\mathcal{D})_n \to N_{dg}(\mathcal{D})_m$ is given by

$$(\{X_i\}_{0\leq i\leq n}, f_I) \mapsto (\{X_{\alpha(j)}\}, \{g_J\})$$

where

$$g_J = \begin{cases} f_{\alpha(J)} & \text{if } \alpha|_J \text{ is injective} \\ \mathrm{Id}_{X_i} & \text{if } J = \{j, j'\} \text{ with } \alpha(j) = \alpha(j') = i, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 8.11. *Let $\mathcal{D}$ be a DG category. Then $N_{dg}(\mathcal{D})$ is an ∞-category. The homotopy categories of $\mathcal{D}$ and $N_{dg}(\mathcal{D})$ are equivalent.*

If $\mathcal{A}$ does not have enough projectives or injectives, we can localise the ∞-category corresponding to the DG category of complexes over $\mathcal{A}$.

Why do we prefer infinity categories to DG categories? Because for infinity categories we have descent.

Let X be a scheme. Denote by $\widetilde{\mathrm{QCoh}}(X)$ the $(\infty, 1)$-enhancement of $\mathcal{D}(\mathrm{QCoh}(X))$. Then

$$\widetilde{\mathrm{QCoh}}(X) = \lim_{U\subset X} \widetilde{\mathrm{QCoh}}(U)$$

where the limit is taken in the category of (presentable) ∞-categories. Presentability is some finiteness condition.

Any pointed ∞-category $\mathcal{C}$ has the universal stable ∞-category; the category of spectrum objects. It is the homotopy inverse limit of the tower

$$\ldots \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}.$$

If we take $\mathcal{C}$ to be the category $\mathcal{S}$ of spaces, the resulting stable ∞-category will be an ∞-enhancement for the stable homotopy theory.

8.6. **Derived algebraic geometry.** This part is based on [Toë14].

Derived algebraic geometry is based on 2 principles

(1) The smooth algebraic varieties or more generally smooth schemes and smooth maps are good. A non-smooth variety, scheme or a map between schemes must be replaced by the best possible approximation by smooth objects.
(2) Approximations of varieties, schemes and maps of schemes are expressed in terms of simplicial resolutions. The simplicial resolutions must only be considered up to the notion of weak equivalence, and are controlled by higher categorical or homotopical structures.

Definition 8.12. A *derived scheme* consists of a pair $(X, \mathcal{A}_X)$ where X is a topological space and $\mathcal{A}_X$ is a sheaf of commutative simplicial rings on X such that

(1) The ringed space $(X, \pi_0(\mathcal{A}_X))$ is a scheme,
(2) For all $i > 0$, $\pi_i(\mathcal{A}_X)$ is quasi-coherent.

We would like derived schemes to form a nice category.

To this end we consider the infinity category of simplicial commutative rings **sComm**, or simply the category of derived rings, which is the localisation of the category of simplicial commutative rings with respect to weak homotopy equivalences. For a topological space X there is an ∞-category of $\mathbf{sComm}(X)$ of stacks on X with coefficients in the ∞-category of derived rings. These are prestack satisfying descent condition.

We can introduce the category of derived ringed spaces **dRgSp** whose objects are pairs $(X, \mathcal{O}_X)$ of a topological space X and $\mathcal{O}_X \in \mathbf{sComm}(X)$. Any derived ringed space has a truncation to a ringed space $(X, \pi_0(\mathcal{O}_X))$ and we can consider the full subcategory of **dRgSp** consisting of derived ringed spaces whose truncations are locally ringed spaces.

Definition 8.13. The ∞-category of derived schemes is the full subcategory of **dRgSp** consisting of $(X, \mathcal{O}_X)$ such that

(1) The ringed space $(X, \pi_0(\mathcal{O}_X))$ is a scheme,
(2) For all $i > 0$, $\pi_i(\mathcal{O}_X)$ is quasi-coherent.

A ring can be considered as a constant simplicial ring, which defines a functor $\mathbf{Comm} \to \mathbf{sComm}$. This inclusion has right adjoint π_0. This adjunction extends to an adjunction between the category of schemes and derived schemes. The inclusion is fully faithful.

For a derived scheme we define the derived category of quasi-coherent sheaves as a limit of such categories for open affine subsets. An affine $U \subset X$ corresponds to some simplicial commutative ring A, which can be normalised to a DG ring. The category of quasi-coherent sheaves is then the localisation of the category of DG modules in quasi-isomorphisms.

In the world of derived schemes the base change holds without flatness assumption. There is also a better behaved version of a cotangent complex.

In characteristic zero the category of simplicial commutative rings is equivalent to the category of non-positively graded DG algebras which simplifies the picture a bit.

8.7. **Why do we care about derived algebraic geometry.**

Except for descent, the derived point of view on algebraic geometry helps us to understand better the 'standard facts'. Let us mention the one regarding deformation theory.

Let X be a scheme. Consider the deformation functor $X^\wedge$ which to a local commutative Artinian algebra A assigns a deformation of X over A, i.e. $\overline{X} \to \operatorname{Spec} A$ such that the fiber over the closed point is isomorphic to X (in fact, we should fix this isomorphism). Deformations over dual numbers $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, the *first order deformations*, are known to be equivalent to $H^1(X, T_X)$. To every first order deformation η_1 of X we can assign an obstruction class $\theta \in H^2(X, T_X)$ which vanishes if and only if η extends to a deformation over $\mathbb{C}[\varepsilon]/(\varepsilon^3)$.

The proof of the last statement is usually an ad hoc argument. However, we can extend the deformation functor to the category of local commutative non-positively graded Artinian DG rings. We can then consider $\mathbb{C}[\delta] := \mathbb{C} \oplus \mathbb{C}[1]$, the square zero extension with ε in degree -1. Then $H^2(X, T_X) \simeq \pi_0 X^\wedge(\mathbb{C}[\delta])$ which gives (classically not known) geometric interpretation of this cohomology group.

The ring $\mathbb{C}[\varepsilon]/(\varepsilon^3)$ fits into a pull-back diagram

$$\begin{array}{ccc} \mathbb{C}[\varepsilon]/(\varepsilon^3) & \longrightarrow & \mathbb{C}[\varepsilon]/(\varepsilon^2) \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{C}[\delta] \end{array}$$

This determines a pull-back square

$$\begin{array}{ccc} X^\wedge(\mathbb{C}[\varepsilon]/(\varepsilon^3)) & \longrightarrow & X^\wedge(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \\ \downarrow & & \downarrow \\ X^\wedge(\mathbb{C}) & \longrightarrow & X^\wedge(\mathbb{C}[\delta]) \end{array}$$

As $X^\wedge(\mathbb{C})$ is a point, we get a fiber sequence of spaces

$$X^\wedge(\mathbb{C}[\varepsilon]/(\varepsilon^3)) \to X^\wedge(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \to X^\wedge(\mathbb{C}[\delta]).$$

In particular, a first order deformation $\eta \in \pi_0 X^\wedge(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ determines an element in $\pi_0(X^\wedge(\mathbb{C}[\delta]) \simeq H^2(X, T_X)$ which vanishes if and only if η can be lifted.

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