1 16.01. Lecture 1. Complex algebraic curves in $\mathbb{C}^2$ and real algebraic curves

Let $P(x,y) \in \mathbb{C}[x,y]$ be a polynomial in 2 variables with complex coefficients. P has *no repeated factors* if it cannot be written as

$$P(x,y) = Q(x,y)^2 R(x,y)$$

for non-constant polynomial Q.

Definition 1.1. Let $P \in \mathbb{C}[x,y]$ be a non-constant polynomial with no repeated factors. The *complex algebraic curve defined by* P is

$$C = \{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\}.$$

We want to assume that curves are given by polynomials with no repeated factors because of the following

Theorem 1.2 (Hilbert's Nullstellensatz). *If $P(x,y)$ and $Q(x,y)$ are polynomials in $\mathbb{C}[x,y]$ then*

$$\{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\} = \{(x,y) \in \mathbb{C}^2 \mid Q(x,y) = 0\}$$

if and only if there exist $m, n \in \mathbb{N}$ such that P divides Q^m and Q divides P^n. In other words, if and only if P and Q have the same irreducible factors possibly occurring with different multiplicities.

It follows from Theorem 1.2 that two polynomials P, Q with no repeated factors define the same complex algebraic curve C if and only if there exists $\lambda \in \mathbb{C}^*$ such that $P = \lambda Q$.

In general, one can define complex algebraic curves as equivalence classes of non-constant polynomials $P \in \mathbb{C}[x,y]$ where $P \sim Q$ if and only if P and Q have the same irreducible factors.

A polynomial with repeated factors defines a curve with multiplicities attached, for $P = x^2 y$ the line $\{y = 0\}$ has multiplicity one while $\{x = 0\}$ comes into $C = \{P(x,y) = 0\}$ with multiplicity two.

Definition 1.3. The *degree* of a curve $C = \{P(x,y) = 0\}$ is the degree of the polynomial $P(x,y) = \sum c_{r,s} x^r y^s$, i.e.

$$d = \max\{r + s \mid c_{r,s} \neq 0\}.$$

Definition 1.4. A point $(a,b) \in C$ is a *singular* point (or *singularity*) of C, if

$$\frac{\partial P}{\partial x}(a,b) = 0, \quad \frac{\partial P}{\partial y}(a,b) = 0.$$

The set of singular points is denoted by $\mathrm{Sing}(C)$. C is called non-singular if $\mathrm{Sing}(C) = \emptyset$.

Definition 1.5. A curve defined by

$$\alpha x + \beta y + \gamma = 0,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ and $(\alpha, \beta) \neq (0,0)$, is a *line*.

We will be mostly interested in curves in projective plane. They are given by homogeneous polynomials

Definition 1.6. A non-zero polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ is *homogeneous* of degree d if

$$P(\lambda x_1, \ldots, \lambda x_n) = \lambda^d P(x_1, \ldots, x_n),$$

for all $\lambda \in \mathbb{C}$. Equivalently P has the form

$$P(x_1, \ldots, x_n) = \sum_{r_1 + \ldots + r_n = d} a_{r_1, \ldots, r_n} x_1^{r_1} \ldots x_n^{r_n}.$$

Lemma 1.7. *If $P \in \mathbb{C}[x, y]$ is homogeneous of degree d then it factors as a product of linear polynomials*

$$P(x, y) = \prod_{i=1}^{d} (\alpha_i x + \beta_i y),$$

for some $\alpha_i, \beta_i \in \mathbb{C}$.

Proof. We have

$$P(x, y) = \sum_{r+s=d} a_{r,s} x^r y^s = \sum_{r=0}^{d} a_{r,d-r} x^r y^{d-r} = y^d \sum_{r=0}^{d} a_{r,d-r} (\frac{x}{y})^r.$$

Polynomial $\widetilde{P}(t) := \sum_{r=0}^{d} a_{r,d-r} t^r \in \mathbb{C}[t]$ is a complex polynomial in one variable, hence it can be factorised as

$$\widetilde{P}(t) = \sum_{r=0}^{d} a_{r,d-r} t^r = a_e \prod_{i=1}^{e} (t - \gamma_i),$$

where $e = \max\{r \mid a_{r,d-r} \neq 0\}$ is the degree of $\widetilde{P}$. Then

$$P(x, y) = y^d \widetilde{P}(\frac{x}{y}) = y^d a_e \prod_{i=1}^{e} ((\frac{x}{y}) - \gamma_i) = a_e y^{d-e} \prod_{i=1}^{e} (x - \gamma_i y).$$

□

Since $P(x, y)$ is a polynomial it has a finite Taylor expansion

$$P(x, y) = \sum_{i,j \geq 0} \frac{\partial^{i+j} P}{\partial x^i \partial y^j}(a, b) \frac{(x-a)^i (y-b)^j}{i! j!}$$

at any point $(a, b) \in \mathbb{C}^2$.

Definition 1.8. The *multiplicity* of curve $C = \{P(x, y) = 0\}$ at point $(a, b) \in C$ is the smallest $m \in \mathbb{N}$ such that $\frac{\partial^m P}{\partial x^i \partial y^j}(a, b) \neq 0$. The polynomial

$$\sum_{i+j=m} \frac{\partial^m P}{\partial x^i \partial y^j}(a, b) \frac{(x-a)^i (y-b)^j}{i! j!} \tag{1}$$

is then homogeneous of degree m and so by Lemma 1.7 can be written as a product of polynomials of the form

$$\alpha(x-a)+\beta(y-b)$$

where $(\alpha,\beta)\in\mathbb{C}^2\setminus\{0\}$. The lines defined by these linear polynomials are *tangent lines* to C at (a,b). The point (a,b) is non-singular if and only if its multiplicity is 1; in this case C has just one tangent line

$$\frac{\partial P}{\partial x}(a,b)(x-a)+\frac{\partial P}{\partial y}(a,b)(y-b)=0.$$

A point $(a,b)\in C$ is a *double point* (respectively *triple point*, etc.) if the multiplicity of C at (a,b) is two (respectively three, etc.). A singular point is *ordinary* if polynomial (1) has no repeated factors, i.e. if C has m distinct tangent lines at (a,b).

Cubic curves

$$C_1=\{y^2-x^3-x^2=0\},\qquad\qquad C_2=\{y^2-x^3=0\}$$

have double points at the origin. Indeed, if $C_i=\{P_i(x,y)=0\}$ then

$$\begin{aligned}
&\frac{\partial P_1}{\partial x}=-3x^2-2x, &&\frac{\partial P_2}{\partial x}=-3x^2,\\
&\frac{\partial P_1}{\partial y}=2y, &&\frac{\partial P_2}{\partial y}=2y,\\
&\frac{\partial^2 P_1}{\partial x^2}=-6x-2, &&\frac{\partial^2 P_2}{\partial x^2}=-6x,\\
&\frac{\partial^2 P_1}{\partial x\partial y}=0, &&\frac{\partial^2 P_2}{\partial x\partial y}=0,\\
&\frac{\partial^2 P_1}{\partial y^2}=2, &&\frac{\partial^2 P_2}{\partial y^2}=2.
\end{aligned}$$

Polynomials (1) are respectively

$$-2\frac{x^2}{2!}+0xy+2\frac{y^2}{2!}=(y-x)(y+x),\qquad 0\frac{x^2}{2!}+0xy+2\frac{y^2}{2!}=y^2.$$

It follows that only for C_1 the double point at the origin is ordinary.

The curve $C_3=\{(x^4+y^4)^2-x^2y^2=0\}$ has a singular point of multiplicity four at the origin which is not ordinary. The curve $C_4=\{(x^4+y^4-x^2-y^2)^2-9x^2y^2\}$ has an ordinary singular point of multiplicity 4 at the origin.

Historically, real algebraic curves were studied first.

Let $P(x,y)\in\mathbb{R}[x,y]$ be a non-constant polynomial in 2 variables with real coefficients. A *real algebraic curve defined by P* is

$$C=\{(x,y)\in\mathbb{R}^2\mid P(x,y)=0\}.$$

The study of real algebraic curves originates in ancient Greece. Ancient Greeks had sophisticated geometrical methods but a relatively primitive understanding of algebra. To them a circle was not defined by an equation

$$(x-a)^2 + (y-b)^2 = r^2$$

but it was the locus of all points having equal distance from a fixed point (a, b). Generalising the above point of view we arrive at the definition of conchoid.

More precisely, consider a fixed point $q \in \mathbb{R}^2$ and a fixed constant $a \in \mathbb{R}_{>0}$. The *conchoid* of C with respect to q and with parameter a is the locus of all points $p \in \mathbb{R}^2$ such that the line through p and q meets C at a distance a from p.

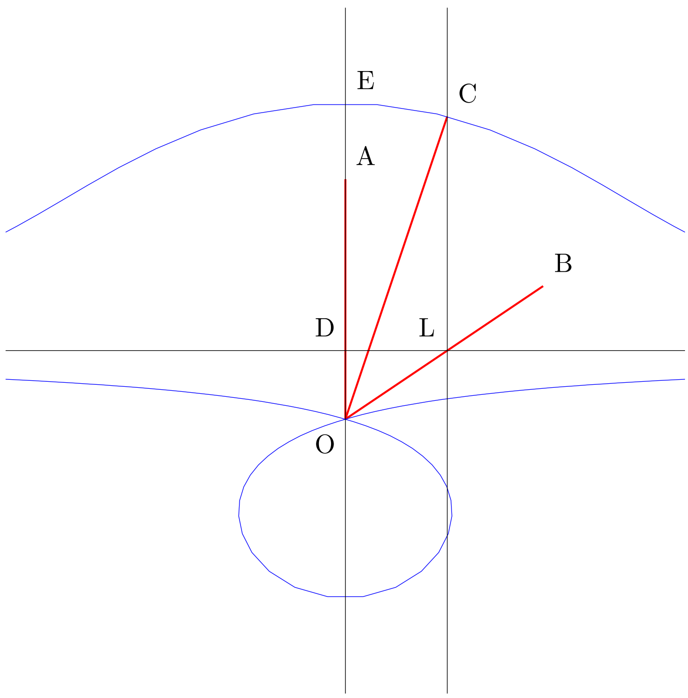
The classical example is the conchoid of Nicomedes (225 B.C.) which is the conchoid of a line with respect to a point not on a line. If the line is

$$x = b$$

and $q = (0, 0)$ is the origin then the conchoid is

$$(x^2 + y^2)(x-b)^2 = a^2x^2.$$

It can be used to trisect an angle α. Let α be the angle AOB and let D be any point



on $|AO|$. Line perpendicular to $|AD|$ in point D intersects line $|OB|$ in point L. We consider the conchoid of line DL with respect to point O with parameter $2|OL|$ (i.e. $|DE| = 2|OL|$). Line parallel to $|OA|$ and passing through L intersect the conchoid in point C. Then the angle AOC is one third of the angle AOB.

One can also consider a path of a fixed point on a circle rolling along a line. The obtained curve, cycloid, was discovered by Bernoulli around 1700. It is a path that a particle takes when sliding from one point to another on a vertical plane. This discovery led to the development of the theory of variations.

2 19.01. Lecture 2. Complex projective spaces and projective transformations

Complex algebraic curves, as defined in the previous lecture, are never compact. For many purposes it is useful to compactify complex algebraic curves by adding points "infinity". For example, curves

$$C_1 = \{y^2 = x^2 - 1\}, \qquad C_2 = \{y = cx\}$$

intersect when $c \neq \pm 1$. If $c = \pm 1$, C_1 and C_2 do not intersect but they are asymptotic as x and y tend to infinity. We would like to "add points at infinity" to say that C_1 and C_2 intersect, possibly at infinity, for any value of $c \in \mathbb{C}$.

To do so, we need the notion of a complex projective space $\mathbb{P}^2$.

Recall that a topological space X is *compact* if any open cover of X has a finite subcover. A metric space (X, d) is compact if (X, d) is complete (any Cauchy sequence has a limit) and universally bounded (for any $\varepsilon > 0$ there exists a finite open cover of X with balls of radius ε). X is a *Hausdorff space* if it satisfies axiom T2, i.e. if for any pair of points x, y of X there exist open subsets $U, V \subset X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Remark 2.1. Recall important properties and topological condition for compactness

(i) A subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ is compact if and only if it is closed and bounded (Heine-Borel theorem).

(ii) If $f\colon X \to Y$ is a continuous map between topological spaces and X is compact, then $f(X)$ is compact.

(iii) If X is compact and $f\colon X \to \mathbb{R}$ is a continuous map, then f is bounded and attains its bounds.

(iv) A closed subset of a compact space is compact.

(v) A compact subset of a Hausdorff space is closed.

(vi) A finite union of compact spaces is compact.

The idea behind constructing $\mathbb{P}^2$ is to view $(x, y) \in \mathbb{C}^2$ as a one-dimensional complex subspace of $\mathbb{C}^3$ spanned by $(x, y, 1)$. Every one-dimensional complex subspace of $\mathbb{C}^3$ not contained in the plane $S = \{(x, y, z) \in \mathbb{C}^3 \mid z = 0\}$ is uniquely defined by a point $(x', y', 1)$. One-dimensional subspaces contained in S provide"points at infinity".

Definition 2.2. Complex projective space $\mathbb{P}^n$ of dimension n is the set of complex one-dimensional subspaces of $\mathbb{C}^{n+1}$. When $n = 1$ we have the projective line $\mathbb{P}^1$, for $n = 2$ we get the projective plane $\mathbb{P}^2$.

To define topology on $\mathbb{P}^n$, we define a quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

First, note that any one-dimensional complex subspace U of $\mathbb{C}^{n+1}$ is spanned by $u \in \mathbb{C}^{n+1} \setminus \{0\}$ and that $u, v \in \mathbb{C}^{n+1} \setminus \{0\}$ span the same U if and only if $u = \lambda v$, for some $\lambda \in \mathbb{C}^*$. Therefore, $\mathbb{P}^n$ is the set of equivalence classes for the equivalence relation $\sim$ on $\mathbb{C}^{n+1} \setminus \{0\}$ such that $u \sim v$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $u = \lambda v$.

Definition 2.3. Any non-zero $(x_0, \dots, x_n) \in \mathbb{C}^{n+1}$ represents an element x of $\mathbb{P}^n$. We call $(x_0, \dots, x_n)$ *homogeneous coordinates* and write

$$x = [x_0 : \ldots : x_n].$$

Then

$$\mathbb{P}^n = \{[x_0 : \ldots : x_n] \,|\, (x_0, \dots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}\}$$

and

$$[x_0 : \ldots : x_n] = [y_0 : \ldots : y_n]$$

if and only if there exists $\lambda \in \mathbb{C}^*$ such that $x_j = \lambda y_j$, for $j = 0, \dots, n$.

We make $\mathbb{P}^n$ into topological space. We define $\Pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ as

$$\Pi(x_0, \dots, x_n) = [x_0 : \ldots : x_n]$$

and endow $\mathbb{P}^n$ with the *quotient topology*, i.e. $U \subset \mathbb{P}^n$ is open if and only if $\Pi^{-1}(U) \subset \mathbb{C}^{n+1} \setminus \{0\}$ is.

Remark 2.4. It follows that

(i) $B \subset \mathbb{P}^n$ is closed if and only if $\Pi^{-1}(B) \subset \mathbb{C}^{n+1} \setminus \{0\}$ is.

(ii) $\Pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is continuous.

(iii) Map $f\colon \mathbb{P}^n \to X$ is continuous if and only if $f \circ \Pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to X$ is. More generally, if $A \subset \mathbb{P}^n$ is any subset then $f\colon A \to X$ is continuous if and only if $f \circ \Pi\colon \Pi^{-1}(A) \to X$ is.

We define subsets $U_i \subset \mathbb{P}^n$ as

$$U_i = \{[x_0 : \ldots : x_n] \,|\, x_i \neq 0\}.$$

Note that it is well-defined: if $x_i \neq 0$ then $\lambda x_i \neq 0$, for $\lambda \in \mathbb{C}^*$. Moreover,

$$\Pi^{-1}(U_i) = \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\} \,|\, x_i \neq 0\}$$

is an open subset of $\mathbb{C}^{n+1} \setminus \{0\}$, hence $U_i \subset \mathbb{P}^n$ is open.

Define $\varphi_0\colon U_0 \to \mathbb{C}^n$ by

$$\varphi_0[x_0 : \ldots : x_n] = (\frac{x_1}{x_0}, \ldots \frac{x_n}{x_0}).$$

It is well-defined with inverse $(y_1, \dots, y_n) \mapsto [1 : y_1 : \ldots : y_n]$.

Definition 2.5. Coordinates $(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$ of $[x_0 : \ldots : x_n] \in U_0$ are called *inhomogeneous coordinates.*

Since the map $\varphi_0 \circ \Pi\colon \Pi^{-1}(U_0) \to \mathbb{C}^n$ is continuous, so is φ_0. The inverse of φ_0 is the composition of Π with the continuous map $\mathbb{C}^n \to \mathbb{C}^{n+1} \setminus \{0\}$, $(y_1, \ldots, y_n) \mapsto (1, y_1, \ldots, y_n)$. It follows that φ_0 is a homeomorphism.

Similarly, $\varphi_j\colon U_j \xrightarrow{\simeq} \mathbb{C}^n$ are homeomorphisms.

The complement $\mathbb{P}^n \setminus U_n$ is the hyperplane

$$\{[x_0 : \ldots : x_n] \,|\, x_n = 0\} \simeq \mathbb{P}^{n-1}.$$

Then $\mathbb{P}^n = U_n \cup \mathbb{P}^{n-1}$ and $U_n \simeq \mathbb{C}^n$.

Any point of $\mathbb{P}^n$ lies in some U_i, hence $\{U_i \,|\, i = 0, \ldots, n\}$ is an open cover of $\mathbb{P}^n$. It follows that $f\colon \mathbb{P}^n \to X$ is continuous if and only if $f \circ \varphi_i^{-1}\colon \mathbb{C}^n \to X$ is continuous, for $i = 0, \ldots, n$. Furthermore, $f\colon X \to \mathbb{P}^n$ is continuous if and only if $f^{-1}(U_i) \subset X$ is open, for any $i = 0, \ldots, n$, and $\varphi_i \circ f\colon f^{-1}(U_i) \to \mathbb{C}^n$ is continuous, for any i.

Proposition 2.6. *The projective space $\mathbb{P}^n$ is compact.*

Proof. Let $S^{2n+1} = \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \,|\, |x_0|^2 + \ldots + |x_n|^2 = 1\}$ be an $2n+1$-dimensional sphere. It is a closed bounded subset of $\mathbb{C}^{n+1}$, hence it is compact by Heine-Borel theorem. The restriction $\Pi|_{S^{2n+1}}\colon S^{2n+1} \to \mathbb{P}^n$ is continuous, hence its image is compact, see 2.1(iii). We shall show that $\Pi(S^{2n+1}) = \mathbb{P}^n$.

Let now $[x_0 : \ldots : x_n] \in \mathbb{P}^n$ be an arbitrary point and let $\lambda = |x_0|^2 + \ldots + |x_n|^2$. Then $[x_0 : \ldots : x_n] = [\sqrt{\lambda} x_0 : \ldots : \sqrt{\lambda} x_n]$ and $[\sqrt{\lambda} x_0 : \ldots : \sqrt{\lambda} x_n] = \Pi|_{S^{2n+1}}(\sqrt{\lambda} x_0, \ldots, \sqrt{\lambda} x_n)$. It follows that $\Pi|_{S^{2n+1}}$ is surjective. □

Definition 2.7. A projective transformation of $\mathbb{P}^n$ is a bijection $f\colon \mathbb{P}^n \to \mathbb{P}^n$ such that there exists a linear isomorphism $\varphi\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ such that $f \circ \Pi = \Pi \circ \varphi|_{\mathbb{C}^{n+1} \setminus \{0\}}$:

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\;f\;} & \mathbb{P}^n \\ \Pi \uparrow & & \uparrow \Pi \\ \mathbb{C}^{n+1} \setminus \{0\} & \xrightarrow{\;\varphi\;} & \mathbb{C}^{n+1} \setminus \{0\} \end{array}$$

Lemma 2.8. *A projective transformation $f\colon \mathbb{P}^n \to \mathbb{P}^n$ is continuous.*

Proof. By definition $f \circ \Pi = \Pi \circ \varphi|_{\mathbb{C}^{n+1} \setminus \{0\}}$, for some $\varphi\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. As both Π and φ are continuous, so is $f \circ \Pi$. It follows from 2.4(iii) that f is continuous. □

Definition 2.9. A *hyperplane* in $\mathbb{P}^n$ is the image under Π of $V \setminus \{0\}$, for a linear subspace $V \subset \mathbb{C}^{n+1}$ of dimension n.

Lemma 2.10. *Given $n+2$ distinct points $p_0, \ldots, p_n$ and q of $\mathbb{P}^n$, no $n+1$ of which lie on a hyperplane, there exists a unique projective transformation f mapping p_i to $[0 : \ldots : 0 : 1 : 0 \ldots, 0]$, where 1 is in the i'th place, and mapping q to $[1 : \ldots : 1]$.*

Proof. Let $u_0, \ldots, u_n$ and v be some elements of $\mathbb{C}^{n+1} \setminus \{0\}$ such that $\Pi(u_i) = p_i$ and $\Pi(v) = q$. By assumption, $u_0, \ldots, u_n$ are linearly independent, hence they form a basis of $\mathbb{C}^{n+1}$ and there exists a unique linear transformation ψ mapping $\langle u_0, \ldots, u_n \rangle$ to the standard basis $\langle e_0, \ldots, e_n \rangle$ of $\mathbb{C}^{n+1}$. Vector v does not lie in a linear subspace generated by any n of $u_0, \dots, u_n$, hence $\psi(v)$ does not lie in a linear subspace generated by any n of $e_0, \dots, e_n$. In other words, $\psi(v) = (\lambda_0, \dots, \lambda_n)$ and $\lambda_i \neq 0$, for all i.

Let φ be the composition of ψ with the linear transformation with matrix $\mathrm{Diag}(1/\lambda_0, \dots, 1/\lambda_n)$. Then φ defines the required projective transformation f. $\square$

Proposition 2.11. *The projective space $\mathbb{P}^n$ is Hausdorff.*

Proof. Let p and q be two distinct points of $\mathbb{P}^n$. If p and q both lie in U_0 then $\varphi_0(p)$ and $\varphi_0(q)$ have disjoint open neighbourhoods V and U in $\mathbb{C}^n$. Then p and q have disjoint neighbourhoods $\varphi_0^{-1}(V)$, $\varphi_0^{-1}(U)$.

If p and q do not lie in U_0 then we can find $p_1, \dots, p_n$ such that $p, p_1, \dots, p_n, q$ satisfy assumption of Lemma 2.10. Thus, there exists a projective transformation f such that $f(p) = [1 : 0 : \ldots : 0]$ and $f(q) = [1 : \ldots : 1]$, i.e. $f(p)$ and $f(q)$ lie in U_0. By the above argument, there exist disjoint open neighbourhoods U' and V' of $f(p)$ and $f(q)$. Then $f^{-1}(U')$ and $f^{-1}(V')$ are disjoint open neighbourhoods of p and q. $\square$

19.01. Homework I

The homework in due on Thursday the 2nd of February. Please hand in your solutions before the lecture.

1. Show that the subset of $\mathbb{C}^2$ consisting of points of the form

$$(t^2, t^3 + 1),\ t \in \mathbb{C}$$

is a complex algebraic curve.

2. The *Hesse pencil* of cubic curves consist of curves

$$C_\lambda = \{[x : y : z] \mid x^3 + y^3 + z^3 + 3\lambda xyz = 0\}.$$

For which values $\lambda \in \mathbb{C}$ is the curve C_λ in $\mathbb{P}^2$ non-singular? List singular points and their multiplicities when they exist.

3. Show that the complex line in $\mathbb{P}^2$ through the points $[0 : 1 : 1]$ and $[t : 0 : 1]$ meets the projective curve

$$C = \{x^2 + y^2 = z^2\}$$

in two points $[0 : 1 : 1]$ and $[2t : t^2 - 1 : t^2 + 1]$. Show that there is a bijection from the complex line defined by $\{y = 0\}$ to C given by

$$[t : 0 : 1] \mapsto [2t, t^2 - 1, t^2 + 1],\ [1 : 0 : 0] \mapsto [0 : 1 : 1].$$

Deduce that the complex solutions to Pythagoras' equation

$$x^2 + y^2 = z^2$$

are

$$x = 2\lambda\mu,\ y = \lambda^2 - \mu^2,\ z = \lambda^2 + \mu^2,$$

for $\lambda, \mu \in \mathbb{C}$.

Solutions to Homework I

1. Any point (t^2, t^3+1) in $\mathbb{C}^2$ satisfies the equation
$$(t^2)^3 = (t^3+1-1)^2 = (t^3+1)^2 - 2(t^3+1) + 1.$$
In other words, any point (t^2, t^3+1) lies on a curve
$$C = \{(x,y) \in \mathbb{C}^2 \,|\, x^3 - y^2 + 2y - 1 = x^3 - (y-1)^2 = 0\}.$$
To show that any point on C is of the form (t^2, t^3+1) for some $t \in \mathbb{C}$ we consider $\psi \colon \mathbb{C} \to C$
$$\psi(t) = (t^2, t^3+1)$$
and check that $\varphi \colon C \to \mathbb{C}$ defined as
$$\varphi(x,y) = \begin{cases} \frac{y-1}{x} & \text{if } (x,y) \neq (0,1), \\ 0 & \text{if } (x,y) = (0,1) \end{cases}$$
is inverse to ψ.

First, we check that φ is well-defined. If $(x,y) \in C$ and $x = 0$ then $(y-1)^2 = 0$ hence $y = 1$. Thus, for points $(x,y) \neq (0,1)$ in C the quotient $(y-1)/x$ is defined.

For $t \in \mathbb{C} \setminus \{0\}$, we have
$$\varphi\psi(t) = \varphi(t^2, t^3+1) = \frac{t^3}{t^2} = t$$
while for $t = 0$:
$$\varphi\psi(0) = \varphi(0,1) = 0.$$
Finally, for $(x,y) \in C \setminus \{(0,1)\}$ we have
$$\psi\varphi(x,y) = \psi(\frac{y-1}{x}) = (\frac{(y-1)^2}{x^2}, \frac{(y-1)^3}{x^3} + 1) = (\frac{x^3}{x^2}, \frac{(y-1)^3}{(y-1)^2} + 1) = (x,y).$$
The last equality follows from the fact that if $(x,y) \in C$ and $y = 1$ then $x^3 = 0$, i.e. $(x,y) = (0,1)$. Hence, for $(x,y) \in C \setminus \{(0,1)\}$, $y - 1 \neq 0$ and $\frac{(y-1)^3}{(y-1)^2} = y - 1$.

For $(0,1) \in C$ we clearly have
$$\psi\varphi(0,1) = \psi(0) = (0,1).$$

2. Assume that $[a:b:c]$ is a singular point of
$$C = \{[x:y:z] \,|\, x^3 + y^3 + z^3 + 3\lambda xyz = 0\}.$$
Then we have
$$P(a,b,c) = a^3 + b^3 + c^3 + 3\lambda abc = 0, \qquad \frac{\partial P}{\partial x}(a,b,c) = 3a^2 + 3\lambda bc = 0,$$
$$\frac{\partial P}{\partial y}(a,b,c) = 3b^2 + 3\lambda ac = 0, \qquad \frac{\partial P}{\partial z}(a,b,c) = 3c^2 + 3\lambda ab = 0.$$

Since the above equations remain unchanged under any permutation of the set $\{a, b, c\}$, points $[a : c : b]$, $[b : a : c]$, $[c : b : a]$, $[b : c : a]$ and $[c : a : b]$ are also singular points of C.

Since either a or b or c in non-zero, we can assume that $c \neq 0$, i.e. that $[\frac{a}{c}, \frac{b}{c} : 1]$ is a singular point of C. Putting $\widetilde{a} = \frac{a}{c}$, $\widetilde{b} = \frac{b}{c}$, we get

$$\widetilde{a}^3 + \widetilde{b}^3 + 1 + 3\lambda\widetilde{a}\widetilde{b} = 0, \qquad \widetilde{a}^2 + \lambda\widetilde{b} = 0,$$
$$\widetilde{b}^2 + \lambda\widetilde{a} = 0, \qquad 1 + \lambda\widetilde{a}\widetilde{b} = 0.$$

From the last equality it follows that

$$\lambda \neq 0$$

and

$$\widetilde{b} = -\frac{1}{\lambda\widetilde{a}}.$$

Putting it into the second and the third equation we get

$$\widetilde{a}^2 - \lambda\frac{1}{\lambda\widetilde{a}} = \widetilde{a}^2 - \frac{1}{\widetilde{a}} = 0, \qquad \frac{1}{\lambda^2\widetilde{a}^2} + \lambda\widetilde{a} = 0,$$

i.e.

$$\widetilde{a}^3 - 1 = 0, \qquad 1 + \lambda^3\widetilde{a}^3 = 0.$$

Since $\widetilde{a}^3 = 1$, we get that

$$1 + \lambda^3 = 0.$$

Let now $\lambda^3 = -1$, i.e. $\lambda \in \{-1, -\varepsilon_3, -\varepsilon_3^2\}$ for $\varepsilon_3 = e^{\frac{2}{3}\pi i}$. Then points

$$[\lambda : -1 : \lambda], \qquad [\lambda\varepsilon_3^2 : -1 : \lambda\varepsilon_3], \qquad [\lambda\varepsilon_3, -1 : \lambda\varepsilon_3^2]$$

and points obtained by any permutation of coordinates are the singular points of C.

If $\lambda = -1$, curve C has 3 singular points:

$$[-1 : -1 : -1] = [1 : 1 : 1],$$
$$[\varepsilon_3^2 : 1 : \varepsilon_3] = [1 : \varepsilon_3 : \varepsilon_3^2] = [\varepsilon_3 : \varepsilon_3^2 : 1]$$
$$[\varepsilon_3 : 1 : \varepsilon_3^2] = [\varepsilon_3^2 : \varepsilon_3 : 1] = [1 : \varepsilon_3^2 : \varepsilon_3].$$

If $\lambda = -\varepsilon_3$, curve C has 3 singular points:

$$[\varepsilon_3 : 1 : \varepsilon_3] = [1 : \varepsilon_3^2 : 1],$$
$$[1 : 1 : \varepsilon_3^2] = [\varepsilon_3 : \varepsilon_3 : 1],$$
$$[\varepsilon_3^2 : 1 : 1] = [1 : \varepsilon_3 : \varepsilon_3].$$

If $\lambda = -\varepsilon_3^2$ curve C has 3 singular points:

$$[\varepsilon_3^2 : 1 : \varepsilon_3^2] = [1 : \varepsilon : 1],$$
$$[\varepsilon_3 : 1 : 1] = [1 : \varepsilon_3^2 : \varepsilon_3^2],$$
$$[1 : 1 : \varepsilon_3] = [\varepsilon_3^2 : \varepsilon_3^2 : 1].$$

Each of the above singular point is of multiplicity two since $\frac{\partial^2 P}{\partial x^2} = 6x$ doesn't vanish on any of them.

3. A projective line is given by an equation

$$Ax + By + Cz = 0.$$

It passes through points $[0 : 1 : 1]$ and $[t : 0 : 1]$ if and only of

$$B + C = 0, \qquad At + C = 0.$$

It follows that line L through $[0 : 1 : 1]$ and $[t : 0 : 1]$ is

$$L_t = \{Ax + Aty - Atz = 0\} = \{x + ty - tz = 0\}.$$

If $t = 0$, L_0 meets

$$C = \{[x : y : z] \,|\, x^2 + y^2 = z^2\}$$

at points $[x : y : z]$ such that $x = 0$ and $y^2 = z^2$, i.e. we have

$$L_0 \cap C = \{[0 : 1 : 1],\ [0 : -1 : 1]\}.$$

For $t \neq 0$ point $[a : b : c]$ lies in $L_t \cap C$ if and only if

$$\begin{cases} x + ty = tz, \\ x^2 + y^2 = z^2 \end{cases}$$

Comparing the square of the first equation with the second equation multiplied by t gives

$$x^2 + 2txy + t^2y^2 = t^2x^2 + t^2y^2,$$

i.e.

$$x(x(1 - t^2) + 2ty) = 0.$$

It follows that we have two solutions, $x = 0$, $ty = tz$, i.e. the point $[0 : 1 : 1]$ and $x = 2t, y = (t^2-1), tz = x+ty = 2t+t^3-t = t(t^2+1)$, i.e the point $[2t : t^2-1, t^2+1]$. Thus

$$L_t \cap C = \{[0 : 1 : 1],\ [2t : t^2 - 1 : t^2 + 1]\}.$$

Let

$$A = \{y = 0\}.$$

For $a \in A \setminus \{[1:0:0]\}$ let L_a be the line via $[0:1:1]$ and a. Then

$$L_a \cap C = \{[0:1:1],\, q\}$$

for $q \neq [0:1:1]$. Hence, the map $\varphi'\colon A \setminus \{[1:0:0]\} \to C$, $\varphi'(a) = q$, can be extended to a map $\varphi\colon A \to C$ which sends $[1:0:0]$ to $[0:1:1]$.

On the other hand, let $c \in C \setminus \{[0:1:1]\}$. Then line L_c via $[0:1:1]$ and c intersects A in some point, hence we get a map $\psi'\colon C \setminus \{[0:1:1]\} \to A$.

The line tangent to C at $[0:1:1]$ is given by $y+z=0$ and it intersects A at the point $[1:0:0]$. Thus, ψ' can be extended to a map $\psi\colon C \to A$.

For $a \in A \setminus \{[1:0:0]\}$, a is a point on $L_{\varphi(a)}$. Since the intersection points of two lines in unique, we have $\psi\varphi(a) = 0$. Similarly, $c \in C \setminus \{[0:1:1]\}$ lies on the line $L_{\psi(c)}$, hence $\varphi\psi(c) = c$. As $\varphi([1:0:0]) = [0:1:1]$ and $\psi([0:1:1]) = [1:0:0]$, it follows that φ and ψ are mutually inverse bijections.

Let now $[\lambda:0:\mu]$ be an arbitrary point of A. Then

$$\varphi([\lambda:0:\mu]) = [2\frac{\lambda}{\mu} : \frac{\lambda^2}{\mu^2} - 1 : \frac{\lambda^2}{\mu^2} + 1] = [2\lambda\mu : \lambda^2 - \mu^2 : \lambda^2 + \mu^2]$$

is an arbitrary point on C. It follows that any complex solution to

$$x^2 + y^2 = z^2$$

is given by

$$x = 2\lambda\mu,\ y = \lambda^2 - \mu^2,\ z = \lambda^2 + \mu^2$$

for some $\lambda, \mu \in \mathbb{C}$.

3 23.01. Lecture 3. Complex projective curves in $\mathbb{CP}^2$

Recall from Definition 1.6 that $P \in \mathbb{C}[x, y, z]$ is homogeneous if $P(\lambda x, \lambda y, \lambda z) = \lambda^d P(x, y, z)$, for any $\lambda \in \mathbb{C}^*$.

On the last lecture we have defined a projective plane $\mathbb{P}^2$ as

$$\mathbb{P}^2 = \{[x : y : z] \mid (x, y, z) \in \mathbb{C}^3 \setminus \{0\}\},$$

with $[x : y : z] = [u : v : w]$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $x = \lambda u$, $y = \lambda v$ and $z = \lambda z$.

It follows that if P is a homogeneous polynomial then $P(x, y, z) = 0$ if and only if $P(u, v, w) = 0$. In other words, the set of zeroes of a homogeneous polynomial P is a well-defined subset of $\mathbb{P}^2$.

Definition 3.1. Let $P \in \mathbb{C}[x, y, z]$ be a non-constant homogeneous polynomial. The *projective curve* defined by P is

$$C = \{[x : y : z] \in \mathbb{P}^2 \mid P(x, y, z) = 0\}. \tag{2}$$

Definition 3.2. The *degree* of a projective curve $C \subset \mathbb{P}^2$ is the degree of P. Curve C is *irreducible* if polynomial P is. An irreducible projective curve $D = \{Q(x, y, z) = 0\}$ is a *component* of C if Q divides P.

Definition 3.3. A point $[a : b : c] \in C$ as in (2) is *singular* if

$$\frac{\partial P}{\partial x}(a, b, c) = 0, \qquad \frac{\partial P}{\partial y}(a, b, c) = 0, \qquad \frac{\partial P}{\partial z}(a, b, c) = 0.$$

The set of singular points of C is denoted by $\mathrm{Sing}(C)$. The curve C is *non-singular* if $\mathrm{Sing}(C) = \emptyset$.

The *multiplicity* of a singular point $[a : b : c]$ is $\min\{m \mid \frac{\partial^m P}{\partial x^i \partial y^j \partial z^{m-i-j}}(a, b, c) \neq 0\}$.

Consider $P_1 = x^2 + y^2 - z^2$ and $P_2 = y^2 z - x^3$. Then

$$\begin{aligned} \frac{\partial P_1}{\partial x} &= 2x, & \frac{\partial P_2}{\partial x} &= -3x^2, \\ \frac{\partial P_2}{\partial y} &= 2y, & \frac{\partial P_2}{\partial y} &= 2yz, \\ \frac{\partial P_1}{\partial z} &= -2z, & \frac{\partial P_2}{\partial z} &= y^2. \end{aligned}$$

Since $[0 : 0 : 0]$ is not a point of $\mathbb{P}^2$, curve $C_1 = \{P_1(x, y, z) = 0\}$ is non-singular. Curve $C_2 = \{P_2(x, y, z) = 0\}$ has singular point $[0 : 0 : 1]$.

Lemma 3.4. *Let $C = \{P = 0\} \subset \mathbb{P}^2$ be an algebraic projective curve and let $f \colon \mathbb{P}^2 \to \mathbb{P}^2$ be a projective transformation. Then $f(C)$ is an algebraic curve and $p \in C$ is a singular point if and only if $f(p) \in f(C)$ is.*

Proof. By definition there exists a linear isomorphism $\varphi\colon \mathbb{C}^3 \to \mathbb{C}^3$ defining f. Let $\psi = \varphi^{-1}$. It defines $g\colon \mathbb{P}^2 \to \mathbb{P}^2$ such that $fg = \mathrm{Id} = gf$. Then

$$f(C) = \{f(p) \in \mathbb{P}^2 \,|\, p \in C\} = \{q \in \mathbb{P}^2 \,|\, g(q) \in C\} = \{q \in \mathbb{P}^2 \,|\, P \circ g(q) = 0\}.$$

As g is linear in the homogeneous coordinates of q, the composition Pg is a homogeneous polynomial, hence $f(C)$ is an algebraic curve.

A point $p \in C$ is singular if and only if any first order derivative of P, $\frac{\partial P}{\partial v}$, for $v = \alpha x + \beta y + \gamma z$ vanishes. It happens if and only if any first order derivative of $P \circ g$ vanishes, i.e. if and only if $f(p)$ is a singular point of $f(C)$. □

Definition 3.5. A projective curve defined by a linear equation

$$\alpha x + \beta y + \gamma z = 0$$

where $(\alpha, \beta, \gamma) \in \mathbb{C}^3 \setminus \{0\}$ is a *projective line.*

The *tangent line* to a projective curve C as in (2) at a non-singular point $[a : b : c]$ is the line

$$\frac{\partial P}{\partial x}(a, b, c)x + \frac{\partial P}{\partial y}(a, b, c)y + \frac{\partial P}{\partial z}(a, b, c)z = 0.$$

We endow $C \subset \mathbb{P}^2$ with the induced topology. Then

Lemma 3.6. *A projective curve $C \subset \mathbb{P}^2$ as in (2) is compact and Hausdorff.*

Proof. Since $\mathbb{P}^2$ is compact, see Proposition 2.6, it suffices to check that $C \subset \mathbb{P}^2$ is closed (use Remark 2.1(iv)). By Remark 2.4(i), C is closed if and only if $\Pi^{-1}(C) \subset \mathbb{C}^3 \setminus \{0\}$ is closed. The set $\Pi^{-1}(C) = \{(x, y, z) \in \mathbb{C}^3 \setminus \{0\} \,|\, P(x, y, z) = 0\}$ is the preimage under P of $0 \in \mathbb{C}$. Hence, $\Pi^{-1}(C)$ is closed, as P is continuous.

Any closed subset of a Hausdorff space is Hausdorff, hence Proposition 2.11 and the above imply that C is Hausdorff. □

Complex algebraic curves as in Definition 1.1 are sometimes called *affine* to distinguish them from projective curves defined in 3.1. Any affine curve C yields a projective curve $\widetilde{C}$ obtained by "adding points at infinity". More precisely, we can identify $\mathbb{C}^2$ with the open set

$$U = \{[x : y : z] \,|\, z \neq 0\} \subset \mathbb{P}^2$$

via $\varphi\colon U \to \mathbb{C}^2$, $\varphi[x : y : z] = (\frac{x}{z}, \frac{y}{z})$ with inverse $(x, y) \mapsto [x : y : 1]$.

Let $\widetilde{P}(x, y, z)$ be a non-constant homogeneous polynomial of degree d. Under the identification of U with $\mathbb{C}^2$ the restriction of $\widetilde{C} = \{\widetilde{P} = 0\}$ to U is given by polynomial $P \in \mathbb{C}[x, y]$ defined as $P(x, y) = \widetilde{P}(x, y, 1)$.

Conversely, given a polynomial $Q \in \mathbb{C}[x, y]$ of degree d one can define a homogeneous polynomial $\widetilde{Q} \in \mathbb{C}[x, y, z]$ via $\widetilde{Q}(x, y, z) = z^d Q(\frac{x}{z}, \frac{y}{z})$. If $\widetilde{C}_Q = \{\widetilde{Q} = 0\} \subset \mathbb{P}^2$ is the projective curve defined by $\widetilde{Q}$, then the intersection $\widetilde{C}_Q \cap U \subset U$, under identification $U \simeq \mathbb{C}^2$, is equal to $C_Q = \{Q = 0\}$.

If $Q = \sum_{r+s\leq d} a_{r,s}x^r y^s$, then $\widetilde{Q} = \sum_{r+s\leq d} a_{r,s}x^r y^s z^{d-r-s}$. Since d is the degree of Q, there exists $a_{r,s} \neq 0$ such that $r + s = d$. It follows that $\widetilde{Q}$ is not divisible by z, i.e. the "line at infinity" $z = 0$ is not contained in $\widetilde{C}_Q$.

In this way we get a bijective correspondence between affine curves in $\mathbb{C}^2$ and projective curves in $\mathbb{P}^2$ not containing the line at infinity.

If $\widetilde{C}$ is non-singular, then so is its restriction $C = \widetilde{C} \cap U$. However, $\widetilde{C}$ might have singular points even if C does not. More precisely, we have

Lemma 3.7. *Let $[a : b : c]$ be a point of the projective curve*

$$\widetilde{C} = \{[x : y : z] \in \mathbb{P}^2 \mid \widetilde{P}(x, y, z) = 0\}.$$

If $c \neq 0$ then $[a : b : c]$ is a non-singular point of $\widetilde{C}$ if and only if $(\frac{a}{c}, \frac{b}{c})$ is a non-singular point of the affine curve

$$C = \{(x, y) \in \mathbb{C}^2 \mid \widetilde{P}(x, y, 1) = 0\}.$$

Moreover, the intersection of $\mathbb{C}^2 \simeq U = \{[x : y : z] \mid z \neq 0\}$ and the projective tangent line at $[a : b : c]$ to $\widetilde{C}$ is the tangent line at $(\frac{a}{c}, \frac{b}{c})$ to C in $\mathbb{C}^2$.

In the proof and many times afterwards we shall need *Euler's relation*:

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z} = (\deg R)R. \tag{3}$$

To prove it we note that for a homogeneous polynomial $R \in \mathbb{C}[x, y, z]$ of degree m, we have $R(\lambda x, \lambda y, \lambda z) = \lambda^m R(x, y, z)$. Differentiating the identity with respect to λ, we get

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z} = m\lambda^{m-1}R.$$

Putting $\lambda = 1$, we get (3).

Proof of Lemma 3.7. Note that $(\frac{a}{c}, \frac{b}{c})$ is a singular point of C if and only if

$$P(\frac{a}{c}, \frac{b}{c}, 1) = \frac{\partial P}{\partial x}(\frac{a}{c}, \frac{b}{c}, 1) = \frac{\partial P}{\partial y}(\frac{a}{c}, \frac{b}{c}, 1) = 0.$$

In view of (3), the above condition is equivalent to

$$P(\frac{a}{c}, \frac{b}{c}, 1) = \frac{\partial P}{\partial x}(\frac{a}{c}, \frac{b}{c}, 1) = \frac{\partial P}{\partial y}(\frac{a}{c}, \frac{b}{c}, 1) = \frac{\partial P}{\partial z}(\frac{a}{c}, \frac{b}{c}, 1) = 0,$$

i.e. to the condition that $[\frac{a}{c} : \frac{b}{c} : 1] = [a : b : c]$ is a singular point of $\widetilde{C}$.

The projective tangent line, defined as

$$x\frac{\partial P}{\partial x}(a, b, c) + y\frac{\partial P}{\partial y}(a, b, c) + z\frac{\partial P}{\partial z}(a, b, c) = 0,$$

intersects $\mathbb{C}^2 \simeq U$ in

$$x\frac{\partial P}{\partial x}(a,b,c) + y\frac{\partial P}{\partial y}(a,b,c) + \frac{\partial P}{\partial z}(a,b,c) = 0. \tag{4}$$

For simplicity, let us assume that $c = 1$. Euler's relation (3) implies that

$$\frac{\partial P}{\partial z}(a,b,c) = -a\frac{\partial P}{\partial x}(a,b,c) - b\frac{\partial P}{\partial y}(a,b,c).$$

The "missing" term $\deg P \cdot P(a,b,c)$ vanishes as $[a:b:c]$ is a point of the curve. The above equality together with equation (4) and the assumption $c = 1$ imply that the intersection of the projective tangent line with U is

$$(x-a)\frac{\partial P}{\partial x}(a,b,1) + (y-b)\frac{\partial P}{\partial y}(a,b,1) = 0,$$

i.e. it is the affine tangent line. $\square$

4 26.01. Lecture 4. Bezout theorem and basic properties of resultants

We shall study some algebraic properties of complex algebraic curves. We shall first describe how two projective curves C and D in $\mathbb{P}^2$ can intersect. If C is of degree n and D is of degree m and C and D have no common components, then $C \cap D$ has at most mn points. It has exactly nm points if every point p of $C \cap D$ is a non-singular point both of C and D and if tangent lines to C and D at p are distinct. We will in fact show a more general result about the number of points in $C \cap D$ once we have defined the intersection multiplicity $I_p(C, D)$. It if defined to be infinity if p lies on a common component of C and D. Otherwise $I_p(C, D) \in \mathbb{N}$ and $I_p(C, D) = 0$ if and only if $p \notin C \cap D$. Moreover, we shall see that $I_p(C, D) = 1$ if and only if p is a non-singular point of both C and D and the tangent lines to C at p and to D at p are distinct.

We aim to prove

Theorem 4.1 (Bezout's theorem). *Let C and D be two projective curves of degree n and m in $\mathbb{P}^2$ which have no common components. Then*

$$\sum_{p \in C \cap D} I_p(C, D) = \sum_{p \in \mathbb{P}^2} I_p(C, D) = nm.$$

We need some preparations to define $I_p(C, D)$ and to prove Theorem 4.1. On this lecture we shall define a *resultant* of a polynomial and prove some of its properties.

Definition 4.2. Let

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n, Q(x) = b_0 + b_1 x + \ldots + b_m x^m \in \mathbb{C}[x]$$

be polynomials with $a_n \neq 0$ and $b_m \neq 0$. The *resultant* $R_{P,Q}$ of P and Q is the determinant of the $m+n$ by $m+n$ matrix

$$M_{P,Q} = \begin{pmatrix} a_0 & a_1 & \ldots & a_n & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & a_0 & a_1 & \ldots & a_n & 0 & \ldots & \ldots & \ldots & 0 \\ & & & & & & & & & \\ & & & & & & & & & \\ 0 & 0 & \ldots & 0 & a_0 & a_1 & \ldots & \ldots & \ldots & a_n \\ b_0 & b_1 & \ldots & \ldots & \ldots & b_m & 0 & \ldots & \ldots & 0 \\ 0 & b_0 & b_1 & \ldots & \ldots & \ldots & b_m & 0 & \ldots & 0 \\ & & & & & & & & & \\ 0 & \ldots & 0 & b_0 & b_1 & \ldots & \ldots & \ldots & \ldots & b_m \end{pmatrix}$$

If

$$P(x, y, z) = a_0(y, z) + a_1(y, z)x + \ldots + a_n(y, z)x^n,$$
$$Q(x, y, z) = b_0(y, z) + b_1(y, z)x + \ldots + b_m(y, z)x^m,$$

then the resultant $R_{P,Q}(y, z)$ is defined as the determinant of an analogous matrix of polynomials. Note that $R_{P,Q}(y, z)$ is a polynomial in y, z such that $R_{P,Q}(b, c)$ is the resultant of $P(x, b, c)$ and $Q(x, b, c)$ provided $a_n(b, c) \neq 0$ and $b_m(b, c) \neq 0$.

Lemma 4.3. *Let $P(x)$, $Q(x)$ be polynomials in x. Then $P(x)$ and $Q(x)$ have a non-constant common factor if and only if $R_{P,Q} = 0$.*

Proof. Let

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n, \qquad Q(x) = b_0 + b_1 x + \ldots + b_m x^m.$$

P and Q have non-constant common factor $R(x)$ if and only if

$$P(x) = R(x)\varphi(x), \qquad Q(x) = R(x)\psi(x).$$

It happens if and only if there exist non-zero polynomials

$$\varphi(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_{n-1} x^{n-1}, \qquad \psi(x) = \beta_0 + \beta_1 x + \ldots + \beta_{m-1} x^{m-1}.$$

of degrees at most $n-1$ and $m-1$ such that

$$P(x)\psi(x) = Q(x)\varphi(x).$$

Comparing the coefficients in front of x^j in the above equality, we get

$$\begin{aligned}
&a_0\beta_0 = b_0\alpha_0,\\
&a_0\beta_1 + a_1\beta_0 = b_0\alpha_1 + b_1\alpha_0,\\
&a_0\beta_2 + a_1\beta_1 + a_2\beta_0 = b_0\alpha_2 + b_1\alpha_1 + b_2\alpha_0,\\
&\ldots\\
&a_n\beta_{m-1} = b_m\alpha_{n-1}.
\end{aligned}$$

In other words, such $(\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{m-1})$ exist if and only if

$$[\beta_0, \ldots, \beta_{m-1}, -\alpha_0, \ldots, -\alpha_{n-1}] M_{P,Q} = 0,$$

i.e. if and only if $\det M_{P,Q} = 0$. By definition of $R_{P,Q}$ the existence of $(\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{m-1})$ is equivalent to the vanishing of $R_{P,Q}$. □

To prove the next statement we need to recall some facts from algebra. Let R be a commutative ring. Then $R[x]$ denotes the ring of polynomials in x with coefficients in R. A polynomial $f(x) \in R[x]$ is *primitive* if the only common factors in R of all coefficients of f are units. It follows that every monic polynomial is primitive. As before, $f(x)$ is irreducible if $f(x) = g(x)h(x)$ implies that either $g(x)$ or $h(x)$ is constant.

Recall that R is a *unique factorisation domain* if it is an integral domain (product of non-zero elements is non-zero) and every non-zero non-unit element can be written as a product of prime elements, uniquely up to order and units. If R is an UFD then so is $R[x]$.

The field of fractions K of R is a field with elements $\frac{a}{b}$, for $a, b \in R$, $b \neq 0$.

Lemma 4.4. *Let R be a unique factorisation domain and K its field of fractions. Any $f(x) \in R[x]$ can be written as*

$$f(x) = \lambda f_1(x) \ldots f_k(x),$$

where $\lambda \in R$ and $f_1(x), \ldots, f_k[x]$ are primitive in $R[x]$ and irreducible in $K[x]$.

Corollary 4.5. *Let R and K be as above. Then $f(x), g(x) \in R[x]$ have a non-constant common factor as elements in $R[x]$ if and only if they have a non-constant common factor as elements of $K[x]$.*

Proof. It follows from the fact that the factorisation in $R[x]$ can be viewed as factorisation in $K[x]$. □

Lemma 4.6. *Let $P(x, y, z)$ and $Q(x, y, z)$ be non-constant homogeneous polynomials such that $P(1, 0, 0) \neq 0 \neq Q(1, 0, 0)$. Then $P(x, y, z)$ and $Q(x, y, z)$ have a non-constant common factor if and only if $R_{P,Q}(y, z)$ is identically zero.*

Let $P(x, y, z)$ be a homogeneous polynomial of degree d. Then

$$P(x, y, z) = \sum_{i=0}^{d} a_i(y, z)x^i,$$

and $\deg(a_i) = d - i$. In particular, $\deg(a_d) = 0$. Since a non-constant homogeneous polynomial a satisfies $a(0, 0) = 0$, condition $P(1, 0, 0) \neq 0$ is equivalent to $a_d \neq 0$. In other words, the degree of P as a polynomial in $\mathbb{C}[y, z][x]$ is equal to the degree of P as a polynomial in $\mathbb{C}[x, y, z]$.

Proof of Lemma 4.6. Without loss of generality we can assume that $P(1, 0, 0) = 1 = Q(1, 0, 0)$. Then we can regard P and Q as monic (i.e. coefficient in front of x^n in P is one, similarly the coefficient in front of x^m in Q) polynomials in x with coefficients in the field $\mathbb{C}(y, z)$ of rational functions, i.e. functions

$$\frac{f(y, z)}{g(y, z)},$$

where $f, g \in \mathbb{C}[y, z]$ and $g(y, z)$ is not identically zero. It follows from Lemma 4.3 that $R_{P,Q}$ vanishes if and only if P and Q have non-constant common factor when considered as elements of $\mathbb{C}(y, z)[x]$. By Corollary 4.5, it happens if and only if P and Q have non-constant common factor when considered as elements of $\mathbb{C}[y, z][x] \simeq \mathbb{C}[x, y, z]$. □

Lemma 4.7. *Let $P(x, y, z)$ and $Q(x, y, z)$ be homogeneous polynomials of degrees n, respectively m. Then $R_{P,Q}(y, z)$ is a homogeneous polynomial of degree nm.*

Proof. By definition, $R_{P,Q}(y, z)$ is the determinant of $n + m$ by $n + m$ matrix whose ijth entry $r_{ij}(y, z)$ is a homogeneous polynomial of degree

$$d_{ij} = \begin{cases} n + i - j & \text{if } 1 \leq i \leq m, \\ i - j & \text{if } m + 1 \leq i \leq n + m, \end{cases}$$

(Note that 0 is a homogeneous polynomial of an arbitrary degree). Then $R_{P,Q}(y, z)$ is a sum of terms of the form

$$\pm \prod_{i=1}^{n+m} r_{i\sigma(i)}(y, z),$$

where σ is a permutation of $\{1,\ldots,n+m\}$. Each non-zero term is a homogeneous polynomial of degree

$$\sum_{i=1}^{n+m} d_{i\sigma(i)} = \sum_{i=1}^{m}(n+i-\sigma(i)) + \sum_{i=m+1}^{m+n}(i-\sigma(i))$$
$$= nm + \sum_{i=1}^{m+n}(i-\sigma(i)) = nm.$$

□

Lemma 4.8. *If* $P(x) = (x-\lambda_1)\ldots(x-\lambda_n)$, $Q(x) = (x-\mu_1)\ldots(x-\mu_m)$, *where* $\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_m \in \mathbb{C}$, *then*

$$R_{P,Q} = \prod_{1\le i\le n, 1\le j\le m}(\mu_j - \lambda_i).$$

In particular,

$$R_{P,QT} = R_{P,Q}R_{P,T}.$$

More generally, if $P,Q,R \in \mathbb{C}[x,y,z]$, *then*

$$R_{P,QT}(y,z) = R_{P,Q}(y,z)R_{P,T}(y,z).$$

Proof. If we regard P and Q as homogeneous polynomials in $x,\lambda_1,\ldots,\lambda_n$ and $x,\mu_1,\ldots,\mu_m$, then the same argument as in the proof of Lemma 4.8 shows that $R_{P,Q}$ is a homogeneous polynomial of degree nm in variables $\lambda_1\ldots,\lambda_n,\mu_1\ldots\mu_m$. By Lemma 4.6, $R_{P,Q}$ vanishes if $\lambda_i = \mu_j$. It follows that $R_{P,Q}$ is divisible by $\prod(\mu_i - \lambda_j)$, i.e. $R_{P,Q} = \lambda\prod(\mu_i - \lambda_j)$.

To calculate λ we evaluate the polynomial $R_{P,Q}(\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_m)$ at a point $\lambda_i \neq 0$ and $\mu_1 = \ldots = \mu_m = 0$. Then $R_{P,Q}$ is the determinant of a matrix such that the last n rows have only one non-zero element, 1. In the row $m+1$'st the element 1 is in the $m+1$'st column, in the row $m+2$'nd it is in the $m+2$'nd column and so on. In other words, the matrix is upper diagonal and its determinant is the product $\prod(-\lambda_1)^m$ of elements on the diagonal. Hence, $\lambda = 1$ and

$$R_{P,Q} = \prod(\mu_i - \lambda_j).$$

It follows that

$$R_{P,QT} = R_{P,Q}R_{P,T}.$$

If P, Q, T are polynomials in x,y,z then

$$R_{P,QT}(b,c) = R_{P,Q}(b,c)R_{P,T}(b,c)$$

for infinitely many $b,c \in \mathbb{C}$. It follows that

$$R_{P,QT}(y,z) = R_{P,Q}(y,z)R_{P,T}(y,z).$$

□

27.01. Workshop I

1. Find the singular points and the tangent lines at the singular points of each of the following curves in $\mathbb{C}^2$:

   - $y^3 - y^2 + x^3 - x^2 + 3y^2x + 3x^2y + 2xy = 0$,
   - $x^4 + y^4 - x^2y^2 = 0$.

2. Find the singular points and the multiplicities of the singular points of the following projective curves:

   - $xy^4 + yz^4 + xz^4 = 0$,
   - $y^2z = x(x-z)(x-\lambda z)$, for $\lambda \in \mathbb{C}$.

3. Let C be a non-singular projective curve of degree two in $\mathbb{P}^2$ defined by a polynomial with rational coefficients. Use the following steps to obtain an algorithm which decides whether or not C has any rational points, i.e. whether there is a point of C which can be represented by rational homogeneous coordinates

   (a) Use the theory of the diagonalisation of quadratic forms to show that there is a projective transformation defined by a matrix with rational coefficients taking C to the curve defined by
   $$ax^2 + by^2 = z^2,$$
   for some $a, b \in \mathbb{Q} \setminus \{0\}$.

   (b) Show that by making additional diagonal transformation we can assume that a and b are integers with no square factors, i.e. each is product of distinct primes. Show that we may also assume that $|a| \geq |b|$.

   (c) Show that if C has a rational point then b is a square modulo p for every prime p dividing a. Deduce from the Chinese reminder theorem that b is a square modulo a, so there are integers m, a_1, such that $|m| \leq |a|/2$ and
   $$m^2 = b + aa_1.$$

   (d) Show that if $m^2 = b + aa_1$ and $ax^2 + by^2 = z^2$ then
   $$a_1(z^2 - by^2)^2 + b(my - z)^2x^2 = (mz - by)^2x^2.$$
   Deduce that C has a rational point if and only if the same is true about the curve defined by
   $$a_1x^2 + by^2 = z^2.$$

   (e) Show that if $|a| > 1$ then $|a_1| < |a|$, and thus the problem is reduced to one of the same form in which $|a| + |b|$ is smaller. Deduce that the argument can be repeated until either b fails to be a square modulo a or we reach the situation $|a| = |b| = 1$, in which case the curve has a rational point if and only if at least one of a and b is positive.

Solutions to workshop I

1. • $P(x,y) = y^3 - y^2 + x^3 - x^2 + 3y^2x + 3x^2y + 2xy$
   Point (a,b) is a singular point if P and its derivatives

$$\frac{\partial P}{\partial x} = 3x^2 - 2x + 3y^2 + 6xy + 2y \qquad \frac{\partial P}{\partial y} = 3y^2 - 2y + 6xy + 3x^2 + 2x$$

   vanish at (a,b) i.e. if

$$\begin{cases} b^3 - b^2 + a^3 - a^2 + 3ab^2 + 3a^2b + 2ab = 0, \\ 3a^2 - 2a + 3b^2 + 6ab + 2b = 0, \\ 3b^2 - 2b + 6ab + 3a^2 + 2a = 0. \end{cases}$$

   Multiplying the first equation by 3 and subtracting a times the second equation and b times the third equation we get:

$$\begin{aligned} &3b^3 - 3b^2 + 3a^3 - 3a^2 + 9ab^2 + 9a^2b + 6ab - 3a^3 \\ &+ 2a^2 - 3ab^2 - 6a^2b - 2ab - 3b^3 + 2b^2 - 6ab^2 - 3a^2b - 2ab = \\ &- b^2 - a^2 - 2ab = -(a-b)^2 = 0. \end{aligned}$$

   Thus, we have $a = b$. Then the first equation gives

$$a^3 - a^2 + a^3 - a^2 + 3a^3 + 3a^3 + 2a^2 = 8a^3 = 0$$

   Thus, the only singular point of $C = \{P(x,y) = 0\}$ is $(0,0)$. We have

$$\frac{\partial^2 P}{\partial x^2} = 6x - 2 + 6y, \qquad \frac{\partial^2 P}{\partial x \partial y} = 6y + 6x + 2, \qquad \frac{\partial^2 P}{\partial y^2} = 6y - 2 + 6x.$$

   It follows that $\frac{\partial^2 P}{\partial x^2}(0,0) = -2$, $\frac{\partial^2 P}{\partial x \partial y}(0,0) = 2$ and $\frac{\partial^2 P}{\partial y^2}(0,0) = -2$. Thus tangent lines are linear factors of the polynomial

$$\frac{-2}{2!}x^2 + \frac{2}{1!1!}xy - \frac{-2}{2!}y^2 = -(x-y)^2.$$

   Thus C has one tangent line $x - y = 0$ at the point $(0,0)$.

   • $P(x,y) = x^4 + y^4 - x^2y^2$.
   Point (a,b) is a singular point if P and its derivatives

$$\frac{\partial P}{\partial x} = 4x^3 - 2xy^2, \qquad \frac{\partial P}{\partial y} = 4y^3 - 2x^2y,$$

   vanish at (a,b) i.e. if

$$\begin{cases} a^4 + b^4 - a^2b^2 = 0, \\ 4a^3 - 2ab^2 = 0, \\ 4b^3 - 2a^2b = 0. \end{cases}$$

Clearly $(0,0)$ is a singular point. To find other singular points we multiply the second equation by $a/2$ to get $2a^4 = a^2b^2$. Then the first equation gives

$$2a^4 + 2b^4 - 2a^2b^2 = a^2b^2 + 2b^4 - 2a^2b^2 = b^2(2b^2 - a^2) = 0,$$

i.e. $b = 0$ or $(\sqrt{2}b - a) = 0$ or $(\sqrt{2}b + a) = 0$.
The first possibility gives point $(0,0)$ as the first equation yields $a^4 = 0$. In the remaining cases, $a = \pm\sqrt{2}b$, the first equation gives

$$4b^4 + b^4 - 2b^4 = 3b^4 = 0,$$

i.e. again $b = 0$ and $a = 0$. It follows that $(0,0)$ is the only singular point of $C = \{P(x,y) = 0\}$.
The second derivatives

$$\frac{\partial^2 P}{\partial x^2} = 12x^2 - 2y^2, \qquad \frac{\partial^2 P}{\partial x \partial y} = -4xy, \qquad \frac{\partial^2 P}{\partial y^2} = 12y^2 - 2x^2,$$

vanish at $(0,0)$.
The third derivatives

$$\frac{\partial^3 P}{\partial x^3} = 24x, \qquad \frac{\partial^3 P}{\partial x^2 \partial y} = -4y \qquad \frac{\partial^3 P}{\partial x \partial y^2} = -4x, \qquad \frac{\partial^3 P}{\partial y^3} = 24y$$

vanish at $(0,0)$.
The fourth derivatives

$$\frac{\partial^4 P}{\partial x^4} = 24, \quad \frac{\partial^4 P}{\partial x^3 \partial y} = 0, \quad \frac{\partial^4 P}{\partial x^2 \partial y^2} = -4, \quad \frac{\partial^4 P}{\partial x \partial y^3} = 0, \quad \frac{\partial^4 P}{\partial y^4} = 24$$

are not all zero at $(0,0)$.
It follows that tangent lines at $(0,0)$ are linear factors of the polynomial

$$\frac{24}{4!}x^4 - \frac{4}{2!2!}x^2y^2 + \frac{24}{4!}y^4 = x^4 - x^2y^2 + y^4.$$

First, we note that

$$t^2 - tw + w^2 = t^2 - 2t\frac{w}{2} + \frac{w^2}{4} + \frac{3w^2}{4} = (t - \frac{w}{2})^2 - (\frac{i\sqrt{3}w}{2})^2 =$$
$$(t - \frac{w}{2} - \frac{i\sqrt{3}w}{2})(t - \frac{w}{2} + \frac{i\sqrt{3}w}{2}) = (t - w(\frac{1+i\sqrt{3}}{2}))(t - w(\frac{1-i\sqrt{3}}{2})).$$

Putting $\lambda_1 = \frac{1+i\sqrt{3}}{2}$ and $\lambda_2 = \frac{1-i\sqrt{3}}{2}$ we get

$$x^4 - x^2y^2 + y^4 = (x^2 - \lambda_1 y^2)(x^2 - \lambda_2 y^2) =$$
$$(x - \sqrt{\lambda_1}y)(x + \sqrt{\lambda_2}y)(x - \sqrt{\lambda_2}y)(x + \sqrt{\lambda_2}y).$$

Thus the tangent lines at $(0,0)$ are

$$x - \sqrt{\lambda_1}y = 0, \quad x + \sqrt{\lambda_2}y = 0, \quad x - \sqrt{\lambda_2}y = 0, \quad x + \sqrt{\lambda_2}y = 0.$$

2. • Consider the curve $C = \{P(x,y,z) = 0\}$ for

$$P(x,y,z) = xy^4 + yz^4 + xz^4.$$

A point $[a : b : c]$ is a singular point of C if partial derivatives

$$\frac{\partial P}{\partial x} = y^4 + z^4, \qquad \frac{\partial P}{\partial y} = 4xy^3 + z^4, \qquad \frac{\partial P}{\partial z} = 4yz^3 + 4xz^3$$

vanish at (a,b,c). It follows from Euler's relation that then also $P(a,b,c) = 0$. We thus want to find solutions of

$$\begin{cases} b^4 + c^4 = 0, \\ 4ab^3 + c^4 = 0, \\ 4bc^3 + 4ac^3 = 0. \end{cases}$$

From the last equation it follows that either $a + b = 0$ or $c = 0$.
We consider the case $a + b = 0$ first. Then the first and the second equations imply that

$$-4b^4 - b^4 = 0,$$

i.e. $b = 0$ hence also $a = 0$. Then $c^4 = 0$ thus we get $a = b = c = 0$ which is not a point in $\mathbb{P}^2$.
Let us now look at the case $c = 0$. The first equation implies that $b = 0$. Then a is arbitrary, thus $[1 : 0 : 0]$ is a singular point of C.
To find its multiplicity we calculate

$$\frac{\partial^2 P}{\partial x^2} = 0, \qquad \frac{\partial^2 P}{\partial x \partial y} = 4y^3, \qquad \frac{\partial^2 P}{\partial x \partial z} = 4z^3,$$
$$\frac{\partial^2 P}{\partial y^2} = 12xy^2, \qquad \frac{\partial^2 P}{\partial y \partial z} = 4z^3, \qquad \frac{\partial^2 P}{\partial z^2} = 12yz^2 + 12xz^2$$

and note that all of them vanish at $[1 : 0 : 0]$. The same is true about third derivatives

$$\frac{\partial^3 P}{\partial x^3} = 0, \qquad \frac{\partial^3 P}{\partial x^2 \partial y} = 0, \qquad \frac{\partial^3 P}{\partial x^2 \partial z} = 0 \qquad \frac{\partial^3 P}{\partial y^3} = 24xy,$$
$$\frac{\partial^3 P}{\partial y^2 \partial x} = 12y^2, \qquad \frac{\partial^3 P}{\partial y^2 \partial z} = 0, \qquad \frac{\partial^3 P}{\partial z^3} = 24yz + 24xz, \qquad \frac{\partial^3 P}{\partial z^2 \partial x} = 12z^2,$$
$$\frac{\partial^3 P}{\partial z^2 \partial y} = 12z^2, \qquad \frac{\partial^3 P}{\partial x \partial y \partial z} = 0$$

The multiplicity of $[1 : 0 : 0]$ is four as

$$\frac{\partial^4 P}{\partial y^4} = 24x \neq 0.$$

- Let us now consider the curve $C = \{P(x,y,z) = 0\}$ with
$$P(x,y,z) = y^2 z - x(x-z)(x-\lambda z).$$
First, we look for singular points of C such that $z = 1$. As we know these are the singular points of the affine curve given by
$$\widetilde{P}(x,y) = y^2 - x(x-1)(x-\lambda).$$
If (x_0, y_0) is a singular point then $\frac{\partial \widetilde{P}}{\partial y} = 2y$ vanishes at y_0, thus $y_0 = 0$. It follows that x_0 is the common zero of $Q(x) = x(x-1)(x-\lambda)$ and its derivative $Q'(x)$. Such a x_0 exists only if $\lambda = 0$ or $\lambda = 1$. Thus,
  - If $\lambda = 0$ then $[0:0:1]$ is a singular point of C.
  - If $\lambda = 1$ then $[1:0:1]$ is a singular point of C.
  - If $\lambda \neq 0, 1$ then $C \cap \{z \neq 0\}$ is smooth.

  It remains to consider the intersection $C \cap \{z = 0\}$. From $z = 0$ it follows that $x = 0$, hence $C \cap \{z = 0\} = [0:1:0]$. As
$$\frac{\partial P}{\partial z} = (\lambda+1)x^2 - 2\lambda xz - y^2$$
does not vanish at $[0:1:0]$, it is never a singular point of C.
We have
$$P(x,y,z) = y^2 z - (x^2 - xz)(x - \lambda z) = y^2 z - x^3 + (\lambda+1)x^2 z - \lambda x z^2$$
hence
$$\frac{\partial P}{\partial x} = -3x^2 + 2(\lambda+1)zx - \lambda z^2, \quad \frac{\partial P}{\partial y} = 2yz, \quad \frac{\partial P}{\partial z} = (\lambda+1)x^2 - 2\lambda xz$$
and
$$\frac{\partial^2 P}{\partial x^2} = -6x + 2(\lambda+1)z, \qquad \frac{\partial^2 P}{\partial x \partial y} = 0, \qquad \frac{\partial^2 P}{\partial x \partial z} = 2(\lambda+1)x - 2\lambda z,$$
$$\frac{\partial^2 P}{\partial y^2} = 2z, \qquad \frac{\partial^2 P}{\partial y \partial z} = 2y, \qquad \frac{\partial^2 P}{\partial z^2} = -2\lambda x.$$
Thus $\frac{\partial^2 P}{\partial y^2}(0,0,1) = 2 = \frac{\partial^2 P}{\partial y^2}(1,0,1)$, i.e. singular points $[0:0:1]$ and $[1:0:1]$ are of multiplicity two.

3. The curve C is given by a symmetric matrix $A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \in M_{3\times 3}(\mathbb{Q})$. A point $[u:v:w] \in \mathbb{P}^2$ lies on C if and only if $(u,v,w)A(u,v,w)^T = 0$. There exists a change of basis $(u,v,w) \to (x,y,z)$ of $\mathbb{Q}^3$ such that
$$(u,v,w)A(u,v,w)^T = 0 \Leftrightarrow (x,y,z)B(x,y,z)^T = 0$$

for $B = \begin{pmatrix} n & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & k \end{pmatrix}$. If one of the m, n, k is zero, say k then $[0 : 0 : 1]$ is a singular point of C. It follows that $n, m, k \in \mathbb{Q} \setminus 0$. Dividing by $-k$ we can thus write

$$C = \{n'x^2 + m'y^2 = z^2\}$$

for $n', m' \in \mathbb{Q} \setminus \{0\}$.

If $n' = \alpha/\beta$ and $m' = \gamma/\delta$ then projective transformation $(x, y, z) \mapsto (x/\beta, y/\delta, z)$ takes C to a curve

$$\{\alpha\beta x^2 + \gamma\delta y^2 = z^2\}.$$

Finally, if $\alpha\beta = p^2\varepsilon$ and $\gamma\delta = q^2\eta$ then the projective transformation $(x, y, z) \mapsto (x/p, y/q, z)$ takes C to the curve

$$\{\varepsilon x^2 + \eta y^2 = z^2\}.$$

Exchanging x and y if necessary we can thus assume that

$$C = \{ax^2 + by^2 = z^2\},$$

for square-free integers a, b such that $|b| \leq |a|$. If x, y and z are integers and $a = \prod_{i=1}^r p_i$ then b is congruent to $(z/y)^2$ modulo any p_i.

Let now $m \in [0, a-1]$ be such that m is congruent to z/y modulo any p_i. Such an m exists and is unique by the Chinese remainder theorem. It follows that b is congruent to m^2 modulo a hence we have

$$m^2 = b + aa_1.$$

Moreover, as m^2 is congruent to $(a-m)^2$ modulo m we can assume that $|m| \leq |a|/2$. We thus have $m^2 = b + aa_1$ and $ax^2 + by^2 = z^2$. It follows that

$$\begin{aligned}
&a_1(z^2 - by^2)^2 + b(my - z)^2x^2 - (mz - by)^2x^2 = \\
&a_1(z^2 - by^2)^2 + b(m^2y^2 - 2myz + z^2)x^2 - (m^2z^2 - 2mbyz + b^2y^2)x^2 = \\
&a_1(z^2 - by^2)^2 + bm^2x^2y^2 - 2bmx^2yz + bx^2z^2 - m^2x^2z^2 + 2mbx^2yz - b^2x^2y^2 = \\
&a_1(z^2 - by^2)^2 + b(b + aa_1)x^2y^2 + bx^2z^2 - (b + aa_1)x^2z^2 - b^2x^2y^2 = \\
&a_1(z^2 - by^2)^2 + b^2x^2y^2 + aa_1bx^2y^2 + bx^2z^2 - bx^2z^2 - aa_1x^2z^2 - bx^2y^2 = \\
&a_1(z^2 - by^2)^2 + aa_1x^2(by^2 - z^2) = a_1(ax^2)^2 - aa_1x^2(ax^2) = 0.
\end{aligned}$$

In other words $[(z^2 - by^2) : (my - z)x : (mz - by)x]$ is a rational point of the curve

$$C' = \{a_1x^2 + by^2 = z^2\}.$$

Since the role of a and a_1 can be exchanged it follows that C has a rational point if and only if C' does.

If $|a| > 1$ then $|a| \leq |a|^2/2$, hence

$$|aa_1| = |m^2 - b| \leq |a|^2/4 + |a| \leq \frac{3|a|^2}{4} < |a|^2,$$

i.e. $|a_1| < |a|$. We have thus reduced the question whether $C = \{ax^2 + by^2 = z^2\}$ has a rational point to the question whether $C' = \{a'x^2 + b'y^2 = z^2\}$ has a rational point with $|a'| + |b'| < |a| + |b|$.

Continuing this way we either get to the case when b is not a square modulo a or $|a| = |b| = 1$. If $|a| = |b| = 1$ the curve C has a rational point if and only if at least one of a, b is positive.

5 30.01. Lecture 5. Weak Bezout theorem and its applications

Using Lemmas from the last lecture we show

Theorem 5.1. *Any two projective curves in C and D in $\mathbb{P}^2$ intersect in at least one point.*

Proof. Let C and D be defined by homogeneous polynomials P and Q. By Lemma 4.7 the resultant $R_{P,Q}(y,z)$ is a homogeneous polynomial of degree nm in y and z. Therefore, by Lemma 1.7 R_{PQ} is either identically zero or it is a product of nm linear polynomials. In either case there exists $(b,c) \in \mathbb{C}^2 \setminus 0$ such that $R_{P,Q}(b,c) = 0$. This means that the resultant of polynomials $P(x,b,c)$, $Q(x,b,c)$ in x is zero, i.e. they have a common root a. Then $[a:b:c] \in C \cap D$. □

Theorem 5.2 (Weak version of Bezout's theorem)**.** *If two projective curves C and D in $\mathbb{P}^2$ of degrees n and m have no common components then they intersect in at most nm points.*

Proof. Suppose that C and D have at least $nm+1$ points of intersection. We shall show that then they have a common component. Choose any set S of $nm+1$ distinct points in $C \cap D$. Then we can choose a point in $\mathbb{P}^2$ which does not lie on C nor on D nor on any line which passes through any two points of S. By applying projective transformation we can assume that this point is $[1:0:0]$. Then C and D are given by $P(x,y,z)$, $Q(x,y,z)$ respectively, and $P(1,0,0) \neq 0 \neq Q(1,0,0)$.

By Lemma 4.7 the resultant $R_{P,Q}(y,z)$ is a homogeneous polynomial of degree nm. If $R_{P,Q}(b,c) = 0$ then $R_{P(x,b,c),Q(x,b,c)} = 0$, i.e. polynomials $P(x,b,c)$, $Q(x,b,c)$ have a common root, see Lemma 4.3. Thus, there exists a such that $P(a,b,c) = 0 = Q(a,b,c)$, i.e. $[a:b:c] \in C \cap D$.

If $[a:b:c] \in S$ then $P(a,b,c) = 0 = Q(a,b,c)$ and $(b,c) \neq (0,0)$, because $[1:0:0] \notin S$. It follows that $bz - cy$ is a factor of $R_{P,Q}(y,z)$. Moreover, if $[\alpha,\beta,\gamma]$ is a different point in S then $\beta z - \gamma y$ is not a scalar multiple of $bz - cy$ (if it was then $[a:b:c]$, $[\alpha:\beta:\gamma]$ and $[1:0:0]$ would lie on $bz = cy$). It shows that $R_{P,Q}$ has at least $nm+1$ distinct linear factors, hence it is identically zero. By Lemma 4.6, C and D have a common component. □

We shall now use the above theorems to prove some properties of algebraic curves.

Corollary 5.3. *(i) A non-singular projective curve C in $\mathbb{P}^2$ is irreducible.*

(ii) An irreducible projective curve C in $\mathbb{P}^2$ has at most finitely many singular points.

Proof. (i) Let

$$C = \{[x:y:z] \in \mathbb{P}^2 \mid P(x,y,z)Q(x,y,z) = 0\}$$

be a reducible curve in $\mathbb{P}^2$. By Theorem 5.1 there is at least one $[a:b:c] \in \mathbb{P}^2$ such that $P(a,b,c) = 0 = Q(a,b,c)$. As

$$\frac{\partial PQ}{\partial x} = \frac{\partial P}{\partial x}Q + P\frac{\partial Q}{\partial x}, \qquad \frac{\partial PQ}{\partial y} = \frac{\partial P}{\partial y}Q + P\frac{\partial Q}{\partial}, \qquad \frac{\partial PQ}{\partial z} = \frac{\partial P}{\partial z}Q + P\frac{\partial Q}{\partial z}$$

$[a:b:c]$ is a singular point of C.

(ii) Let C be defined by a homogeneous polynomial $P(x, y, z)$ of degree n. Without loss of generality we may assume that $[1 : 0 : 0] \notin C$, i.e. that $P(1, 0, 0) \neq 0$ so the coefficient in front of x^n in P is not equal to zero. This ensures that $Q(x, y, z) = \frac{\partial P}{\partial x}(x, y, z)$ is a homogeneous polynomial of degree $n - 1$ which is not identically zero, hence defines a curve in D in $\mathbb{P}^2$.

Any singular point of C belongs to the intersection $C \cap D$. By Theorem 5.2 $|C \cap D| \leq n(n - 1)$ is finite. The statement follows.

□

Definition 5.4. A *conic* is a curve of degree two in $\mathbb{C}^2$ or $\mathbb{P}^2$.

Corollary 5.5. *Any irreducible projective conic C in $\mathbb{P}^2$ is equivalent under projective transformation to the conic*

$$x^2 = yz$$

and in particular is non-singular.

Proof. By Corollary 5.3, C has finitely many singular points. Hence, by applying a suitable projective transformation, we may assume that $[0 : 1 : 0]$ is a non-singular point of C and that the tangent line to C at $[0 : 1 : 0]$ is $z = 0$. Then C must be defined by a polynomial of the form

$$ayz + bx^2 + cxz + dz^2.$$

Indeed, the coefficient in front of y^2 must vanish and $\frac{\partial}{\partial x}(0, 1, 0) = 0$, i.e. the coefficient in front of xy must be zero.

Since C is irreducible, both a and b are non-zero. Then

$$[x, y, z] \mapsto [\sqrt{b}x, ay + cx + dz, -z]$$

takes C to the conic $x^2 = yz$. Since this conic is non-singular, so is C. □

Let C be a non-singular conic defined by $x^2 = yz$. There is a homeomorphism

$$f \colon \mathbb{P}^1 \to C, \qquad f([x : y]) = [xy : y^2 : x^2],$$

with inverse

$$g \colon C \to \mathbb{P}^1, \qquad g([x : y : z]) = \begin{cases} [x : y] & \text{if } y \neq 0, \\ [z : x] & \text{if } z \neq 0. \end{cases}$$

Note that if $[x : y : z] \in C$ then $x^2 = yz$ and if $y \neq 0$ and $z \neq 0$ then $x \neq 0$ and

$$[x : y] = [x^2, xy] = [yz : xy] = [z : x].$$

Thus, by Corollary 5.5 any irreducible conic is homeomorphic to $\mathbb{P}^1$.

Proposition 5.6. *It two projective curves C and D of degrees n in $\mathbb{P}^2$ intersect in exactly n^2 points and if exactly nm of these points lie on an irreducible curve E of degree $m < n$ then the remaining $n(n - m)$ points lie on a curve of degree at most $n - m$.*

Proof. Let C, D and E be defined by homogeneous polynomials

$$P(x, y, z), Q(x, y, z), R(x, y, z).$$

Choose a point $[a : b : c]$ on E which does not lie on $C \cap D$. Then the curve of degree n defined by

$$C' = \lambda P(x, y, z) + \mu Q(x, y, z) = 0,$$

with $\lambda = Q(a, b, c)$, $\mu = -P(a, b, c)$ meets E in at least $nm + 1$ points, namely in $[a : b : c]$ and nm of points of $C \cap D$ which lie on E. Then by Theorem 5.2 C' and E must have a common component which must be E, as E is irreducible. Thus

$$\lambda P(x, y, z) + \mu Q(x, y, z) = R(x, y, z)S(x, y, z),$$

for some homogeneous polynomial S of degree $n - m$. Hence, if $[u : v : w] \in C \cap D$ then either $R(u, v, w)$ of $S(u, v, w)$ vanishes. Therefore, the $n(n - m)$ points of $C \cap D$ which do not lie on E must lie on $\{S(x, y, z) = 0\}$. $\square$

A hexagon in $\mathbb{P}^2$ is determined by six distinct points $p_1, \ldots, p_6$ – vertices and lines joining p_1 to p_2, p_2 to p_3 and so on till p_6 to p_1. The side opposite the line joining p_1 to p_2 is the line joining p_4 to p_5 and so on. A hexagon is inscribed in a conic if $p_1, \ldots, p_6$ lie on a conic.

Corollary 5.7 (Pascal's mystic hexagon)**.** *The pairs of opposite sides of a hexagon inscribed in an irreducible conic in* $\mathbb{P}^2$ *meet in 3 collinear points.*

Proof. Let the successive sides of the hexagon be lines given by polynomials $L_1, \ldots, L_6$. Two projective curves defined by $L_1L_2L_5$ and $L_2L_4L_6$ intersect in the six vertices of the hexagon and three points of intersection of the opposite sides. By assumption, the first 6 points lie on a curve of degree $2 \leq 3$. Thus, by Proposition 5.6, the remaining 3 points lie of a curve of degree at most $3 - 2 = 1$. $\square$

6 02.02. Lecture 6. Intersection multiplicity and the proof of Bezout Theorem

To prove the strong form of Bezout's theorem we must first define the intersection multiplicity $I_p(C, D)$ at a point $p = [a : b : c]$ of two curves C and D in $\mathbb{P}^2$. We shall define the intersection multiplicity by using the resultant of the polynomials defining C and D in a suitable coordinate system. In order to show that the definition is independent of the choice we show that $I_p(C, D)$ is uniquely determined by the properties listed below

Theorem 6.1. *There exists a unique intersection multiplicity $I_p(C, D)$ defined for all projective curves C, D in $\mathbb{P}^2$ such that*

(i) $I_p(C, D) = I_p(D, C)$.

(ii) $I_p(C, D) = \infty$ *if p lies on a common component of C and D and otherwise* $I_p(C, D) \in \mathbb{Z}_{\geq 0}$.

(iii) $I_p(C, D) = 0$ *if and only if* $p \notin C \cap D$.

(iv) Two distinct lines meet with intersection multiplicity one at their unique point of intersection

(v) If $C_1 = \{P_1(x, y, x) = 0\}$, $C_2 = \{P_2(x, y, z) = 0\}$ *and* $C = \{P_1(x, y, z)P_2(x, y, z) = 0\}$ *then* $I_p(C, D) = I_p(C_1, D) + I_p(C_2, D)$, *for any projective curve D and any point p.*

(vi) If $C = \{P(x, y, z) = 0\}$ *and* $D = \{Q(x, y, z) = 0\}$ *with* $\deg P = n$ *and* $\deg Q = m$ *and* $E = \{PR + Q = 0\}$ *with* $\deg R = m - n$ *then*

$$I_p(C, D) = I_p(C, E).$$

Moreover if C and D have no common component and we choose projective coordinates so that the conditions

(a) $[1 : 0 : 0]$ *does not belong to* $C \cup D$,

(b) $[1 : 0 : 0]$ *does not lie on any line containing two distinct points of* $C \cap D$,

(c) $[1 : 0 : 0]$ *does not lie on the tangent line to C or D and any point of* $C \cap D$,

are satisfied then the intersection multiplicity $I_p(C, D)$ of C and D at any

$$p = [a : b : c] \in C \cap D$$

is the largest integer k such that $(bz - cy)^k$ divides the resultant $R_{P,Q}(y, z)$.

Proof. To simplify the notation we shall write $I_p(P,Q)$ instead of $I_p(C,D)$ for curves $C = \{P = 0\}$, $D = \{Q = 0\}$.

First we show that conditions (i)-(vi) determine $I_p(P,Q)$ completely. Since the conditions are independent of the choice of coordinates we might assume that $p = [0 : 0 : 1]$. We may assume that $I_p(P,Q)$ is finite, greater than zero and that P and Q are irreducible. Thus, $I_p(P,Q) = k$, for some $k > 0$. By induction on k we may assume that any intersection multiplicity strictly less than k can be calculated using only conditions (i)-(vi).

Consider polynomials $P(x,0,1)$ and $Q(x,0,1)$. Assume that they are of degree r and s respectively. By (i) we can assume that $r \leq s$. There are 2 cases to consider

$r = 0$ First, we consider an example. Assume $P(x,y,z)$ has degree 3. Then it is a linear combination of $x^3, y^3, z^3, x^2y, x^2z, y^2z, xy^2, xz^2, yz^2$ and xyz. $P(x,0,1) = a_{x^3}x^3 + a_{x^2z}x^2 + a_{xz^2}x + a_{z^3}$. Since $P(0,0,1) = 0$, we know that $a_{z^3} = 0$. Because $P(x,0,1)$ is constant (since its degree is zero!), we know that it is identically zero, hence $a_{x^3} = a_{x^2z} = a_{xz^2} = a_{z^3} = 0$. It follows that $P(x,y,z)$ is spanned by $y^3, x^2y, y^2z, xy^2, yz^2$ and xyz, i.e. $P(x,y,z) = yR(x,y,z)$.

The above works for P of any degree, hence $P(x,y,z) = yR(x,y,z)$. Moreover, by dividing the monomials into those which do or do not contain powers of y, we can present Q as

$$Q(x,y,z) = Q(x,0,z) + yS(x,y,z).$$

Because $Q(0,0,1) = 0$, we have $Q(x,0,z) = x^qT(x,z)$, for some $q > 0$, where $T(x,z)$ is a polynomial such that $T(0,1) \neq 0$. It implies that $[0 : 0 : 1]$ does not lie on the curve $\{T(x,z) = 0\}$, hence $I_p(y, T(x,z)) = 0$, by (iii).

Condition (v) gives

$$I_p(P,Q) = I_p(y,Q) + I_p(R,Q).$$

By (vi), $I_p(y,Q) = I_p(y, x^qT) = I_p(y,x^q) + I_p(y,T) = I_p(y,x^q)$. Condition (v) implies that $I_p(y,x^q) = qI_p(y,x) = q$, by (iv). It follows that

$$I_p(P,Q) = I_p(R,Q) + q,$$

for some $q > 0$. By induction, $I_p(R,Q) < I_p(P,Q)$ is uniquely determined by conditions (i)-(vi).

$r > 0$ We want to construct a polynomial S such that $S(x,0,1) = Q(x,0,1) - x^{s-r}P(x,0,1)$ has degree $t < s$.

First, we note that multiplying P and Q by scalars, we can assume that $Q(x,0,1)$ and $P(x,0,1)$ are monic. We define

$$S(x,y,z) = z^{n+s-r}Q(x,y,z) - x^{s-r}z^mP(x,y,z).$$

Since P and Q are irreducible and distinct, S is not identically zero. By (vi) we have

$$I_p(P,S) = I_p(P, z^{n+s-r}Q) = I_p(P, z^{n+s-r}) + I_p(P,Q) = I_p(P,Q),$$

where $I_p(P, z^{n+s-r}) = 0$ because p does not lie on $\{z = 0\}$. Now, we can replace P and Q with P and S and continue till we reach the situation $r = 0$.

This completes the uniqueness part of the proof. To prove existence, we define $I_p(C, D)$ as follows:

- If p lies on a common component of C and D then $I_p(C, D) = \infty$,
- If p does not belong to $C \cap D$ then $I_p(C, D) = 0$,
- If $p \in C \cap D$ but does not lie on a common component, first remove all common components of C and D and then choose coordinates such that (a)-(c) are satisfied. If $p = [a : b : c]$ in these coordinates then $I_p(C, D)$ is the largest k such that $(bz-cy)^k$ divides the resultant $R_{P,Q}(y, z)$.

  It remains to show that (i)-(vi) are satisfied:

  (i) is a direct consequence that interchanging two rows changes the sign of the determinant, $R_{P,Q} = \pm R_{Q,P}$.

  (ii) follows from the definition and Lemma 4.6.

  (iii) If $p = [a : b : c] \in C \cap D$ then $P(x, b, c)$ and $Q(x, b, c)$ have a common root, hence their resultant vanishes by Lemma 4.3. By Lemma 4.6, $R_{P,Q}(y, z)$ vanishes when $y = b$ and $z = c$, i.e. it is divisible by $bz - cy$, which shows that $I_p(C, D) > 0$ (because by (a), $(b, c) \neq (0, 0)$).

  (iv) is a straightforward computation of 2 by 2 determinant

  (v) Follows from Lemma 4.8

  (vi) holds because determinant is unchanged by the addition of a scalar multiple of one row to another. The resultant of P and $PR + Q$ is the determinant of a matrix (s_{ij}) obtained from the matrix (r_{ij}) defining the resultant of P and Q by addition of scalar multiples of the first n rows to the last m rows.
  Let us consider an example $P = x + y$, $Q = x^2 + xy + yz$, $R = x + z$. Then

$$R_{P,Q} = \det \begin{pmatrix} y & 1 & 0 \\ 0 & y & 1 \\ yz & y & 1 \end{pmatrix}$$

  while $R_{P,Q+PR}$ is a determinant of a matrix with last row $z[y, 1, 0] + 1[0, y, 1] + [yz, y, 1]$. Indeed $RP + Q = x(x + y) + z(x + y) + x^2 + xy + yz = 2x^2 + x(2y + z) + 2yz$.
  In general, if $R(x, y, z) = \rho_0(y, z) + x\rho_1(y, z) + \ldots + x^{n-m}\rho_{n-m}(y, z)$, then

$$s_{ij} = \begin{cases} r_{ij} & \text{if } i \leq m, \\ r_{ij} + \sum_{k=i-m}^{i-n} \rho_{i-n-k} r_{kj} & \text{if } i > m. \end{cases}$$

□

Now, we are ready to prove

Theorem (4.1 Bezout's theorem). *Let C and D be two projective curves of degree n and m in $\mathbb{P}^2$ which have no common components. Then*

$$\sum_{p\in C\cap D} I_p(C,D) = \sum_{p\in\mathbb{P}^2} I_p(C,D) = nm.$$

Proof. Let C and D be curves with no common component. We may choose coordinates so that conditions (a)-(c) are satisfied. Let $C = \{P(x,y,z) = 0\}$, $D = \{Q(x,y,z) = 0\}$ be polynomials in this coordinate system. Then by Lemmas 4.6 and 4.7, $R_{P,Q}(y,z)$ is a non-zero homogeneous polynomial of degree nm. Hence, by Lemma 1.7 it can be expressed as a product of linear factors

$$R_{P,Q}(y,z) = \prod_{i=1}^{k} (c_i z - b_i y)_i^e$$

where $e_1 + \ldots + e_k = nm$.

It means that the resultant $R_{P,Q}(b_i, c_i) = 0$, i.e. polynomials $P(x, b_i, c_i)$, $Q(x, b_i, c_i)$ have a common root a_i. Then $p_i = [a_i : b_i : c_i]$ belongs to $C \cap D$. On the other hand, if $p = [a : b : c] \in C \cap D$ then $(b,c) \neq (0,0)$ because $[1 : 0 : 0] \notin C \cap D$. It follows that $bz - cy$ is a factor of $R_{P,Q}(y,z)$, i.e. $p = p_i$, for some i. Any two points $[a : b : c]$, $[\alpha : \beta : \gamma]$ of $C \cap D$ give different factors of $R_{P,Q}(y,z)$. Indeed, if $bz - cy$ was proportional to $\beta z - \gamma y$ then points $[a : b : c]$, $[\alpha : \beta : \gamma]$ and $[1 : 0 : 0]$ would be collinear which contradicts (c). To sum up,

$$C \cap D = \{[a_i : b_i : c_i] \mid 1 \leq i \leq k\}.$$

By the definition of $I_p(C,D)$, we get

$$\sum_{p\in C\cap D} I_p(C,D) = \sum_{i=1}^{k} e_i = nm.$$

□

Lemma 6.2. *If $p \in C \cap D$ is a singular point of C then $I_p(C,D) > 1$.*

Proof. We may assume that C and D have no common component, and hence we may choose coordinates such that $p = [0 : 0 : 1]$ and the conditions (a)-(c) of Theorem 6.1 are satisfied.

We wish to show that y^2 divides the resultant $R_{P,Q}(y,z)$, where $C = \{P = 0\}$ and $D = \{Q = 0\}$. We have

$$\frac{\partial P}{\partial x}(0,0,1) = \frac{\partial P}{\partial y}(0,0,1) = \frac{\partial P}{\partial z}(0,0,1) = 0,$$

hence $P(x,y,z)$ cannot have terms xz^{n-1}, yz^{n-1} and z^n. It follows that $P(x,y,z)$ is a sum of monomials of degree at least two in x and y:

$$P(x,y,z) = a_0(y,z) + xa_1(y,z) + \ldots + x^n a_n(y,z)$$

where y^2 divides a_0 and y divides a_1. Also $Q(0,0,1) = 0$, so

$$Q(x, y, z) = b_0(y, z) + xb_1(y, z) + \ldots + x^m b_m(y, z)$$

and b_0 is divisible by y. It follows that the first column of the matrix defining $R_{P,Q}(y, z)$ is divisible by y.

We want to check that either the first column is divisible by y^2 or that both the first and the second column are divisible by y. For this, we write

$$b_0 = b_{01} y z^{m-1} + y^2 c_0(y, z), \qquad b_1 = b_{10} z^{m-1} + y c_1(y, z).$$

If $b_{01} = 0$ then the first column is divisible by y^2. If $b_{01} \neq 0$ then we can subtract b_{10}/b_{01} times the first column from the second one to get that the second column is divisible by y. As it does not influence the determinant, we get that y^2 divides $R_{P,Q}(y, z)$. □

We can now describe when the intersection multiplicity is one

Proposition 6.3. *Let C and D be projective curves in $\mathbb{P}^2$ and let p be any point in $C \cap D$. Then $I_p(C, D) = 1$ if and only if p is a non-singular point of both C and D and the tangent lines to C and D at p are distinct.*

Proof. We may assume that C and D have no common component, hence we may assume that $p = [0 : 0 : 1]$ and conditions (a)-(c) of Theorem 6.1 are satisfied. By Lemma 6.2 we know that p is a non-singular point of C and D. Let $C = \{P = 0\}$, $D = \{Q = 0\}$. We show that the resultant $R_{P,Q}(y, z)$ is divisible by y^2 if and only if tangent lines to C and D at p coincide. Since $R_{P,Q}$ is a homogeneous polynomial in y and z divisible by y, the first condition is equivalent to

$$\frac{\partial R_{P,Q}}{\partial y}(0, 1) = 0.$$

By (c), $[1 : 0 : 0]$ does not lie on the tangent line

$$x\frac{\partial P}{\partial x}(0,0,1) + y\frac{\partial P}{\partial y}(0,0,1) + z\frac{\partial P}{\partial z}(0,0,1) = 0,$$

hence $\frac{\partial P}{\partial x}(0,0,1) \neq 0$. The implicit function theorem for $P(x, y, 1)$ implies that there are open neighbourhoods U, V of $0 \in \mathbb{C}$ and a holomorphic function $\lambda_1 \colon U \to V$ such that

$$\lambda_1(0) = 0$$

and if $x \in V$, $y \in U$ then

$$P(x, y, 1) = 0 \Leftrightarrow x = \lambda_1(y).$$

Moreover,

$$P(x, y, 1) = (x - \lambda_1(y)) l(x, y),$$

where $l(x, y)$ is a polynomial in x whose coefficients are holomorphic functions of y. If we assume that the coefficient $P(1, 0, 0)$ of x^n in $P(x, y, z)$ is one, then

$$P(x, y, 1) = \prod_{i=1}^{n} (x - \lambda_i(y)).$$

Similarly, if U and V are sufficiently small, there exists $\mu_1 \colon U \to V$ such that

$$\mu_1(0) = 0, \qquad Q(x, y, 1) = 0, \Leftrightarrow x = \mu_1(y),$$

for $x \in V$, $y \in U$. We can write

$$Q(x, y, 1) = \prod_{i=1}^{m} (x - \mu_i(y)).$$

The tangent lines to C and D at point p are defined by

$$x = \lambda_1'(0)y, \qquad x = \mu_1'(0)y.$$

If $y \in U$ then by Lemma 4.8,

$$R_{P,Q}(y, 1) = (\mu_1(y) - \lambda_1(y))S(y),$$

where

$$S(y) = \prod_{(i,j) \neq (1,1)} (\mu_i(y) - \lambda_j(y)).$$

It is a product of holomorphic functions, hence it is holomorphic.

Since $\lambda_1(0) = 0 = \mu_1(0)$, we have

$$\frac{\partial R_{P,Q}}{\partial y}(0, 1) = (\mu_1'(0) - \lambda_1'(0))S(0).$$

As $\frac{\partial P}{\partial x}(0, 0, 1) \neq 0$, polynomial P does not have repeated zeroes at p, similarly for Q. It shows that $\lambda_i(0) \neq 0$ and $\mu_i(0) \neq 0$, for $i > 1$. Moreover, if $\lambda_i(0) = \mu_j(0)$ for some pair i, j, then $[0 : 0 : 1]$ and $[\mu_j(0) : 0 : 1] = [\lambda_j(0) : 0 : 1]$ are distinct points of $C \cap D$ lying on a line $y = 0$ which contradicts (b). It shows that $S(0) \neq 0$. Thus $\frac{\partial R_{P,Q}}{\partial y}(0, 1) = 0$ if and only if $\lambda_1'(0) = \mu_1'(0)$ if and only if tangent lines to C and D at p coincide. □

Corollary 6.4. *Let C and D be projective curves in $\mathbb{P}^2$ of degrees n and m. Assume that every point $p \in C \cap D$ is a non-singular point of C and D and that the tangent lines to C and D at p are distinct. Then $C \cap D$ has exactly nm points.*

Theorem 6.5 (Implicit function theorem)**.** *Let $A(z, w)$ be a polynomial with complex coefficients. Suppose that*

$$A(z_0, w_0) = 0 \neq \frac{\partial A}{\partial w}(z_0, w_0).$$

Then there is a holomorphic function $f \colon U \to V$, where U and V are open neighbourhoods of z_0 and w_0 in $\mathbb{C}$ such that

$$f(z_0) = w_0$$

and if $z \in U$, $w \in V$ then $A(z, w) = 0$ if and only if $f(z) = w$. Moreover, on $U \times V$

$$A(z, w) = (w - f(z))B(z, w),$$

where $B(z, w)$ is a polynomial in w whose coefficients are holomorphic functions of z.

02.02. Homework II

The homework in due on Thursday the 16th of February. Please hand in your solutions before the lecture.

1. Prove that a non-singular cubic curve C in $\mathbb{P}^2$ has exactly 9 inflection points [Hint: use Remark 7.6 and Corollary 6.4].

2. Prove the following converse to Pascal's theorem: if the intersections of the opposite sides of a hexagon lie on a straight line then the vertices lie on a conic. More precisely, let $p_1, \ldots, p_6$ be distinct points of $\mathbb{P}^2$, no three of which lie on a line. If $i \neq j$ let L_{ij} be the line through p_i and p_j. Show that if three points of intersection $L_{12} \cap L_{45}$, $L_{23} \cap L_{56}$, $L_{34} \cap L_{16}$ are collinear then $p_1, \ldots, p_6$ lie on a conic.

3. Let C be a non-singular cubic curve defined by

$$x^3 + y^3 + z^3 + \lambda xyz = 0$$

where $\lambda^3 + 27 \neq 0$. Show that the points of inflection of C are the points of intersection of C with a different curve of the same form, and deduce that they satisfy

$$x^3 + y^3 + z^3 = 0 = xyz.$$

Deduce that C has exactly nine points of inflection and that a line through any two of them meets C again at the third point of inflection.

Solutions to Homework II

1. Let C be a non-singular cubic curve in $\mathbb{P}^2$ defined by $P \in \mathbb{C}[x, y, z]$ and let D be a cubic curve defined by the Hessian $\mathcal{H}_P \in \mathbb{C}[x, y, z]$. It follows from Corollary 6.4 that in order to prove that C has nine inflection points we need to show that if $p \in C \cap D$ then p is a non-singular point of D and the tangent lines to C and D at p are distinct.

   By Remark 7.6 we can assume that $p = [0 : 1 : 0]$ and

   $$P(x, y, z) = y^2z - x(x - z)(x - \lambda z) = y^2z - x^3 + (\lambda + 1)x^2z - \lambda xz^2.$$

   Then

   $$\frac{\partial P}{\partial x} = -3x^2 + 2(\lambda + 1)xz - \lambda z^2, \quad \frac{\partial P}{\partial y} = 2yz, \quad \frac{\partial P}{\partial z} = y^2 + (\lambda + 1)x^2 - 2\lambda xz$$
   $$\frac{\partial P}{\partial x}(0, 1, 0) = 0, \quad \frac{\partial P}{\partial y}(0, 1, 0) = 0, \quad \frac{\partial P}{\partial z}(0, 1, 0) = 1.$$

   We have

   $$\frac{\partial^2 P}{\partial x^2} = -6x + 2(\lambda + 1)z, \quad \frac{\partial^2 P}{\partial x \partial y} = 0, \quad \frac{\partial^2 P}{\partial x \partial z} = 2(\lambda + 1)x - 2\lambda z,$$
   $$\frac{\partial^2 P}{\partial y^2} = 2z, \quad \frac{\partial^2 P}{\partial y \partial z} = 2y, \quad \frac{\partial^2 P}{\partial z^2} = -2\lambda x.$$

   hence D is given by a polynomial

   $$\mathcal{H}_P(x, y, z) = \det \begin{pmatrix} -6x + 2(\lambda + 1)z & 0 & 2(\lambda + 1)x - 2\lambda z \\ 0 & 2z & 2y \\ 2(\lambda + 1)x - 2\lambda z & 2y & -2\lambda x \end{pmatrix} =$$
   $$(24\lambda + 16\lambda(\lambda + 1))xz^2 - (8\lambda(\lambda + 1) + 8\lambda^2)z^3 - 8(\lambda + 1)^2x^2z - 8(\lambda + 1)y^2z + 24xy^2.$$

   We have

   $$\frac{\partial \mathcal{H}_P}{\partial x} = (24\lambda + 16\lambda(\lambda + 1))z^2 - 16(\lambda + 1)^2xz + 24y,$$
   $$\frac{\partial \mathcal{H}_P}{\partial x}(0, 1, 0) = 24,$$
   $$\frac{\partial \mathcal{H}_P}{\partial y} = -16(\lambda + 1)yz + 24x,$$
   $$\frac{\partial \mathcal{H}_P}{\partial y}(0, 1, 0) = 0,$$
   $$\frac{\partial \mathcal{H}_P}{\partial z} = 2(24\lambda + 16\lambda(\lambda + 1))xz - 3(8\lambda(\lambda + 1) + 8\lambda^2)z^2 - 8(\lambda + 1)^2x^2 - 8(\lambda + 1)y^2,$$
   $$\frac{\partial \mathcal{H}_P}{\partial z}(0, 1, 0) = -8(\lambda + 1).$$

   It follows that the line

   $$z = 0$$

tangent to C at p is distinct from the line

$$24x - 8(\lambda+1)z = 0$$

tangent to D at p. The statement follows.

2. Let $L_{ij} = \{[x : y : z] \mid P_{ij}(x,y,z) = 0\}$ be lines through points p_i, p_j. Consider two cubics
$$Q_1 = \{P_{12}P_{34}P_{56} = 0\}, \quad Q_2 = \{P_{23}P_{45}P_{61} = 0\}.$$
Cubics Q_1 and Q_2 intersect in

$$\begin{array}{lll} p_1 = L_{12} \cap L_{61}, & p_2 = L_{12} \cap L_{23}, & p_3 = L_{34} \cap L_{23}, \\ p_4 = L_{34} \cap L_{45}, & p_5 = L_{56} \cap L_{45}, & p_6 = L_{56} \cap L_{61}, \\ q_1 = L_{12} \cap L_{45}, & q_2 = L_{23} \cap L_{56}, & q_3 = L_{24} \cap L_{16}. \end{array}$$

If q_1, q_2, q_3 lie on a line, i.e. a curve of degree one then, by Proposition 5.6 the remaining six points $p_1, \ldots, p_6$ lie on a curve C of degree at most two. Since no three of the points $p_1, \ldots, p_6$ lie on a line, the curve C has degree two, i.e. it is a conic.

3. Let
$$P(x,y,z) = x^3 + y^3 + z^3 + \lambda xyz.$$
Then

$$\begin{array}{lll} \dfrac{\partial P}{\partial x} = 3x^2 + \lambda yz, & \dfrac{\partial P}{\partial y} = 3y^2 + \lambda xz, & \dfrac{\partial P}{\partial z} = 3z^2 + \lambda xy, \\ \dfrac{\partial^2 P}{\partial x^2} = 6x, & \dfrac{\partial^2 P}{\partial x \partial y} = \lambda z, & \dfrac{\partial^2 P}{\partial x \partial z} = \lambda y, \\ \dfrac{\partial^2 P}{\partial y^2} = 6y, & \dfrac{\partial^2 P}{\partial y \partial z} = \lambda x, & \dfrac{\partial^2 P}{\partial z^2} = 6z, \end{array}$$

and

$$\mathcal{H}_P(x,y,z) = \det \begin{pmatrix} 6x & \lambda z & \lambda y \\ \lambda z & 6y & \lambda x \\ \lambda y & \lambda x & 6z \end{pmatrix} =$$
$$216xyz + \lambda^3 xyz + \lambda^3 xyz - 6\lambda^2 y^3 - 6\lambda^2 x^3 - 6\lambda^2 z^3 = -6\lambda^2(x^3+y^3+z^3) + (216+2\lambda^3)xyz.$$

If $\lambda = 0$ then inflection points of

$$C = \{[x : y : z] \mid P(x,y,z) = 0\}$$

satisfy

$$x^3 + y^3 + z^3 = 0, \qquad 216xyz = 0.$$

If $\lambda \neq 0$, inflection points of C satisfy

$$x^3 + y^3 + z^3 + 3\lambda xyz = 0, \qquad x^3 + y^3 + z^3 - \frac{108 + 2\lambda^3}{3\lambda^2} xyz = 0.$$

The above cubics coincide if and only if $\lambda = -\frac{108+2\lambda^3}{3\lambda^2}$, if and only if $4\lambda^3 = -108$, i.e. $\lambda^3 = -27$. By assumption $\lambda^3 \neq -27$, hence two cubic are different. Note that the only pair $(a, b) \in \mathbb{C}^2$ such that $a + \lambda b = 0$ and $a + \mu b = 0$ with $\mu \neq \lambda$ is the pair $(a, b) = (0, 0)$. Thus, the inflection points need to satisfy

$$x^3 + y^3 + z^3 = 0, \qquad xyz = 0.$$

Let $\varepsilon_3 = e^{\frac{2}{3}\pi i}$ be the primitive third root of unity. Then the inflection points of C with coordinate x equal to zero are

$$[0 : 1 : -\varepsilon], \qquad [0 : 1 : -\varepsilon^2], \qquad [0 : 1 : -1].$$

Similarly we can write down the inflection points with coordinate y or z equal to zero. To sum up C has nine inflection points

$$\begin{aligned} q_1 &= [0 : 1 : -\varepsilon], & q_2 &= [0 : 1 : -\varepsilon^2], & q_3 &= [0 : 1 : -1], \\ q_4 &= [1 : 0 : -\varepsilon], & q_5 &= [1 : 0 : -\varepsilon^2], & q_6 &= [1 : 0 : -1], \\ q_7 &= [1 : -\varepsilon : 0], & q_8 &= [1 : -\varepsilon^2 : 0], & q_9 &= [1 : -1 : 0]. \end{aligned}$$

We have

$$\begin{aligned} &q_1, q_2, q_3 \in \{x = 0\}, && q_4, q_5, q_6 \in \{y = 0\}, && q_7, q_8, q_9 \in \{z = 0\}, \\ &q_1, q_4, q_9 \in \{\varepsilon x + \varepsilon y + z = 0\}, && q_2, q_6, q_7 \in \{\varepsilon x + y + \varepsilon z = 0\}, && q_3, q_5, q_8 \in \{x + \varepsilon y + \varepsilon z = 0\}, \\ &q_1, q_5, q_7 \in \{\varepsilon^2 x + \varepsilon y + z = 0\}, && q_2, q_4, q_8 \in \{\varepsilon^2 x + y + \varepsilon z = 0\}, && q_3, q_6, q_9 \in \{x + y + z = 0\}, \\ &q_1, q_6, q_8 \in \{x + \varepsilon y + z = 0\}, && q_2, q_5, q_9 \in \{x + y + \varepsilon z = 0\}, && q_3, q_4, q_7 \in \{\varepsilon x + y + z = 0\}. \end{aligned}$$

It follows that a line through any pair of inflection points of C intersects C in another inflection point.

7 06.02. Lecture 7. Cubic curves: classification and group structure

We show that every non-singular projective curve of degree greater than two has at least one and at most finitely many inflection points. As a corollary, we get that every smooth projective cubic curve is of the form

$$y^2z = x(x-z)(x-\lambda z).$$

Definition 7.1. Let $P \in \mathbb{C}[x,y,z]$ be a homogeneous polynomial. The *Hessian* $\mathcal{H}_P$ of P is the polynomial defied by

$$\mathcal{H}_P = \det \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix}.$$

Definition 7.2. A non-singular point $[a:b:c]$ of a projective curve C in $\mathbb{P}^2$ defined by $P(x,y,z)$ is called a *point of inflection* (of flex) of C if

$$\mathcal{H}_P(a,b,c) = 0.$$

Note that if $\deg P = d$ then the second partial derivatives of P are homogeneous of degree $d-2$. It follows that $\mathcal{H}_P$ is homogeneous of degree $3d-6$, hence the inflection point is well-defined.

In order to prove

Lemma 7.3. *Let $C = \{P(x,y,z) = 0\}$ be an irreducible projective curve of degree d. Then every point of C is an inflection point if and only if $d = 1$.*

we need to know that if $\deg P = d > 1$ then

$$z^2\mathcal{H}_P(x,y,z) = (d-1)^2 \det \begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & P_y \\ P_x & P_y & dP/(d-1) \end{pmatrix}. \tag{5}$$

Indeed, Euler's relation gives

$$\begin{aligned}
&dP(x,y,z) = xP_x(x,y,z) + yP_y(x,y,z) + zP_z(x,y,z),\\
&(d-1)P_x(x,y,z) = xP_{xx}(x,y,z) + yP_{yx}(x,y,z) + zP_{zx}(x,y,z),\\
&(d-1)P_y(x,y,z) = xP_{xy}(x,y,z) + yP_{yy}(x,y,z) + zP_{zy}(x,y,z),\\
&(d-1)P_z(x,y,z) = xP_{xz}(x,y,z) + yP_{yz}(x,y,z) + zP_{zz}(x,y,z).
\end{aligned}$$

Then

$$z\det\begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix} =$$

$$\det\begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ xP_{xx}+yP_{yx}+zP_{zx} & xP_{xy}+yP_{yy}+zP_{zy} & xP_{xz}+yP_{yz}+zP_{zz} \end{pmatrix} =$$

$$(d-1)\det\begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{x} & P_{y} & P_{z} \end{pmatrix}$$

Applying the same to the columns of the above matrix we get the result.

Now if $\frac{\partial P}{\partial y}$ is non-zero the equation

$$P(x, y, 1) = 0$$

locally defines y as a holomorphic function of x (by the implicit function theorem). Then

$$\frac{\partial P}{\partial x} + \frac{dy}{dx}\frac{\partial P}{\partial y} = 0,$$
$$\frac{\partial^2 P}{\partial x^2} + (\frac{dy}{dx})^2\frac{\partial^2 P}{\partial y^2} + 2\frac{dy}{dx}\frac{\partial^2 P}{\partial x \partial y} + \frac{d^2 y}{dx^2}\frac{\partial P}{\partial y} = 0,$$

i.e. (as $\frac{dy}{dx} = -\frac{\partial P}{\partial x}/\frac{\partial P}{\partial y}$)

$$\frac{d^2 y}{dx^2} = (P_y)^{-3}\det\begin{pmatrix} P_{xx} & P_{xy} & P_{x} \\ P_{yx} & P_{yy} & P_{y} \\ P_{x} & P_{y} & 0 \end{pmatrix}$$

It follows that

$$\frac{d^2 y}{dx^2} = \frac{\mathcal{H}_P(x, y, 1)}{(d-1)^2 (P_y)^3} \tag{6}$$

Proof of Lemma 7.3. Since the second derivatives of a polynomial of degree 1 vanish, one implication is clear.

Let now C be a curve whose every point is an inflection point and assume that $d = \deg C$ is greater than 1. By applying a suitable projective transformation we can assume that

$$P(0, 0, 1) = 0 \neq \frac{\partial P}{\partial y}(0, 0, 1).$$

The implicit function theorem tells us that there are open neighbourhoods of U and V of $0 \in \mathbb{C}$ and $g\colon U \to V$ such that $g(0) = 0$ and, for $x \in V$, $y \in U$, $P(x, y, 1) = 0$ if and only if $y = g(x)$. Formula (6) implies that $g'' = 0$, hence $g(x) = \lambda x$. It follows that $P(x, \lambda x, 1)$ vanishes identically.

Then $P(x, y, z)$ is divisible by $y - \lambda x$. Indeed, let us first consider an example

$$P(x, y, z) = ax^2 + bxy + cy^2 + dxz + ez^2 + fyz.$$

Then $P(x, \lambda x, 1) = ax^2 + b\lambda x^2 + c\lambda^2 x^2 + dx + e + f\lambda x = 0$ if and only if $e = 0$, $d = -f\lambda$ and $a = -b\lambda - c\lambda^2$. It follows that

$$P(x, y, z) = -b\lambda x^2 + bxy - c\lambda^2 x^2 + cy^2 - f\lambda xz + fyz = (y - \lambda x)(bx + c(y + \lambda x) + fz).$$

In general, if $P = \sum_{i_1+i_2+i_3=n} a_{i_1,i_2,i_3} x^{i_1} y^{i_2} z^{i_3}$, we get

$$a_{j,0,n-j} + \lambda a_{j-1,1,n-j} + \ldots + \lambda^j a_{0,j,n-j} = 0.$$

Hence, for any monomial we have:

$$\begin{aligned}
&- \lambda^i a_{j-i,i,n-j} x^j z^{n-j} - a_{j-i,i,n-j} x^{j-i} y^i z^{n-j} \\
&= a_{j-i,i,n-j} x^{j-i} z^{n-j} (y^i - \lambda^i x^i) \\
&= a_{j-i,i,n-j} x^{j-i} z^{n-j} (y - \lambda x)(y^{i-1} + \lambda x y^{i-2} + \ldots + \lambda^{i-1} x^{i-1}).
\end{aligned}$$

Since C was assumed to be irreducible, it follows that $C = \{x - \lambda y = 0\}$ has degree one. □

Proposition 7.4. *Let C be a non-singular projective curve in $\mathbb{P}^2$ of degree d.*

(i) If $d \geq 2$ then C has at most $3d(d-2)$ points of inflection.

(ii) If $d \geq 3$ then C has at least one point of inflection.

Proof. $\mathcal{H}_P$ is homogeneous of degree $3d - 6$, so provided it is not constant (when $d > 2$ this means not identically zero) it defines a projective curve in $\mathbb{P}^2$ (possibly with multiple components). By Lemma 7.3 we know that $\mathcal{H}_P$ is not the zero polynomial.

Statement (i) follows from the weak Bezout's theorem 5.2 provided P and $\mathcal{H}_P$ have no common factor. Since a non-singular curve is irreducible, Corollary 5.3, the common component would be the whole curve C. Thus, to finish the proof of (i) we need to show that the only curve whose every point is an inflection point has degree one. It is the statement of Lemma 7.3 below.

To prove (ii) we note that curve C given by a polynomial P of degree $d \geq 3$ intersects curve given by polynomial $\mathcal{H}_P$ of degree greater than or equal to 1 in at least one point, see Theorem 5.1. □

Corollary 7.5. *Let C be a non-singular cubic curve in $\mathbb{P}^2$. Then C is equivalent under a projective transformation to the curve defined by*

$$y^2 z = x(x - z)(x - \lambda z),$$

for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

Proof. By Proposition 7.4, C has a point of inflection. We may assume that $[0:1:0]$ is the inflection point and that the tangent line to C at $[0:1:0]$ is $z=0$. Then $C=\{P(x,y,z)=0\}$ and

$$P(0,1,0)=0, \qquad \frac{\partial P}{\partial x}(0,1,0)=0, \qquad \frac{\partial P}{\partial y}(0,1,0)=0,$$
$$\mathcal{H}_P(0,1,0)=0, \qquad \frac{\partial P}{\partial z}(0,1,0)\neq 0,$$

as C is non-singular. Applying (5) with y and z exchanged, we get

$$y^2\mathcal{H}_P(x,y,z)=4\det\begin{pmatrix} P_{xx} & P_x & P_{xz} \\ P_x & \frac{3}{2}P & P_z \\ P_{zx} & P_z & P_{zz}\end{pmatrix}$$

so

$$0=4\det\begin{pmatrix} P_{xx} & 0 & P_{xz} \\ 0 & 0 & P_z \\ P_{zx} & P_z & P_{zz}\end{pmatrix}=-4P_z^2P_{xx}.$$

It follows that $P_{xx}(0,1,0)=0$. The above conditions imply that the coefficients in front of monomials x^2y ($P_{xx}=0$), xy^2 ($P_x=0$) and y^3 ($P_y=0$) in P must vanish. Hence, P can be written as

$$P(x,y,z)=yz(\alpha x+\beta y+\gamma z)+\varphi(x,z),$$

for some homogeneous polynomial φ of degree three. Moreover,

$$\beta=\frac{\partial P}{\partial z}(0,1,0)\neq 0.$$

Projective transformation

$$[x:y:z]\mapsto[x:y+\frac{\alpha x+\gamma z}{2\beta}:z]$$

maps C to a curve defined by

$$\beta y^2z+\psi(x,z)=0,$$

where $\psi(x,z)=\varphi(x,z)+z\frac{\alpha^2x^2+2\alpha\gamma xz+\gamma^2z^2}{4\beta^2}$.

$\psi(x,z)$ is a product of three linear factors. Moreover, as C is irreducible, ψ is not divisible by z, hence the coefficient of x^3 in ψ is not zero. After a suitable diagonal transformation C is defined by

$$y^2z=(x-az)(x-bz)(x-cz).$$

and a,b,c are distinct (otherwise C would be singular). Projective transformation

$$[x:y:z]\mapsto[\frac{x-az}{b-a},\eta y,z],$$

where $\eta^2=(b-a)^3$ puts C into form

$$y^2z=x(x-z)(x-\lambda z),$$

for some $\lambda\in\mathbb{C}\setminus\{0,1\}$. □

Remark 7.6. We have in fact proved that for any non-singular cubic curve C and an inflection point p there exists a projective transformation which maps C to the curve given by $y^2z = x(x-z)(x-\lambda z)$ and p to the point $[0:1:0]$.

Lemma 7.7. *A line L in $\mathbb{P}^2$ meets a non-singular cubic C either*

(a) in three distinct points p, q, r each with intersection multiplicity one (i.e. L is not the tangent line to C at p, q and r),

(b) in two points, p with multiplicity one and q with intersection multiplicity two (i.e. L is the tangent line to C at q but not at p and q is not a point of inflection on C),

(c) in one point p with intersection multiplicity three (i.e. L is a tangent line to C at p and p is the inflection point).

Theorem 7.8. *Given any non-singular projective cubic C in $\mathbb{P}^2$ and a point of inflection p_0 on C there is a unique group structure on C such that p_0 is the zero element and three points of C add up to zero if and only if they are the three points of intersection of C with some line in $\mathbb{P}^2$.*

Proof. To check uniqueness, not first that additive inverses are uniquely determined since $-p_0 = p_0$ and if $p \neq p_0$ then $-p$ is the third point on the intersection of C with the line through p and p_0. Also if p and q are any points then $p+q=-r$, where r is the third point of intersection of C with the line through p and q or the tangent line to C at $p=q$.

It remains to show that it is an additive group structure. Commutativity comes directly from the definition of the group structure. At this point we cannot show associativity. We just check that $p+p_0=p$. We have $p+p_0=-r$ where r is the third point of the intersection of the line via p and p_0 with C. This point is not p_0 as any the line tangent to C to p_0 does not intersect it in any other point, see Lemma 7.7(c). It follows that $-r$ is the third point of the intersection with the line through r and p_0 with C, which is of course p.

By definition $-p$ is the third point of the intersection of a line through p and p_0 with C. Then the third point on the line through p and $-p$ which lies on C is p_0, hence $p+(-p)=p_0$. □

8 09.02. Lecture 8. Topology of non-singular plane curves and branched covers of $\mathbb{CP}^1$

As a subset of $\mathbb{P}^2$ a complex projective curve has a natural topology. In fact C is a sphere with g handles. This number g is called the genus of the curve. We shall show that we have a degree-genus formula

$$g = \frac{1}{2}(d-1)(d-2).$$

Lemma 8.1. *A complex projective line L in $\mathbb{P}^2$ is homeomorphic to the two-dimesnional sphere*

$$S^2 = \{(u, v, w) \in \mathbb{R}^2 \mid u^2 + v^2 + w^2 = 1\}.$$

Proof. By applying a projective transformation which is a homeomorphism by Lemma 2.8 we may assume that $L = \{z = 0\}$. We define $\varphi\colon S^2 \to L$ via

$$\varphi(u, v, w) = [u + iv, 1 - w, 0].$$

its inverse is given by

$$\psi([x : y : 0]) = (\frac{2\mathrm{Re}\,(x\overline{y})}{|x|^2 + |y|^2}, \frac{2\mathrm{Im}\,(x\overline{y})}{|x|^2 + |y|^2}, \frac{|x|^2 - |y|^2}{|x|^2 + |y|^2}).$$

□

To describe the approach we are taking to determine genus of a curve, we first consider an example of a projective curve

$$C = \{[x : y : z] \mid x^3 + y^3 + z^3 = 3yz^2\}.$$

Its affine part is given by

$$x^3 + y^3 + 1 = 3y.$$

Then

$$y = \left(\frac{-(x^3+1) + \sqrt{x^6 + 2x^3 - 3}}{2}\right)^{\frac{1}{3}} + \left(\frac{-(x^3+1) - \sqrt{x^6 + 2x^3 - 3}}{2}\right)^{\frac{1}{3}}$$

for an appropriate choice of the cube roots. If $x \notin S = \{1, \omega, \overline{\omega}, -\sqrt[3]{3}, -\omega\sqrt[3]{3}, -\overline{\omega}\sqrt[3]{3}\}$ then the value of y is unique (the set S is the set of roots of $x^6 + 2x^3 - 3$). If we cut $\mathbb{C}$ along straight line segments connecting points of S, i.e. if we remove $[1, -\omega\sqrt[3]{3}]$, $[-\omega\sqrt[3]{3}, \overline{\omega}]$, $[\overline{\omega}, -\sqrt[3]{3}]$, $[-\sqrt[3]{3}, \omega]$ and $[\omega, -\overline{\omega}\sqrt[3]{3}]$ then on the remaining open set D there are three functions f_1, f_2, f_3 satisfying

$$f_j(x)^3 + x^3 + 1 = 3f_j(x).$$

If we add three points at the infinity we can construct C by gluing three copies of $D \cup \{\infty\}$. We can read off the gluing from the behaviour of branches f_1, f_2, f_3 when we move around points of S.

We will see that the map which assigns to (x, y) in the affine part of C the coordinate x extends to a map $\varphi\colon C \to \mathbb{P}^1$ such that for all but finitely many points of $\mathbb{P}^1$, $\varphi^{-1}(p)$ has three elements. In general, we will show that any smooth projective curve admits a map $\varphi\colon C \to \mathbb{P}^1$ which is a homeomorphism outside of finitely many points on C and $\mathbb{P}^1$ – called ramification points and the branch locus. On the next lecture we will construct a triangulation of $\mathbb{P}^1$ and the induced triangulation of C. They will tell us how to glue C from pieces of $\mathbb{P}^1$. It will allow us to relate the genus and degree of C.

Note that curve $C = \{y^2 = xz\}$ admits a surjection $\varphi\colon C \to \mathbb{P}^1$ defined $\varphi[x : y : z] = [x : z]$ such that if $[x : z] \in \mathbb{P}^1$ then $\varphi^{-1}([x : z])$ consists of exactly two points unless $x = 0$ or $z = 0$. Such a map $\varphi\colon C \to \mathbb{P}^1$ is called a double cover of $\mathbb{P}^1$ branched over $[0 : 1]$ and $[1 : 0]$. We can use φ to visualise C as two copies of $\mathbb{P}^1$ cut and glued together.

We shall see that any non-singular projective curve C of degree $d > 1$ can be viewed as a branched cover of $\mathbb{P}^1$ in a similar way.

Let C be a non-singular projective curve in $\mathbb{P}^2$ defined by a homogeneous polynomial $P(x, y, z)$ of degree $d > 1$. By applying a suitable projective transformation we may assume that $[0 : 1 : 0] \notin C$. Then we have a well-defined continuous map $\varphi\colon C \to \mathbb{P}^1$ given by

$$\varphi[x : y : z] = [x : z].$$

Definition 8.2. The *ramification index* $\nu_\varphi[a : b : c]$ of φ at a point $[a : b : c] \in C$ is the order of zero of the polynomial $P(a, y, c)$ at point $y = b$. Point $[a : b : c]$ is a *ramification point* of φ if $\nu_\varphi[a : b : c] > 1$.

Remark 8.3. (i) $\nu_\varphi[a : b : c] > 0$ for any $[a : b : c] \in C$.

(ii) $\nu_f[a : b : c] > 1$ if and only if

$$P(a, b, c) = 0 = \frac{\partial P}{\partial y}(a, b, c) = 0,$$

i.e. if and only if $[a : b : c] \in C$ and the tangent line to C at $[a : b : c]$ contains the point $[0 : 1 : 0]$.

(iii) $\nu_f[a : b : c] > 2$ if and only if

$$P(a, b, c) = 0 = \frac{\partial P}{\partial y}(a, b, c) = \frac{\partial^2 P}{\partial y^2}(a, b, c) = 0.$$

This happens if and only if the tangent line to C at $[a : b : c]$ contains $[0 : 1 : 0]$ and $[a : b : c]$ is the point of inflection on C. To prove this note that $[a : b : c] \neq [0 : 1 : 0]$ so $a \neq 0$ or $c \neq 0$. Let us assume that $c \neq 0$. If $P(a, b, c) = 0 = \frac{\partial P}{\partial y}(a, b, c)$ then formula (5) from the last lecture gives

$$\mathcal{H}_P(a, b, c) = \frac{(d-1)^2}{c^2} \det \begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & 0 \\ P_x & 0 & 0 \end{pmatrix} = -\frac{(d-1)^2}{c^2}(P_x)^2 P_{yy}.$$

By Euler's relation, if $P_x(a, b, c) = 0$ then $P_z(a, b, c) = 0$ and the point would be singular. It follows that $P_x \neq 0$, hence $\mathcal{H}_P(a, b, c) = 0$ if and only if $P_{yy}(a, b, c) = 0$.

Lemma 8.4. *The inverse image $\varphi^{-1}([a:c])$ of any $[a:c] \in \mathbb{P}^1$ under φ contains exactly*

$$d - \sum_{p \in \varphi^{-1}([a:c])} (\nu_\varphi(p) - 1)$$

points. In particular $\varphi^{-1}([a:c])$ contains d points if and only of $\varphi^{-1}([a:c])$ contains no ramification points of φ.

Proof. A point of C lies in $\varphi^{-1}([a:c])$ if and only if it is of the form $[a:b:c]$ where $P(a,b,c) = 0$. By assumption $P(0,1,0) \neq 0$ and we may assume that $P(0,1,0) = 1$. Then $P(a,y,c)$ is a monic polynomial of degree d:

$$P(a,y,c) = \prod_{1 \leq i \leq r} (y - b_i)^{m_i},$$

where $b_1, \ldots, b_r$ are distinct complex numbers and $m_1 + \ldots + m_r = d$. Thus

$$\varphi^{-1}([a:c]) = \{[a, b_i, c] \,|\, 1 \leq i \leq r\}$$

and

$$\nu_\varphi[a : b_i : c] = m_i.$$

Then

$$r = d - \sum_{i=1}^{r} m_i + r = d - \sum_{i=1}^{r} (m_i - 1).$$

□

Note that the above proof does not require that C is non-singular.

Definition 8.5. Let R be the set of ramification points of φ. The image $\varphi(R)$ of R under φ is the *branch locus* of φ and $\varphi\colon C \to \mathbb{P}^1$ is a *branch cover* of $\mathbb{P}^1$.

Lemma 8.6. *(i) φ has at most $d(d-1)$ ramification points.*

(ii) If $\nu_\varphi[a:b:c] \leq 2$ for all $[a:b:c] \in C$ then C has exactly $d(d-1)$ ramification points.

Proof. Since C is non-singular it is irreducible, Corollary 5.3. By assumption $[0:1:0] \notin C$ so the coefficient $P(0,1,0)$ of y^d in P is non-zero. Thus, the homogeneous polynomial $\frac{\partial P}{\partial y}(x,y,z)$ is homogeneous of degree $d-1$. It cannot be divisible by P, so C and the curve $D = \{\frac{\partial P}{\partial y} = 0\}$ have no common component. It follows from Remark 8.3(ii) that any ramification point of φ lies on $C \cap D$. By Theorem 5.2 $C \cap D \leq d(d-1)$ which finishes the proof of part (i).

Now suppose that $\nu_\varphi[a:b:c] \leq 2$, for all $[a:b:c] \in C$. By Corollary 6.4, in order to prove (ii) it suffices to check that for any $p \in C \cap D$, p is a non-singular point of D and the tangent line to C and D at p are distinct. If not then $p = [a:b:c]$ satisfies

$$P(a,b,c) = 0 = P_y(a,b,c)$$

and the tangent vector to D

$$(P_{xy}(a,b,c), P_{yy}(a,b,c), P_{zy}(a,b,c))$$

is either zero or the scalar multiple of the tangent vector to C:

$$(P_x(a,b,c), P_y(a,b,c), P_z(a,b,c)).$$

It follows that $P_{yy}(a,b,c) = 0$, i.e. $\nu_\varphi[a:b:c] \geq 2$. □

Lemma 8.7. *By applying a suitable projective transformation to C we may assume that*

$$\nu_\varphi[a:b:c] \leq 2,$$

for all $[a:b:c] \in C$.

Proof. We know that C has at most $3d-6$ points of inflection, see Proposition 7.4. Thus by applying a suitable projective transformation we may assume that $[0:1:0]$ does not lie on C nor on any of the tangent lines to C at its inflection points. We conclude by Remark 8.3(iii). □

10.02. Workshop II

1. Let $p = [0 : 0 : 1]$. Calculate $I_p(x^2 + 2yz, y^3 + x^2z)$.

2. Prove Lemma 7.7: A line L in $\mathbb{P}^2$ meets a non-singular cubic C either

   (a) in three distinct points p, q, r each with intersection multiplicity one (i.e. L is not the tangent line to C at p, q and r),

   (b) in two points, p with multiplicity one and q with intersection multiplicity two (i.e. L is the tangent line to C at q but not at p and q is not a point of inflection on C),

   (c) in one point p with intersection multiplicity three (i.e. L is a tangent line to C at p and p is the inflection point).

   Reduce to the case when $L = \{y = 0\}$ and $[1 : 0 : 0] \notin C$. Describe points in $C \cap L$. Calculate tangent line to C at $p \in C \cap L$ and check when it coincides with L. Check that p is an inflection point if and only if $|C \cap L| = \{p\}$.

3. Use Bezout's theorem to show that if a projective curve C in $\mathbb{P}^2$ of degree d has strictly more than $d/2$ singular points all lying on a line L then L is a component of C.

4. Prove Pappus' theorem: if L and M are two projective lines in $\mathbb{P}^2$ and p_1, p_2, p_3 lie on $L \setminus L \cap M$ and q_1, q_2, q_3 lie on $M \setminus L \cap M$ then if L_{ij} is the line joining p_i and q_j the three points of intersection of the pairs of lines L_{ij} and L_{ji} are collinear.

Solutions to Workshop II

1. Since p does not lie on $\{z = 0\}$, we have

$$I_p(x^2 + 2yz, y^3 + x^2z) = I_p(z(x^2 + 2yz), y^3 + x^2z)$$

We use property (vi) to get

$$I_p(zx^2+2yz^2, y^3+x^2z) = I_p(zx^2+2yz^2-y^3-x^2z, y^3+x^2z) = I_p(y(2z^2-y^2), y^3+x^2z).$$

Using (v) we get

$$I_p(y(2z^2 - y^2), y^3 + x^2z) = I_p(y, y^3 + x^2z) + I_p(2z^2 - y^2, y^3 + x^2z).$$

Since p does not lie on the curve $\{2z^2 - y^2 = 0\}$, the second intersection is zero. Using (vi) and (v) again, we get

$$I_p(y, y^3 + x^2z) = I_p(y, x^2z) = 2I_p(y, x) + I_p(y, z) = 2 \cdot 1 + 0 = 2.$$

2. Since C is irreducible, it does not contain L so we can assume that $L = \{y = 0\}$ and that $[1 : 0 : 0] \notin C$. If

$$C = \{[x : y : z] \mid P(x, y, z) = 0\}$$

then

$$C \cap L = \{[x : 0 : z] \mid P(x, 0, z) = 0\}.$$

By Lemma 1.7, $P(x, 0, z)$ can be written as

$$P(x, 0, z) = \mu(x - \lambda_1 z)(x - \lambda_2 z)(x - \lambda_3 z).$$

Then

$$C \cap L = \{[\lambda_i : 0 : 1] \mid i = 1, 2, 3\}.$$

The tangent line to L at $[\lambda_i : 0 : 1]$ is

$$L_i = \{\frac{\partial P}{\partial x}(\lambda_i, 0, 1)x + \frac{\partial P}{\partial y}(\lambda_i, 0, 1)y + \frac{\partial P}{\partial z}(\lambda_i, 0, 1)z = 0\}.$$

The line L_i is L if $\frac{\partial P}{\partial x}(\lambda_i, 0, 1) = 0 = \frac{\partial P}{\partial z}(\lambda_i, 0, 1)$.

We know that

$$\lambda_i \frac{\partial P}{\partial x}(\lambda_i, 0, 1) + \frac{\partial P}{\partial z}(\lambda_i, 0, 1) = 3P(\lambda_i, 0, 1) = 0.$$

Hence,

$$\frac{\partial P}{\partial x}(\lambda_i, 0, 1) = 0 \Leftrightarrow \frac{\partial P}{\partial z}(\lambda_i, 0, 1) = 0.$$

It follows that L_i is L if and only if λ_i is a repeated root of the polynomial

$$P(x, 0, 1) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

In other words, if $P(x,0,1)$ has three pairwise different irreducible factors then L intersects C in three different points and is not tangent to C at any of these points.

If L is tangent to C at point $[\lambda_i : 0 : 1]$ we need to check that $P(x,0,y) = (x-\lambda_i)^3$ if and only if $[\lambda_i : 0 : 1]$ is the inflection point of C.

From the lecture we know that

$$z^2\mathcal{H}_P(x,y,z) = (d-1)^2 \det \begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & P_y \\ P_x & P_y & dP/(d-1) \end{pmatrix}.$$

It follows that

$$\mathcal{H}_P(\lambda_i,0,1) = 4\det \begin{pmatrix} P_{xx} & P_{xy} & 0 \\ P_{yx} & P_{yy} & P_y \\ 0 & P_y & 0 \end{pmatrix} (\lambda_i,0,1) = -4(P_y)^2 P_{xx}(\lambda_i,0,1).$$

Since L is assumed to be tangent to C at $[\lambda_i : 0 : 1]$, derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial z}$ vanish when evaluated at $(\lambda_i,0,1)$. Since C is non-singular, it follows that $\frac{\partial P}{\partial y}(\lambda_i,0,1) \neq 0$. It shows that $[\lambda_i : 0 : 1]$ is an inflection point of C if and only if $\frac{\partial^2 P}{\partial x^2}(\lambda_i,0,1) = 0$, i.e. if and only if $P(x,0,z) = (x-\lambda_i z)^3$.

3. Let C be curve of degree d and let L be a line which contains e singular points of C. For any $p \in C \cap L$ we have $I_p(C,L) \geq 2$. It follows that $\sum_{p\in C\cap L} I_p(C,L) \geq 2e$.

   If $e > |d|/2$ then $\sum_{p\in C\cap L} I_p(C,L) > d$. It follows from Bezout's theorem that L is a component of C.

4. Let L_{ij} be a line through p_i and q_j and let $r_1 = L_{12}\cap L_{21}$, $r_2 = L_{13}L_{31}$, $r_3 = L_{23}\cap L_{32}$. Consider a line E through r_1 and r_2. We shall show that $r_3 \in E$.

   Let $\Sigma = \{p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2\}$. We show that the space of cubics which contain Σ is two-dimensional. A cubic is given by

   $$P(x,y,z) = Ax^3 + By^3 + Cz^3 + Dx^2y + Ex^2z + Fxy^2 + Gy^2z + Hxz^2 + Iyz^2 + Jxyz,$$

   i.e. the space V_3 of cubics in 10 dimensional. We have the evaluation map given by Σ:

   $$\psi\colon V_3 \to \mathbb{C}^8, \quad \psi(P) = (P(p_1), \ldots, P(r_2)).$$

   The space of cubics which contain Σ is the kernel of ψ. It is two dimensional if ψ is surjective, i.e. if for any point of Σ there exists a cubic which does not contain this point and contains the remaining 7 points of Σ. This is indeed the case. For p_1 it is the cubic $M \cup L_{21} \cup L_{31}$. Similarly, one can construct cubics for the remaining 7 points of Σ.

   Let C_1 be a cubic $L_{12}\cup L_{23}\cup L_{31}$ and C_2 the cubic $L_{21}\cup L_{32}\cup L_{13}$. The corresponding polynomials P_1 and P_2 form a basis of cubics containing Σ. It follows that the cubic $C = M \cup L \cup E$ corresponds to a polynomial $P = \lambda P_1 + \mu P_2$. In particular, $C_1 \cap C_2 = \Sigma \cup \{r_3\}$ is contained in C which finishes the proof.

9 13.02. Lecture 9. Covering projections, triangulations, and Euler characteristic

Definition 9.1. A continuous map $\pi\colon Y \to X$ between topological spaces is a *covering projection* if each $x \in X$ has an open neighbourhood $U \subset X$ such that $\pi^{-1}(U)$ is a disjoint union of open subsets of Y each of which is mapped homeomorphically onto U by π.

We have seen an example of a covering projection last time. Namely, let C be a projective algebraic curve of degree d which does not contain $[0:1:0]$. The map

$$\varphi\colon C \to \mathbb{P}^1,\ f([x:y:z]) = [x:z]$$

is well-defined. If we denote by $S \subset C$ the (finite!!) set of ramification points and by $B = \varphi(S)$ the branch locus, φ induces a map $\pi\colon C \setminus S \to \mathbb{P}^1 \setminus B$ such that the preimage of every point has exactly d points. The same is true about a small open subset U of $\mathbb{P}^1 \setminus B$, $\pi^{-1}(U)$ is a disjoint union of d copies of U.

Another example, which we shall use later is given by a lattice (i.e. an additive subgroup isomorphic to $\mathbb{Z}^2$) $\Lambda \subset \mathbb{C}$. The natural projection $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ is a covering projection of a torus.

Lemma 9.2. *Let $\pi\colon Y \to X$ be a covering projection and $f\colon [0,1] \to X$ a continuous map. Given $y \in Y$ such that $\pi(y) = f(0)$ there exists a unique map $F\colon [0,1] \to Y$ such that $F(0) = y$ and $\pi \circ F = f$.*

Proof. X has an open cover $X = \bigcup U_i$ such that $\pi^{-1}(U_i)$ is a disjoint union of open subsets of Y mapped homeomorphically onto U_i. We may assume that $f(0) \in U_0$. Since $[0,1]$ is compact, there exist $0 = t_0 < t_1 < \ldots < t_{n-1} < 1 = t_n$ such that $f[t_{j-1}, t_j]$ is contained in one of U_i's.

Let $\pi^{-1}(U_0) = V_0 \cup \ldots \cup V_d$. Fixed element y belongs to one of the V_k's, without loss of generality we may assume $y \in V_0$. Let $\psi\colon U \to V_0$ be a homeomorphism such that $\pi \circ \psi = \mathrm{Id}_U$. Then $F_1 := \psi \circ f|_{[0,t_1]}\colon [0,t_1] \to Y$ is a map such that $F_1(0) = y$ and $\pi \circ F_1 = f|_{[0,t_1]}$.

Map F_1 gives a point $F_1(t_1) \in \pi^{-1}(t_1)$. Point t_1 lies in some element of the open cover of X, let us denote it by U_{i_1}. As before we have $\pi^{-1}(U_{i_1}) = V'_0 \cup \ldots \cup V'_d$ and without loss of generality we may assume that $F_1(t_1) \in V'_0$. A homeomorphism $\psi'\colon U_{i_1} \to V'_0$ such that $\pi \circ \psi' = \mathrm{Id}_{U_{i_1}}$ gives a map $\psi' \circ f|_{[t_1,t_2]}\colon [t_1,t_2] \to Y$. Since $\psi' \circ f(t_1) = F_1(t_1)$, maps $\psi' \circ f|_{[t_1,t_2]}$ and F_1 can be glued to give $F_2\colon [0,t_2] \to Y$ such that $F_2(0) = y$ and $\pi \circ F_2 = f|_{[0,t_2]}$.

Analogous argument gives maps $F_3, \ldots, F_n$. By definition $F_n = F$ is the desired map $[0,1] \to Y$. Since at every step the construction of F_i was unique, map F is unique. $\square$

We consider a closed simplex

$$\Delta = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\},$$

its interior

$$\Delta^0 = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y < 1\},$$

and boundary

$$\partial\Delta = \Delta \setminus \Delta^0.$$

Definition 9.3. A topological space X is *simply connected* if any continuous map $g\colon \partial\Delta \to X$ can be extended to a continuous map $g\colon \Delta \to X$.

Lemma 9.2 can be generalised to

Lemma 9.4. *(i) Let $\pi\colon Y \to X$ be a covering projection and let $f\colon A \to X$ be a continuous map. Suppose that A is simply connected, path connected and locally path connected (i.e. every point in A has arbitrarily small path connected open neighbourhood in A). Then given any $a \in A$ and $y \in Y$ such that $f(a) = \pi(y)$ there is a unique continuous map $F\colon A \to Y$ such that $F(a) = y$ and $\pi \circ F = f$.*

(ii) If f is a homeomorphism onto its image then F is a homeomorphism onto a connected component of $\pi^{-1}f(A)$.

A non-singular complex algebraic curve is given cut out by a polynomial from $\mathbb{P}^2$, i.e. it has a complex dimension one. When considered over the field of real numbers, C is of dimension two, i.e. it is a surface. We shall introduce Riemann surfaces later, for now on let us just say that a surface X is a topological space such that every point $x \in X$ has an open neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^2$.

Definition 9.5. Let X be a surface. A *triangulation* of X is the following data

(a) A finite nonempty set V of points called vertices,

(b) A finite nonempty set E of continuous maps $e\colon [0,1] \to X$ called edges,

(c) A finite nonempty set F of continuous maps $f\colon \Delta \to X$ called faces,

satisfying

(i) $V = \{e(0) \mid e \in E\} \cup \{e(1) \mid e \in E\}$, i.e. vertices are the end points of edges;

(ii) If $e \in E$ then the restriction of e to the open interval $(0,1)$ is a homeomorphism onto its image, and this image contains no points in V or in the image of any other $\widetilde{e} \in E$;

(iii) If $f \in F$ then the restriction of f to Δ^0 is a homeomorphism onto a connected component K_f of $X \setminus \Gamma$, where

$$\Gamma = \bigcup_{e \in E} e([0,1]).$$

If $r\colon [0,1] \to [0,1]$ and $\sigma_i\colon [0,1] \to \Delta$ are defined by

$$r(t) = 1-t, \qquad \sigma_1(t) = (t,0), \qquad \sigma_2(t) = (1-t,t), \qquad \sigma_3(t) = (0,1-t)$$

then either $f \circ \sigma_i$ or $f \circ \sigma_i \circ r$ is an edge $e_f^i \in E$, for $1 \leq i \leq 3$;

(iv) The mapping $f \mapsto K_f$ from F to the set of connected components of $X \setminus \Gamma$ is a bijection;

(v) for every $e \in E$ there is exactly one face $f_e^+ \in F$ such that $e = f_e^+ \circ \sigma_i$, for some i and exactly one face $f_e^- \in F$ such that $e = f_e^- \circ \sigma_i \circ r$, for some i.

Note that the boundary of Δ is oriented counter clock-wise.
The *Euler number* $\chi_T(X)$ with respect to the triangulation T is

$$\chi_T(X) = |V| - |E| + |F|.$$

By Lemma 8.1 a complex projective line in $\mathbb{P}^2$ is homemomorphic to a sphere. Thus it has a triangulation with three vertices, three edges and two faces. It follows that

$$\chi_T(\mathbb{P}^1) = 2.$$

Theorem 9.6. *$\chi_T(X)$ depends only on X, not on the choice of triangulation.*

In order to prove Theorem 9.6 we introduce complex vector spaces

$$C_0^T(X), C_1^T(X), C_2^T(X)$$

with bases V, E and F. Thus

$$C_0^T(X) = \{\sum_{v\in V} \lambda_v v \,|\, \lambda_v \in \mathbb{C}\, v \in V\}.$$

We also have linear maps

$$C_2^T(X) \xrightarrow{\partial_2^T} C_1^T(X) \xrightarrow{\partial_1^T} C_0^T(X) \xrightarrow{\partial_0^T} \mathbb{C}$$

$$\begin{aligned}
\partial_0^T(\sum \lambda_v v) &= \sum \lambda_v, \\
\partial_1^T(\sum \lambda_e e) &= \sum \lambda_e (e(1) - e(0)), \\
\partial_2^T(\sum \lambda_f f) &= \sum \lambda_f (\pm e_f^1 \pm e_f^2 \pm e_f^3),
\end{aligned}$$

where the sign in front of e_f^i is positive if and only if $f \circ \sigma_i = e_f^i$. We have

$$\partial_0^T \partial_1^T(e) = \partial_0^T(e(1) - e(0)) = 0$$

and

$$\partial_1^T \partial_2^T(f) = f(1,0) - f(0,0) + f(0,1) - f(1,0) + f(0,0) - f(0,1) = 0.$$

It follows that

$$\operatorname{im} \partial_1^T \subset \ker \partial_0^T, \qquad \operatorname{im} \partial_2^T \subset \ker \partial_1^T.$$

Lemma 9.7. *We have*

$$\chi_T(X) = \dim(\frac{\ker \partial_0^T}{im\, \partial_1^T}) - \dim(\frac{\ker \partial_1^T}{im\, \partial_2^T}) + k + 1$$

where k is the number of connected components of X.

Proof. We have

$$\begin{aligned} |V| &= \dim \ker \partial_0^T + \dim \operatorname{im} \partial_0^T, \\ |E| &= \dim \ker \partial_1^T + \dim \operatorname{im} \partial_1^T, \\ |F| &= \dim \ker \partial_2^T + \dim \operatorname{im} \partial_2^T. \end{aligned}$$

Then

$$\chi_T(C) = \dim(\frac{\ker \partial_0^T}{\operatorname{im} \partial_1^T}) - \dim(\frac{\ker \partial_1^T}{\operatorname{im} \partial_2^T}) + \dim \operatorname{im} \partial_0^T + \dim \ker \partial_2^T.$$

Since $V \neq \emptyset$ map ∂_0^T is non-zero, hence surjective. It follows that $\dim \operatorname{im} \partial_0^T = 1$. It thus suffices to show that $\dim \ker \partial_2^T = k$.

By point (vi) of the definition of the triangulation, we have

$$\partial_2^T(\sum \lambda_f f) = \sum (\lambda_{f_e^+} - \lambda_{f_e^-}) e.$$

It vanishes if and only if $\lambda_{f_e^+} = \lambda_{f_e^-}$ for all $e \in E$.

Let $X_1, \ldots, X_k$ be connected components of X. Since Δ is connected, $f(\Delta)$ is contained in one of the $X_i's$. Moreover, by (v), $X = \bigcup_{f\in F} f(\Delta)$. Finally, $f(\Delta) \cap \widetilde{f}(\Delta)$ if and only if there exists $e \in E$ such that $\{f, \widetilde{f}\} = \{f_e^+, f_e^-\}$. It follows that $f(\Delta)$ and $\widetilde{f}(\Delta)$ lie in the same connected component of X if and only if there is a sequence

$$f = f_0, f_1, \ldots, f_n = \widetilde{f}$$

such that

$$\{f_i, f_{i+1}\} = \{f_{e_i}^+, f_{e_i}^-\}.$$

Therefore $\sum \lambda_f f$ satisfies $\lambda_{f_e^+} = \lambda_{f_e^-}$ if and only if there exist $\mu_1, \ldots, \mu_k \in \mathbb{C}$ such that $\lambda_f = \mu_i$ if and only if $f(\Delta) \subset X_i$. It follows that $\dim \ker \partial_2^T = k$ which finishes the proof. □

To prove Theorem 9.6 it suffices to check that $\dim(\frac{\ker \partial_0^T}{\operatorname{im} \partial_1^T})$ and $\dim(\frac{\ker \partial_1^T}{\operatorname{im} \partial_2^T})$ are independent of the triangulation.

We introduce infinite dimensional spaces $C_0(X)$, $C_1(X)$ and $C_2(X)$. The basis of $C_0(X)$ are points of X, i.e.

$$C_0(X) = \{\sum_{x\in X} \lambda_x\, x \,|\, x \in X,\ \lambda_x \in \mathbb{C},\ \lambda_x = 0 \text{for all but finitely many } x \in X\}.$$

The basis of $C_1(X)$ are continuous maps $[0,1] \to X$ and the basis of $C_2(X)$ are continuous maps $\Delta \to X$. We have

$$C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \mathbb{C}, \tag{7}$$

for ∂_2, ∂_1 and ∂_0 defined analogously as ∂_2^T, ∂_1^T, ∂_0^T. One checks that (7) is a complex. It is a part of *singular cochain* of X.

Proposition 9.8. $\frac{\ker \partial_0^T}{im \partial_1^T} \simeq \frac{\ker \partial_0}{im \partial_1}$ *and* $\frac{\ker \partial_1^T}{im \partial_2^T} \simeq \frac{\ker \partial_1}{im \partial_2}$.

The proof is based on Lemmas

Lemma 9.9. *If $\xi \in C_0(X)$ then there exists $\eta \in C_1(X)$ such that $\xi - \partial_1 \eta \in C_0^T(X)$.*

Lemma 9.10. *If $\xi \in C_1(X)$ and $\partial_1 \xi \in C_0^T(X)$ then there exists $\eta \in C_2(X)$ such that $\xi - \partial_1 \eta \in C_1^T(X)$.*

Lemma 9.11. *If $\xi \in C_2(X)$ and $\partial_2 \xi \in C_1^T(X)$ then there exists $\eta \in C_2(X)$ such that $\partial_2 \xi = \partial_2^T \eta$.*

Proof of Proposition 9.8. It suffices to show that for $j = 0, 1$,

$$\ker \partial_j = \mathrm{im}\partial_{j+1} + \ker \partial_j^T, \qquad \mathrm{im}\partial_{j+1} \cap \ker \partial_j^T = \mathrm{im}\partial_{j+1}^T$$

as then inclusions $\ker \partial_j^T \subset \ker \partial_j$ and $\mathrm{im}\partial_j^T \subset \mathrm{im}\partial_j$ will give isomorphisms

$$\frac{\ker \partial_j^T}{\mathrm{im}\partial_j^T} \simeq \frac{\ker \partial_j}{\mathrm{im}\partial_j}.$$

Indeed, any $\xi \in \ker \partial_j$ can be then written as $\zeta + \partial_j \eta$ for $\zeta \in \ker \partial_j^T$, hence ζ and ξ are equivalent in $\frac{\ker \partial_j}{\mathrm{im}\partial_{j+1}}$. Moreover, the second isomorphism implies that $\frac{\ker \partial_j^T}{\mathrm{im}\partial_j} \simeq \frac{\ker \partial_j^T}{\mathrm{im}\partial_j^T}$.

Inclusions

$$\mathrm{im}\partial_{j+1} + \ker \partial_j^T \subset \ker \partial_j, \qquad \mathrm{im}\partial_{j+1}^T \subset \mathrm{im}\partial_{j+1}^T \cap \ker \partial_j^T$$

are clear.

Let now $\xi \in \ker \partial_j$. There exists $\eta \in C_{j+1}(X)$ such that $\xi - \partial_{j+1}\eta = \zeta \in C_j^T(X)$. Moreover,

$$\partial_j^T(\zeta) = \partial_j(\zeta) = \partial_j \xi + \partial_j \partial_{j+1} \eta = 0,$$

so $\xi = \zeta + \partial_{j+1}\eta \in \ker \partial_j^T + \mathrm{im}\partial_{j+1}$.

Next, suppose that $\zeta \in \ker \partial_0^T$ and $\zeta = \partial_1 \xi$, for some $\xi \in C_1(X)$. There exists $\eta \in C_2(X)$ such that $\xi - \partial_2 \eta = \chi \in C_1^T(X)$. Then

$$\zeta = \partial_1 \xi = \partial_1 \chi = \partial_1^T \chi,$$

i.e. $\zeta \in \mathrm{im}\partial_1^T$.

Finally, suppose that $\zeta \in \ker \partial_1^T$, $\xi = \partial_2(\xi)$ for some $\xi \in C_2(X)$. There exists $\eta \in C_2^T(X)$ such that $\partial_2^T(\eta) = \partial_2 \xi = \zeta$, i.e. $\zeta \in \mathrm{im}\partial_2^T$.

Since the three objects above were arbitrary, we get the required isomorphisms. □

10 16.02. Lecture 10. Triangulation of plane curves and the degree–genus formula

Last time we defined a triangulation and its Euler number $\chi_T(X)$. We proved that $\chi_T(X)$ depends only on X, not on the triangulation. Today, we show

Theorem 10.1. *Every nonsingular projective curve C in $\mathbb{P}^2$ has a triangulation. Moreover, if C is of degree d and r is a positive integer such that $r \geq d(d-1)$ and $r \geq 3$ then C has a triangulation with $rd - d(d-1)$ vertices, $3(r-2)d$ edges and $2(r-2)d$ faces.*

Using map $\varphi\colon C \to \mathbb{P}^1$ we will relate the Euler number of C to the Euler number of $\mathbb{P}^1$ and degree of C. As a corollary, we will obtain the degree-genus formula.

First we relate the Euler number of a surface X to its genus.

Lemma 10.2. *Let X be a sphere with g handles. Then its Euler number is $2-2g$.*

Proof. Picture!!! □

Definition 10.3. The *genus* of a nonsingular projective curve is

$$g = \frac{1}{2}(2-\chi),$$

where χ is the Euler number of C.

As a corollary of Theorem 10.1 we get

Corollary 10.4. *The Euler number χ and genus g of a non-singular projective curve of degree d in $\mathbb{P}^2$ are given by*

$$\chi = rd - d^2 + d - 3rd + 6d + 2rd - 4d = d(3-d), \qquad g = \frac{1}{2}(d-1)(d-2).$$

For the proof of Theorem 10.1 we need

Lemma 10.5. *Let $p_1 \dots p_r$ be any set of at least 3 distinct points in $\mathbb{P}^1$. Then there is a triangulation of $\mathbb{P}^1$ with $p_1, \dots, p_r$ as vertices, $3r-6$ edges and $2r-4$ faces.*

Proposition 10.6. *Let C be a nonsingular projective curve in $\mathbb{P}^2$ not containing $[0:1:0]$ and let $\varphi\colon C \to \mathbb{P}^1$ be the branched cover defined by $\varphi([x:y:z]) = [x:z]$. Suppose that (V,E,F) is a triangulation of $\mathbb{P}^1$ such that V contains the branch locus of φ. Then there is a triangulation $(\widetilde{V}, \widetilde{E}, \widetilde{F})$ of C such that*

$$\begin{aligned}
\widetilde{V} &= \varphi^{-1}(V),\\
\widetilde{E} &= \{\widetilde{e}\colon [0,1] \to C \,|\, \widetilde{e} \text{ is continuous } \varphi \circ \widetilde{e} \in E\},\\
\widetilde{F} &= \{\widetilde{e}\colon \Delta \to C \,|\, \widetilde{f} \text{ is continuous } \varphi \circ \widetilde{f} \in F\}.
\end{aligned}$$

Moreover, if $\nu_\varphi(p)$ is the ramification index of φ at p and d is the degree of C then

$$|\widetilde{V}| = d|V| - \sum_{p\in R}(\nu_f(p) - 1),$$
$$|\widetilde{E}| = d|E|,$$
$$|\widetilde{F}| = d|F|,$$

where $R \subset C$ is the set of ramification points for φ.

It gives *Riemann-Hurwitz formula*

$$\chi(C) = d\chi(\mathbb{P}^1) - \sum_{p\in R}(\nu_\varphi(p) - 1).$$

Proof of Theorem 10.1. Let $P(x, y, z)$ be a homogeneous polynomial of degree d defining the curve C. By Lemma 8.7 after applying a suitable projective transformation we can assume that map

$$\varphi\colon C \to \mathbb{P}^1,\ \varphi([x : y : z]) = [x : z]$$

is well-defined and the ramification index of φ at every point of C is less than or equal to two. Then, by Lemma 8.6 φ has exactly $d(d-1)$ ramification points. We denote the set of ramification points by R. By Lemma 10.5 if $r \geq 3$ and $r \geq d(d-1)$ then we can choose a triangulation (V, E, F) of $\mathbb{P}^1$ such that $\varphi(R) \subset V$ and $|V| = r$, $|E| = 3r - 6$, $|F| = 2r - 4$. Therefore, by Proposition 10.6, there is a triangulation $(\widetilde{V}, \widetilde{E}, \widetilde{F})$ of C with

$$|\widetilde{V}| = d|V| - \sum_{p\in R}(\nu_\varphi(p) - 1) = rd - d(d-1),$$
$$|\widetilde{E}| = d|E| = 3(r-2)d,$$
$$|\widetilde{F}| = d|F| = 2(r-2)d.$$

□

Proof of Lemma 10.5. We proceed by induction on r.

We know that $\mathbb{P}^1 \simeq S^2$ so it has a triangulation with 3 vertices, 3 edges and 2 faces.

Assume now that a triangulation with vertices $p_1, \ldots, p_{r-1}$ exists. By inductive assumption it has $3r - 9$ edges and $2r - 6$ faces.

Point p_r might lie in an interior of a face or of an edge. In the first case we consider the subdivision of the triangle. We add 3 edges and 2 faces. If p_r lies on an edge e, we need to subdivide two faces adjacent to e. We add 3 edges and 2 faces. Hence in both cases we got a triangulation with r vertices, $3r - 6$ edges and $2r - 4$ faces which finishes the proof. □

Sketch of proof of Proposition 10.6. Let $f\colon \Delta \to \mathbb{P}^1$ be an element of F. If $p \in C$ is a point such that $\varphi(p) = f(t)$, for some $t \notin \{(0,0), (0,1), (1,0)\}$, then by Lemma 9.4 there exists a unique $\widetilde{f}\colon \Delta \to C$ such that $\widetilde{f}(t) = p$ and $\varphi \circ \widetilde{f} = f$. Since $\varphi^{-1}(f(t))$ has d points, there are exactly d "faces over" $\widetilde{f}$.

Similarly, any $e\colon [0,1] \to \mathbb{P}^1$ can be lifted in d possible ways to $\widetilde{e}\colon [0,1] \to C$.

By Lemma 8.4, $\widetilde{V}$ has $d|V| - \sum_{p\in R}(\nu_\varphi(p) - 1)$ points.

For a lift $\widetilde{e}$ of e, $\widetilde{e}(0)$ and $\widetilde{e}(1)$ lie in the fibers of φ over $e(0)$ and $e(1)$. As for any e, $e(0)$ and $e(1)$ are points of V, the end points of $\widetilde{e}$ are points of $\widetilde{V}$. Similarly for the edges of $\widetilde{f} \in \widetilde{F}$. It follows that conditions (i)-(iii) are satisfied.

By construction the set $G = \bigcup_{\widetilde{f}\in\widetilde{F}} \widetilde{f}(\Delta)$ contains $C \setminus \varphi^{-1}(V)$. As $\varphi^{-1}(V)$ is finite, G is dense. Moreover, G is a union of compact, hence closed sets. It follows that $G = C$ and (iv) holds.

The proof of (v) is technical and will be skipped. □

16.02. Homework III

The homework in due on Thursday the 9th of March. Please hand in your solutions before the lecture.

1. Let (V, E, F) be a triangulation of a path connected surface X and let $C_0^T(X)$ be a complex vector space with basis V. Let further $C_0(X)$ be a complex vector space whose basis consists of all points of X and $C_1(X)$ a complex vector space with basis of all continuous maps $[0,1] \to X$. Let $\partial_1 \colon C_1(X) \to C_0(X)$ be a linear map which to a basis element $e\colon [0,1] \to X$ assigns $e(1) - e(0) \in C_0(X)$. Prove that for any $\xi \in C_0(X)$ there exists $\eta \in C_1(X)$ such that $\xi - \partial_1\eta \in C_0^T(X)$.

2. Let $\varphi\colon C \to \mathbb{P}^1$ be defined by
$$\varphi([x : y : z]) = [x : z]$$
where $C \subset \mathbb{P}^2$ is a non-singular projective curve not containing $[0 : 1 : 0]$. Show that if C has degree $d > 1$ then φ has at least one ramification point. Show that if $f = 1$ then φ has no ramification points and is a homeomorphism.

3. The theorem of isolated zeros in complex analysis states that for an open $U \subset \mathbb{C}$ and a non-constant holomorphic function $f\colon U \to \mathbb{C}$ the set $\{u \in U \mid f(u) = 0\}$ is totally disconnected. Use it to show that if $f\colon R \to S$ is a non-constant holomorphic map between connected Riemann surfaces then every $x \in R$ has an open neighbourhood $U \subset R$ such that $f(y) \neq f(x)$ for all $y \in U \setminus \{x\}$.

Solutions to Homework III

1. Let $\xi = \sum_{i=1}^r \lambda_i x_i$ and let v be a vertex of the triangulation. Since X is path connected, there exist paths connecting each of x_i with v. Let $e_i\colon [0,1] \to X$ be such a path, i.e. assume that $e_i(0) = x_i$ and $e_i(1) = v$.

   Define $\eta = \sum_{i=1}^r \lambda_i e_i \in C_1(X)$ and $v' = (\sum_{i=1}^r \lambda_i)v \in C_0^T(X)$. Then

   $$\xi - \partial_1 \eta = \sum_{i=1}^r \lambda_i x_i - \sum_{i=1}^r (\lambda_i e_i(0) - \lambda_i e_i(1)) = \sum_{i=1}^r \lambda_i v = v',$$

   i.e. η is the required element of $C_1(X)$.

2. Ramification points on φ are the points $[a : b : c] \in C$ such that $\frac{\partial P}{\partial y}(a, b, c) = 0$. If $d > 1$ then $\deg \frac{\partial P}{\partial y} \geq 1$, hence ramification points are the intersection points of C with $D = \{[x : y : z] \mid \frac{\partial P}{\partial y}(x, y, z) = 0\}$. By Bezout's theorem curves C and D intersect, hence φ has at least one ramification point.

   If $C = \{[x : y : z] \mid P(x, y, z) = 0\}$ is of degree one, then for any $[a : b : c] \in C$ the polynomial $P(a, y, c)$ is of degree one, hence it cannot have multiple zeroes. By definition φ has no ramification points.

   Let now q be any point in $\mathbb{P}^1$. We know that $\varphi^{-1}(q)$ has precisely

   $$1 - \sum_{p \in \varphi^{-1}(q)} (\nu_\varphi(p) - 1) = 1$$

   point. Thus, φ is surjective. Moreover, φ is injective as $\varphi(p) = \varphi(p')$ would imply that the preimage of $\varphi(p)$ would have at least two points. It follows that φ is bijective. We know that φ is continuous. To prove that φ is a homeomorphism we need to check that φ^{-1} is continuous.

   If $C = \{\alpha x + \beta y + \gamma z = 0\}$ then the condition $[0 : 1 : 0] \notin C$ implies that $\beta \neq 0$. Thus

   $$\varphi^{-1}([x : z]) = [x : \frac{-\alpha x - \gamma z}{\beta} : z]$$

   is continuous, as map $\psi\colon \mathbb{C}^2 \to \mathbb{C}^3$, $\psi(x, z) = (x, \frac{-\alpha x - \gamma z}{\beta}, z)$ is continuous and $\mathbb{P}^1$, $\mathbb{P}^2$ have quotient topology.

3. Let $x \in R$ be any point, $\varphi_\alpha\colon U_\alpha \to V_\alpha$, $\psi_\beta\colon W_\beta \to X_\beta$ atlases on R and S such that $x \in U_\alpha$ and $f(x) \in W_\beta$.

   Function $\psi_\beta \circ f \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha \cap f^{-1}W_\beta) \to X_\beta$ is holomorphic, hence so is

   $$g := \psi_\beta \circ f \circ \varphi_\alpha^{-1} - \psi_\beta \circ f(x)\colon \varphi_\alpha(U_\alpha \cap f^{-1}W_\beta) \to X'_\beta,$$

   where $X'_\beta = X_\beta - \psi_\beta \circ f(x)$ is the image of X_β under the homeomorphism $\mathbb{C} \to \mathbb{C}$, $z \mapsto z - \psi_\beta \circ f(x)$.

Theorem on isolated zeros implies that $\varphi_\alpha(x)$ has an open neighbourhood $V' \subset V_\alpha$ such that g is not equal to zero on $V' \setminus \{\varphi_\alpha(x)\}$. Since φ_α is a homeomorphism, $U' = \varphi_\alpha^{-1}(V')$ is an open neighbourhood of x.

Let y be a point in U'. If $f(y) = f(x)$ then $\psi_\beta \circ f \circ \varphi_\alpha^{-1}(\varphi_\alpha(y)) = \psi_\beta \circ f \circ \varphi_\alpha^{-1}(\varphi_\alpha(x))$ hence $\varphi_\alpha(y) \in V'$ is a zero of g. The contradiction with the choice of V' implies that $f(x) \neq f(y)$.

11 27.02. Lecture 11. Holomorphic atlases, Riemann surfaces and nonsingular plane curves

We define Riemann surfaces and show that every non-singular projective curve is a Riemann surface.

Definition 11.1. A *surface* is a Hausdorff topological space S which is locally homeomorphic to $\mathbb{C}$ (or equivalently to $\mathbb{R}^2$).

It means that any $x \in S$ has an open neighbourhood which is homeomorphic to an open subset in $\mathbb{C}$.

A homeomorphism $\varphi\colon U \to V$ between an open subset U of S and an open subset V of $\mathbb{C}$ is called a *chart* on S. An *atlas* Φ for S is a collection of charts

$$\Phi = \{\varphi_\alpha\colon U_\alpha \to V_\alpha \,|\, \alpha \in A\}$$

such that

$$S = \bigcup_{\alpha \in A} U_\alpha.$$

If $\varphi_\alpha\colon U_\alpha \to V_\alpha$ and $\varphi_\beta\colon U_\beta \to V_\beta$ are charts then

$$\varphi_\alpha(U_\alpha \cap U_\beta)$$

is an open subset of $\mathbb{C}$. If Φ is an atlas for S then the homeomorphisms

$$\varphi_{\alpha\beta} = \varphi_\alpha \varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

are called *transition functions*. An atlas Φ is *holomorphic* if all of its transition functions are holomorphic.

If U is an open subset then Id_U is a holomorphic atlas on U.

Proposition 11.2. *If C is a complex algebraic curve in $\mathbb{C}^2$ defined by a polynomial $P(x, y)$ then $C \setminus Sing(C)$ has a holomorphic atlas.*

Proof. Suppose that $(a, b) \in C$ and $\frac{\partial P}{\partial y}(a, b) \neq 0$. The implicit function theorem tells us that there are open neighbourhoods V and W of a and b in $\mathbb{C}$ and a holomorphic function $g\colon V \to W$ such that $g(a) = b$ and if $x \in V$, $y \in W$ then

$$P(x, y) = 0 \Leftrightarrow y = g(x).$$

Since $C \setminus \mathrm{Sing}(C)$ is open in C, shrinking V and W if necessary, we may assume that

$$U = \{(x, y) \in C \,|\, x \in V,\, y \in W\}$$

is an open neighbourhood of (a, b) in $C \setminus \mathrm{Sing}(C)$. Then $\varphi\colon U \to V$

$$\varphi(x, y) = x$$

is a continuous function with a continuous inverse

$$x \mapsto (x, g(x)).$$

Similarly, if $\frac{\partial P}{\partial x}(a,b) \neq 0$ there is an open neighbourhood U of (a,b) in $C \setminus \mathrm{Sing}(C)$ such that the map $\psi\colon U \to \mathbb{C}$,

$$\psi(x,y) = y$$

is a homeomorphism onto an open subset V of $\mathbb{C}$ with the inverse given by

$$y \mapsto (h(y), y).$$

Thus there is an atlas on $C \setminus \mathrm{Sing}(C)$ such that every chart is one of these two φ and ψ. The transition functions are either identity or

$$x \mapsto (x, g(x)) \mapsto g(x), \qquad\qquad y \mapsto (h(y), y) \mapsto h(y).$$

It follows that the atlas is holomorphic. □

Proposition 11.3. *If $C = \{P = 0\}$ is a projective curve in $\mathbb{P}^2$ then $C \setminus Sing(C)$ has a holomorphic atlas.*

Proof. Suppose that $[a:b:c] \in C$ and that $\frac{\partial P}{\partial y}(a,b,c) \neq 0$. By Euler's relation

$$a\frac{\partial P}{\partial x}(a,b,c) + b\frac{\partial P}{\partial y}(a,b,c) + c\frac{\partial P}{\partial z}(a,b,c) = 0$$

so $a = c = 0$ implies $a = b = c = 0$. It follows that $(a,c) \neq (0,0)$. Assume that $c \neq 0$. Then

$$\frac{\partial P}{\partial y}(a/c, b/c, 1) = c^{-d-1}\frac{\partial P}{\partial y}(a,b,c) \neq 0$$

where $d = \deg P$. By implicit function theorem for $P(x,y,1)$ there exist open neighbourhoods V of a/c and W of b/c and holomorphic $g\colon V \to W$ such that for $x \in V$, $y \in W$,

$$P(x,y,1) = 0 \Leftrightarrow y = g(x).$$

If V and W are small enough,

$$U = \{[x:y:1] \in C \,|\, x \in V,\, y \in W\}$$

is an open neighbourhood of $[a:b:c] = [a/c:b/c:1]$ in $C \setminus \mathrm{Sing}(C)$. The map $\varphi\colon U \to V$

$$\varphi([x:y:z]) = x/z$$

has an inverse

$$x \mapsto [x : g(x) : 1].$$

Similarly, if $a \neq 0$ or the other partial derivatives of P at (a,b,c) do not vanish we can find an open neighbourhood of $[a:b:c]$ in $C \setminus \mathrm{Sing}(C)$ and a homeomorphism to an open subset of $\mathbb{C}$ given by

$$z/x,\ y/z,\ z/y,\ x/y,\ y/x$$

and inverse

$$w \mapsto [1 : g(w) : w],\ [g(w) : w : 1]\ [g(w) : 1 : w],\ [w : 1 : g(w)],\ [1 : w : g(w)].$$

Thus we get an atlas in which all the transition functions are of the form

$$w \mapsto w,\ 1/w,\ g(w),\ 1/g(w),\ w/g(w),\ g(w)/w$$

such that g is holomorphic and the denominator does not vanish on the set where the transition function is defined. It follows that the atlas is holomorphic. □

When we have a holomorphic atlas, we can define a holomorphic function on a surface.

Definition 11.4. Let $\Phi = \{\varphi_\alpha \colon U_\alpha \to V_\alpha \mid \alpha \in A\}$ be a holomorphic atlas on a surface S. A continuous map $f \colon S \to \mathbb{C}$ is *holomorphic* with respect to Φ at $x \in S$ if there is a chart $\varphi_\alpha \colon U_\alpha \to V_\alpha$ in Φ such that $x \in U_\alpha$ and $f \circ \varphi_\alpha^{-1} \colon V_\alpha \to \mathbb{C}$ is holomorphic. Map φ is *holomorphic* if it is holomorphic with respect to Φ at every $x \in S$.

Because Φ is a holomorphic atlas, the above condition is equivalent to requiring that for any U_α such that $x \in U_\alpha$ map $f \circ \varphi_\alpha^{-1}$ is holomorphic.

It implies that

Lemma 11.5. *A continuous function $\varphi \colon S \to \mathbb{C}$ is holomorphic with respect to a holomorphic atlas Φ if and only of $f \circ \varphi_\alpha^{-1} \colon V_\alpha \to \mathbb{C}$ is holomorphic for every chart $\varphi_\alpha \colon U_\alpha \to V_\alpha$ in Φ.*

Definition 11.6. Let S and T be surfaces with holomorphic atlases Φ and Ψ. A continuous map $f \colon S \to T$ is *holomorphic* with respect to Φ and Ψ if

$$\psi_B \circ f \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap f^{-1} W_\beta)} \colon \varphi_\alpha(U_\alpha \cap f^{-1} W_\beta) \to Y_\beta$$

is holomorphic for every chart $\varphi_\alpha \colon U_\alpha \to V_\alpha$ in Φ and every chart $\psi_\beta \colon W_\beta \to Y_\beta$ in Ψ.

Note that $f^{-1}(W_\beta)$ is open because f is continuous, hence $U \cap f^{-1}(W_\beta)$ is open.

Lemma 11.7. *If $f \colon S \to T$ and $g \colon T \to R$ are holomorphic with respect to given holomorphic atlases Φ, Ψ, Θ on the surfaces S, T, R then $g \circ f \colon S \to R$ is holomorphic with respect to atlases Φ and Θ.*

Proof. Choose $x \in S$ and charts $\varphi_\alpha \colon U_\alpha \to V_\alpha$, $\psi_\beta \colon W_\beta \to Y_\beta$, $\theta_\gamma \colon X_\gamma \to Z_\gamma$ such that $x \in U_\alpha$, $f(x) \in V_\beta$, $gf(x) \in X_\gamma$. We need to check that $\theta_\gamma \circ g \circ f \circ \varphi_\alpha^{-1}$ is holomorphic at $\varphi_\alpha(x)$. It follows from the fact that locally we can write

$$\theta_\gamma \circ g \circ f \circ \varphi_\alpha^{-1} = (\theta_\gamma \circ g \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ f \circ \varphi_\alpha^{-1}).$$

□

Definition 11.8. Two holomorphic atlases Φ, Ψ on a surface S are *compatible* if the identity map $\mathrm{Id}_S \colon S \to S$ is holomorphic both as map from S with atlas Φ to S with atlas Ψ and as a map from S with atlas Ψ to S with atlas Φ.

It follows from Lemma 11.7 that compatibility defines equivalence relation on the set of holomorphic atlases on S.

The holomorphic atlases

$$\{\mathrm{Id}_{\mathbb{C}} \colon \mathbb{C} \to \mathbb{C}\}$$

and

$$\{h \colon U \to V \mid U, V \subset \mathbb{C} \text{ open, } h \text{ is holomorphic with a holomorphic inverse}\}$$

on $\mathbb{C}$ are compatible. They are not compatible with the atlas with the single chart

$$\{\varphi \colon \mathbb{C} \to \mathbb{C}\}, \quad \varphi(z) = \overline{z}.$$

Definition 11.9. A *Riemann surface* os a surface S together with an equivalence class $\mathcal{H}$ of holomorphic atlases on S. A *holomorphic map* $f \colon (S, \mathcal{H}) \to (T, \mathcal{F})$ is a continuous map which is holomorphic with respect to any holomorphic atlas $\Phi \in \mathcal{H}$ on S and $\Psi \in \mathcal{F}$ on T.

Definition 11.10. Two Riemann surfaces S and T are *biholomorphic* if there is a holomorphic bijection $f \colon S \to T$ whose inverse is holomorphic.

Examples

(a) Any open subset U of $\mathbb{C}$ with the holomorphic atlas $\{\mathrm{Id}_U \colon U \to U\}$ is a Riemann surface.

(b) If S is a Riemann surface with a holomorphic atlas

$$\Phi = \{\varphi_\alpha \colon U_\alpha \to V_\alpha \mid \alpha \in A\}$$

and $W \subset S$ is an open subset then

$$\Phi|_W = \{f_\alpha|_{U_\alpha \cap W} \colon U_\alpha \cap W \to \varphi_\alpha(U_\alpha \cap W) \mid \alpha \in A\}$$

is a holomorphic atlas on W.

(c) Every nonsingular complex or projective curve can be regarded as a Riemann surface.

(d) The *Riemann sphere* is $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ where $z \in \mathbb{C}$ is identified with $[z : 1] \in \mathbb{P}^1$ and $\infty = [0 : 1]$.

Let $U = \mathbb{P}^1 \setminus \{\infty\}$, $V = \mathbb{P}^1 \setminus \{0\}$. Define $\varphi \colon U \to \mathbb{C}$, $\psi \colon V \to \mathbb{C}$ by

$$\varphi([x : y]) = x/y, \; \psi([x : y]) = y/x.$$

It is a homolomorphic atlas with transition functions

$$\varphi \circ \psi^{-1} = \psi \circ \varphi^{-1} \colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$$

given by

$$z \mapsto 1/z.$$

If U is an open subset of $\mathbb{C}$ then a meromorphic function on U is $f\colon U \to \mathbb{C} \cup \{\infty\}$ such that for any $a \in U$ there exists $\varepsilon > 0$ such that f maps the punctured disc

$$\{z \in \mathbb{C} \,|\, 0 < |z - a| < \varepsilon\}$$

to $\mathbb{C}$ and admits a Laurent series expansion

$$f(z) = \sum_{k \geq -m} c_k (z - a)^k.$$

If this power series is identically zero then $f(a) = 0$. Otherwise we can assume that $c_{-m} \neq 0$. Then

$$f(a) = \begin{cases} \infty & \text{if } m > 0, \\ c_0 & \text{if } m = 0, \\ 0 & \text{if } m = 0. \end{cases}$$

Equivalently, we can write

$$f(z) = (z - a)^{-m} h(z)$$

where $h(z)$ is holomorphic and $h(a) \neq 0$. Thus $f(z)$ is meromorphic in a neighbourhood of a if and only if

$$1/f(z) = (z - a)^m / h(z)$$

is holomorphic. This shows that a meromorphic function is a holomorphic function $f\colon U \to \mathbb{P}^1$. Conversely, any holomorphic $f\colon U \to \mathbb{P}^1$ such that $f(U) \notin \{\infty\}$ is a meromorphic function on U.

If

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n, \qquad q(z) = b_0 + b_1 z + \ldots b_m z^m$$

are polynomials of degree n and m with no common factor there is a holomorphic map $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$ whose restriction to $\mathbb{C}$ is the rational function

$$p(z)/q(z)$$

and whose value at ∞ is

$$f(\infty) = \begin{cases} 0 & \text{if } m > n, \\ a_n/b_n & \text{if } m = n, \\ \infty & \text{if } m < n. \end{cases}$$

It follows that unless f is constant it must take value ∞ at least once.

Lemma 11.11. *Any nonconstant holomorphic map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is rational hence takes the value ∞ at least once.*

Proof. By definition if a meromorphic function has a pole at a, it is holomorphic on some neighbourhood of a. Since the restriction of f to $\mathbb{C}$ and map $\mathbb{C} \to \mathbb{P}^1$ defined by $z \mapsto f(1/z)$ are both meromorphic and $\mathbb{P}^1$ is compact, f can have only finitely many poles. Let us assume that they are at $a_1, \ldots, a_k$.

In the neighbourhood of a_j f has a Laurent expansion

$$f(z) = \sum_{n \geq -m_j} c_n^{(j)} (z - a_j)^n.$$

We consider

$$g(z) = \sum_{j=1}^{k} \sum_{n=-m_j}^{-1} c_n^{(j)} (z - a_j)^n.$$

It is a rational function and it extends to a holomorphic function $g\colon \mathbb{P}^1 \to \mathbb{P}^1$ with $g(\infty) = 0$. Map $f - g\colon \mathbb{P}^1 \to \mathbb{P}^1$ has no poles in $\mathbb{C}$ so

$$f(z) - g(z) = \sum_{n \geq 0} c_n z^n.$$

In the neighbourhood of ∞ with the local chart $w = 1/z$ the difference $f - g$ has Laurent expansion

$$f(w) - g(w) = \sum_{n \geq 0} c_n w^{-n}.$$

It is meromorphic if and only if there exists N such that $c_n = 0$ for $n > N$. It follows that $f(z) = g(z) + \sum_{n=0}^{N} c_n z^n$ is a sum of rational functions, hence it is rational. $\square$

12 02.03. Lecture 12. Weierstrass' elliptic function

Let $W \subset \mathbb{C}$ be open. A function $f\colon W \to \mathbb{C}$ is *holomorphic* if its derivative

$$f'(a) = \lim_{z\to a} \frac{f(z) - f(a)}{z - a}$$

exists at every $a \in W$. A function f is holomorphic on an open disc $\{z \in \mathbb{C} \,|\, |z - a| < r\}$ if and only if it can be expressed as a convergent power series

$$f(z) = \sum_{n\geq 0} c_n (z - a)^n, \; |z - a| < r$$

called the Taylor series of f about a. The coefficients are $c_n = \frac{1}{n!} f^{(n)}(a)$.

A *meromorphic* function on W is a function $f\colon W \to \mathbb{C} \cup \{\infty\}$ such that $f|_{W\setminus f^{-1}(\infty)}$ is holomorphic and f has a *pole* at every point $a \in f^{-1}(\infty)$. It means that near a

$$f(z) = \frac{g(z)}{(z - a)^m},$$

for some holomorphic function g such that $g(a) \neq 0$. Equivalently

$$f(z) = \sum_{n\geq -m} c_n (z - a)^n.$$

Then m is called the *order* or *multiplicity* of the pole. The coefficient c_{-1} is called the *residue* of the pole and is denoted by

$$\mathrm{res}\,\{f(z),\, a\}.$$

The residue at a of $\frac{f'(z)}{f(z)}$ is then $-m$. Similarly, if f has zero of order m in a then the residue of $\frac{f'(z)}{f(z)}$ at a is m.

Residues of meromorphic functions can be calculated by integrals over closed piecewise smooth contours γ, i.e. functions $\gamma\colon [0,1] \to \mathbb{C}$ such that $\gamma(0) = \gamma(1)$, $\gamma(t_1) \neq \gamma(t_2)$ if $t_1 \neq t_2$ and $\{t_1, t_2\} \neq \{0, 1\}$ and γ is smooth except finitely many points. If $\gamma\colon [c, d] \to W$ is smooth, the integral of f over γ is by definition

$$\int_\gamma f(z)dz = \int_c^d f(\gamma(t))\gamma'(t)dt.$$

Theorem 12.1 (Cauchy's theorem)**.** *Let γ be a closed piecewise smooth contour in $\mathbb{C}$ and let f be a function which is holomorphic inside and on γ. Then*

$$\int_\gamma f(z)dz = 0.$$

Theorem 12.2 (Cauchy's residue theorem)**.** *Let γ be a closed piecewise smooth contour in $\mathbb{C}$ and let φ be a meromorphic function inside and on γ with no poles on γ and poles $a_1, \ldots, a_r$ inside γ. Then*

$$\int_\gamma f(z)dz = \pm 2\pi i \sum_{j=1}^{r} res\,\{f(z),\, a_j\},$$

where the sign depends on the orientation of γ.

There is an useful construction of holomorphic functions

Theorem 12.3. *Let $(f_n\colon W \to \mathbb{C})_{n\geq 1}$ be a sequence of holomorphic functions on an open subset W of $\mathbb{C}$ converging uniformly to a function $f\colon W \to \mathbb{C}$. Then f is holomorphic on W and the derivatives f_n' converge uniformly to f' on W.*

Recall that a sequence converges uniformly if for any $\varepsilon > 0$ there exists N such that for any $n, m > N$, $|f_n(z) - f_m(z)| < \varepsilon$ for all $z \in W$.

A direct application is Weierstrass M-test

Theorem 12.4 (Weierstrass M-test)**.** *Let $(f_n\colon W \to \mathbb{C})_{n\geq 1}$ be a sequence of holomorphic functions on an open subset W of $\mathbb{C}$. Suppose there exist positive real numbers M_n such that*

$$\sum_{n\geq 1} M_n$$

converges and

$$|f_n(z)| \leq M_n \, \forall z \in W.$$

Then the series

$$\sum_{n\geq 1} f_n(z)$$

converges uniformly on W to a holomorphic function $f(z)$ such that

$$f'(z) = \sum_{n\geq 1} f_n'(z).$$

Now we are ready to define Weierstrass elliptic $\wp$ function. Let ω_1, ω_2 be complex number linearly independent over $\mathbb{R}$ (i.e. $\omega_1/\omega_2 \notin \mathbb{R}$). We define a lattice

$$\Lambda = \{n\omega_1 + m\omega_2 \,|\, (n,m) \in \mathbb{Z}^2\}$$

which is an additive subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z}^2$.

Proposition 12.5. *There is a meromorphic function $\wp(z)$ on $\mathbb{C}$ defined by*

$$\wp(z) = z^{-2} + \sum_{\omega\in\Lambda\setminus\{0\}} ((z-\omega)^{-2} - \omega^{-2}),$$

with derivative given by

$$\wp'(z) = \sum_{\omega\in\Lambda} -2(z-\omega)^{-3}.$$

Proof. We want to show that function $\wp$ is meromorphic at any point of $\mathbb{C}$. We will show that $\wp(z)$ is a sum of a homolomorphic function and finitely many meromorphic functions.

Note that any point lies in some disc $\{z \in \mathbb{C} \,|\, |z| \leq R\}$. Therefore it is enough to prove that, for any R there exists a finite subset $\Lambda_R \subset \Lambda$ such that $\sum_{\omega\in\Lambda\setminus\Lambda_R}((z-\omega)^{-2} - \omega^{-2})$ is holomorphic. Using Weierstrass M-test we will show that $\sum_{\omega\in\Lambda\setminus\Lambda_R}((z-\omega)^{-2} - \omega^{-2})$ converges uniformly on $\{z \in \mathbb{C} \,|\, |z| \leq R\}$.

We shall need to know that there exists δ such that

$$|x\omega_1 + y\omega_2| \geq \delta\sqrt{x^2+y^2}$$

for any $x, y \in \mathbb{R}$. We prove it below.

Let now

$$\Lambda_R = \{\omega \in \Lambda \,|\, |\omega| < 2R\}.$$

The above inequality implies that

$$\Lambda_R \subset \{n\omega_1 + m\omega_2 \,|\, n^2+m^2 \leq 4R^2\delta^{-2}\}$$

hence it is finite.

Let now $z \in \mathbb{C}$ be such that $|z| \leq R$ and let $\omega \in \Lambda \backslash \Lambda_R$, i.e. $|\omega| > 2R$. Then $|z| \leq \frac{1}{2}|\omega|$, hence $|z - 2\omega| < |z| + 2|\omega| \leq \frac{5}{2}|\omega|$ and $|z-\omega| > \frac{|\omega|}{2}$. It follows that

$$\begin{aligned} |(z-\omega)^{-2} - \omega^{-2}| = |\frac{z(z-2\omega)}{(z-\omega)^2\omega^2}| &\leq \frac{5R|\omega|/2}{|\omega|^4/4} \\ &= 10R/|\omega|^3 \leq 10R\delta^{-3}(n^2+m^2)^{-\frac{3}{2}}. \end{aligned}$$

The result follows from the convergence of the series

$$\sum_{(n,m)\neq(0,0)} (n^2+m^2)^{-\frac{3}{2}} = \sum_{k\geq 1} \sum_{\max(|n|,|m|)=k} (n^2+m^2)^{-\frac{3}{2}} \leq \sum_{k\geq 1} \frac{8}{k^2} \leq \infty.$$

The second estimate comes from the fact that there are exactly $8k$ pairs (n,m) such that $\max\{|n|,|m|\} = k$. For each of them we have $n^2+m^2 \geq k^2$ so $\frac{1}{n^2+m^2} \leq \frac{1}{k^2}$ and $(\frac{1}{n^2+m^2})^{\frac{3}{2}} \leq \frac{1}{k^3}$. □

Lemma 12.6. *There exists $\delta > 0$ such that*

$$|x\omega_1 + y\omega_2| \geq \delta\sqrt{x^2+y^2}$$

for all real numbers x, y.

Proof. Consider $f\colon [0,2\pi] \to \mathbb{R}$, $f(\theta) = |(\cos\theta)\omega_1 + (\sin\theta)\omega_2|$. Interval $[0,2\pi]$ is compact, hence f is bounded and attains its bounds. Moreover, $f(\theta) \neq 0$ as ω_1 and ω_2 are linearly independent over $\mathbb{R}$. It follows that $f(\theta) > \delta$, for some $\delta > 0$.

Since $|\lambda x\omega_1 + \lambda y\omega_2| = \lambda|x\omega_1 + y\omega_2|$, it follows that $|x\omega_1 + y\omega_2| = |(x,y)||x'\omega_1 + y'\omega_2| \geq \delta\sqrt{x^2+y^2}$ where (x', y') is the projection of point $(x,y) \neq (0,0)$ to the unit circle in $\mathbb{R}^2$ and $|(x,y)|$ is the distance $\sqrt{x^2+y^2}$ of the point $(x,y) \in \mathbb{R}^2$ to zero. □

Definition 12.7. $\wp(z)$ is called the Weierstrass elliptic $\wp$-function associated to the lattice Λ.

Lemma 12.8. *We have $\wp(-z) = \wp(z) = \wp(z+\xi)$, for all $z \in \mathbb{C}$ and $\xi \in \Lambda$.*

Proof. We have

$$\wp(-z) = z^2 + \sum_{\Lambda \backslash 0}((z+\omega)^{-2} - \omega^{-2})$$

and we can rearrange, by changing ω to $-\omega$ to get $\wp(z) = \wp(-z)$.

First we note that

$$\wp'(z+\xi) = -2\sum_{\omega\in\Lambda}(z+\xi-\omega)^3.$$

Since the series converges absolutely and since $\Lambda = -\Lambda_\xi$, we have $\wp'(z+\xi) = \wp'(z)$, i.e. $\wp(z+\xi) = \wp(z) + c(\xi)$. Putting $z = -\frac{1}{2}\xi$, we get

$$c(\xi) = \wp(\frac{1}{2}\xi) - \wp(-\frac{1}{2}\xi) = 0.$$

□

Definition 12.9. Functions f on $\mathbb{C}$ with the property that

$$f(z+\xi) = f(z),\, \forall\, z \in \mathbb{C},\, \xi \in \Lambda$$

or equivalently

$$f(z+\omega_1) = f(z) = f(z+\omega_2),\, \forall z \in \mathbb{C}$$

are *doubly periodic* with period lattice Λ.

We use

Theorem 12.10 (Liouville's theorem)**.** *Any bounded holomorphic function on $\mathbb{C}$ is constant.*

to conclude that

Lemma 12.11. *A doubly periodic holomorphic function f is constant.*

Proof. Since f is holomorphic it is continuous, hence it is bounded on any compact subset of $\mathbb{C}$ such as

$$P = \{s\omega_1 + t\omega_2 \,|\, s,t \in [0,1]\}.$$

Given any $z \in \mathbb{C}$ we can find $\xi \in \Lambda$ such that $z+\xi \in P$. It shows that $f(z) = f(z+\xi)$ is bounded. □

Lemma 12.12.

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where

$$g_2 = g_2(\Lambda) = 60 \sum_{\omega\in\Lambda\backslash\{0\}} \omega^{-4}, \qquad g_3 = g_3(\Lambda) = 140 \sum_{\omega\in\Lambda\backslash\{0\}} \omega^{-6}.$$

Proof. Function

$$\wp(z) - z^{-2} = \sum_{\omega \in \Lambda \setminus \{0\}} ((z-\omega)^{-2} - \omega^{-2})$$

vanishes at 0 and restricts to a holomorphic function in an open neighbourhood of zero. Moreover it is even ($\wp(z) - z^{-2} = \wp(-z) - (-z)^{-2}$) so its odd derivatives vanish at zero. Then

$$\wp(z) - z^{-2} = \lambda z^2 + \mu z^4 + z^6 h(z),$$

where $h(z)$ is holomorphic in a neighbourhood of zero. Then

$$\wp'(z) = -2z^{-3} + 2\lambda z + 4\mu z^3 + 6z^5 h(z) + z^6 h'(z).$$

The "meromorphic part" of $\wp(z)^3$ is

$$z^{-6} + 3\lambda z^{-2} + 3\mu$$

while the "meromorphic" part of $\wp'(z)^2$ is

$$4z^{-6} - 8\lambda z^{-2} - 16\mu.$$

Then

$$\wp'(z)^2 - 4\wp(z)^3 = -20\lambda z^{-2} - 28\mu + z\widetilde{h}(z).$$

It follows that

$$k(z) = \wp'(z)^2 - 4\wp(z)^3 + 20\lambda\wp(z) + 28\mu$$

is a holomorphic function in an open neighbourhood of zero which vanishes at zero. Since $\wp(z)$ and $\wp'(z)$ are holomorphic on $\mathbb{C} \setminus \Lambda$ and doubly periodic, $k(z)$ is holomorphic and doubly periodic. By Lemma 12.11 it is constant and equal to $0 = k(0)$.

By definition, we have $2\lambda = \wp^{(2)}(0)$ and $24\mu = \wp^{(4)}(0)$. □

Proposition 12.13. *The Weierstrass $\wp$-function*

$$\wp\colon \mathbb{C} \setminus \Lambda \to \mathbb{C}$$

is surjective. Also

$$\wp(z) = \wp(w)$$

if and only if $w \in \Lambda \pm z$.

Proof. Consider $c \in \mathbb{C}$ and $f(z) = \wp(z) - c$. Let

$$P(a) = \{a + s\omega_1 + t\omega_2 \mid s, t \in [0,1]\}$$

be a parallelogram such that the boundary of $P(a)$ does not pass through any zeroes or poles of f. If we denote by γ the boundary of $P(a)$ then Cauchy's residue theorem implies that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = Z - P$$

where Z is the number of zeroes and P the number of poles of f inside $P(a)$. Since f is periodic, integral along opposite sides of the parallelogram cancel each other, hence

$$Z - P = 0.$$

We know that $\wp z$ has exactly one pole of multiplicity two inside $P(a)$, hence so does f. It follows that

$$Z = 2,$$

i.e. there exists ω_0 such that $f(\omega_0) = 0$, i.e. $\wp(\omega_0) = c$.

Since $\wp(z)$ is even and doubly periodic we have

$$\wp(z) = \wp(\omega_0) = c \quad \forall\, z \in \Lambda \pm \omega_0.$$

There exists $\omega_1 \in \Lambda - \omega_0$ which lies in $P(a)$. Then ω_0 and ω_1 are the two zeroes of f inside $P(a)$. It remains to check that if $\omega_1 = \omega_0$ then $f(z)$ has a double zero at ω_0, i.e. $f'(\omega_0) = \wp'(\omega_0) = 0$ (we want to know that there is no other zero of f).

Equality $\omega_1 = \omega_0$ implies that $\Lambda + \omega_0 = \Lambda - \omega_0$. As $\wp'(z)$ is odd and doubly periodic it follows that

$$\wp'(\omega_0) = -\wp'(-\omega_0) = -\wp'(\omega_0).$$

□

03.03. Workshop III

1. Show that there is a homeomorphism given by

$$[s : t] \mapsto [st^3 : s^4 : t^4]$$

from $\mathbb{P}^1$ onto a quartic curve in $\mathbb{P}^2$. Why doesn't it contradict the degree-genus formula?

2. The identity theorem of complex analysis tells that if $f\colon U \to \mathbb{C}$ and $g\colon U \to \mathbb{C}$ are holomorphic functions from a connected open subset U of $\mathbb{C}$ to $\mathbb{C}$, and if $f(z) = g(z)$ for all z in some non-empty open subset W of U then $f(z) = g(z)$ for all $z \in U$. Use this to show that if $f, g\colon R \to S$ are holomorphic functions between Riemann surfaces with R connected and if $f(z) = g(z)$ for all z in some non-empty open subset W of R then $f(z) = g(z)$ for all $z \in R$.

3. The open mapping theorem of complex analysis says that if $f\colon U \to \mathbb{C}$ is a non-constant holomorphic function from a connected open subset U of $\mathbb{C}$ to $\mathbb{C}$ then $f(U)$ is open in $\mathbb{C}$. Use this to show that if $f\colon R \to S$ is a non-constant holomorphic map between Riemann surfaces and R is connected then $f(R)$ is an open subset of S.

4. Let C be a cubic curve defined as

$$C = \{[x : y : z] \mid y^2z = 4x^3 - g_2xz^2 - g_3z^3\}.$$

Show that C is non-singular if and only if $g_2^2 - 27g_3^2 \neq 0$.

Solutions to Workshop III

1. Let $C = \{[x : y : z] \,|\, x^4 = yz^3\}$ be a quartic curve in $\mathbb{P}^2$. Map $\varphi\colon \mathbb{P}^1 \to C$, $\varphi([s : t]) = [st^3 : s^4 : t^4]$ is well-defined as $(st^3)^4 = s^4 \cdot (t^3)^4$.

   Map φ is continuous since $\psi\colon \mathbb{C}^2 \to \mathbb{C}^3$, $\psi(s,t) = (st^3, s^4, t^4)$ is continuous. Indeed, it follows that $\Pi \circ \psi\colon \mathbb{C}^2 \setminus \{(0,0\} \to \mathbb{P}^2$ is continuous. As $\mathbb{P}^1$ has the quotient topology and diagram

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{P}^2 \\ \uparrow \Pi & & \uparrow \Pi \\ \mathbb{C}^2 \setminus \{(0,0)\} & \xrightarrow{\psi} & \mathbb{C}^3 \setminus \{(0,0,0)\} \end{array}$$

   commutes, it follows that φ considered as a map $\mathbb{P}^1 \to \mathbb{P}^2$ is continuous. Hence it is continuous as a map from $\mathbb{P}^1$ to any subset of $\mathbb{P}^2$ containing $\varphi(\mathbb{P}^1)$, in particular as a map $\mathbb{P}^1 \to C$.

   To show that φ is a bijection, we construct its inverse

$$\psi\colon C \to \mathbb{P}^1, \qquad \psi([x : y : z]) = \begin{cases} [\frac{x}{z} : 1] & \text{if } z \neq 0, \\ [1 : 0] & \text{if } z = 0. \end{cases}$$

   First, note that if $[x : y : z] \in C$ and $z = 0$ then $x = 0$ hence $[x : y : z] = [0 : 1 : 0]$. Then we have

$$\varphi \circ \psi([0 : 1 : 0]) = \varphi([1 : 0]) = [0 : 1 : 0].$$

   If $z \neq 0$ then equality $yz^3 = x^4$ implies that

$$\varphi \circ \psi([x : y : z]) = \varphi([\frac{x}{z} : 1]) = [\frac{x}{z} : \frac{x^4}{z^4} : 1] = [\frac{x}{z} : \frac{yz^3}{z^4} : 1] = [\frac{x}{z} : \frac{y}{z} : 1] = [x : y : z].$$

   On the other hand, we have

$$\psi \circ \varphi([1 : 0]) = \psi([0 : 1 : 0]) = [1 : 0], \quad \psi \circ \varphi([s : 1]) = \psi([s : s^4 : 1]) = [s : 1],$$

   hence $\psi \circ \varphi = \mathrm{Id}$ and $\varphi \circ \psi = \mathrm{Id}$.

   Instead of showing that φ^{-1} is continuous we prove that any continuous bijection from a compact space to a Hausdorff space is always a homeomorphism. First we show that for a continuous $f\colon X \to Y$ with X compact and Y Hausdorff the image $f(Z) \subset Y$ of any closed subset of X is closed. Indeed, let $y \in Y \setminus f(Z)$. By assumption, for any $z \in Z$ there exist open subsets U_z, V_z in Y such that $f(z) \in U_z$, $y \in V_z$ and $U_z \cap V_z = \emptyset$. We thus have an open cover $f(Z) \subset \bigcup_{z \in Z} U_z$. Set Z is a closed subset of a compact space, hence it is compact and so is its image under continuous map. It follows that there exist $z_1, \ldots, z_n$ such hat $f(Z) \subset \bigcup_{i=1}^n U_{z_i}$. Then $V := \bigcap_{i=1}^n V_{z_i}$ is open such that $y \in V$ and $Y \cap f(Z) = \emptyset$, i.e. $f(Z)$ is closed.

   Assume now that $f\colon X \to Y$ as above is a continuous bijection. To prove that f^{-1} is continuous we need to check that f is open, i.e. the image of an open set $U \subset X$

is open. It follows immediately from the above argument as $f(U) = Y \setminus f(X \setminus U)$ and $f(X \setminus U) \subset Y$ is closed.

If $P(x, y, z) = x^4 - yz^3$ then partial derivatives $\frac{\partial P}{\partial x} = 4x^3$, $\frac{\partial P}{\partial y} = -z^3$, $\frac{\partial P}{\partial z} = -3yz^2$ vanish at $[0 : 1 : 0]$, hence C is singular and the degree-genus formula does not hold.

2. Let $\mathcal{W} \subset R$ be the union of all open subsets in R on which $f(z) = g(z)$. By definition $\mathcal{W}$ is open and by assumption it is non-empty. To show that $\mathcal{W} = R$ we need to show that $\mathcal{W}$ is closed.

   Let $z \in R \setminus \mathcal{W}$. If $f(z) \neq g(z)$ then, because S is Hausdorff, there exist open neighbourhoods W_f of $f(z)$ and W_g of $g(z)$ such that $W_f \cap W_g = \emptyset$. For any $z' \in f^{-1}(W_f) \cap g^{-1}(W_g)$ we have $f(z') \neq g(z')$ hence $f^{-1}(W_f) \cap g^{-1}(W_g)$ is an open neighbourhood of z in $R \setminus \mathcal{W}$.

   If $f(z) = g(z)$ but $z \notin \mathcal{W}$ let $\varphi_\alpha \colon U_\alpha \to V_\alpha$ be a chart containing z and $\psi_\beta \colon W_\beta \to X_\beta$ a chart containing $f(z)$. Let $U = f^{-1}(W_\beta) \cap g^{-1}(W_\beta)$ be an open neighbourhood of z. If $U \cap \mathcal{W} \neq \emptyset$ then holomorphic maps $\psi_\beta \circ f \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U)}$ and $\psi_\beta \circ g \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U)}$ agree on an open subset $\varphi_\alpha(U \cap \mathcal{W})$ of $\varphi_\alpha(U)$, hence by the identity theorem, they agree on $\varphi_\alpha(U)$. Then, by definition, $z \in \mathcal{W}$ which contradicts our choice. Thus $U \cap \mathcal{W} = \emptyset$, i.e. U is an open neighbourhood of z in $R \setminus \mathcal{W}$. It proves that $\mathcal{W}$ is closed.

   As the only open and closed subset of a connected space is the space itself, we conclude that $\mathcal{W} = R$, i.e. $f(z) = g(z)$ for all $z \in R$.

3. Let s be a point in $f(R)$. We need to show that there exists an open $W \subset S$ such that $s \in W$ and $W \subset f(R)$. Let $r \in R$ be such that $f(r) = s$ and let $\varphi_\alpha \colon U_\alpha \to V_\alpha$, $\psi_\beta \colon W_\beta \to X_\beta$ be charts on R, respectively on S, such that $r \in U_\alpha$ and $s \in W_\beta$. It follows from the previous exercise that f is non-constant on U_α and any of its open subsets (if it was constant on an open subset of R it would be constant on the whole R). Thus, map $\psi_\beta \circ f \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap f^{-1}W_\beta) \to W_\beta$ is a non-constant holomorphic function. It follows that there exists an open subset $W' \subset \psi_\beta \circ f \circ \varphi_\alpha^{-1}(\varphi_\alpha(U_\alpha \cap f^{-1}(W_\beta)))$ which contains $\psi_\beta(s)$. Then $\psi_\beta^{-1}(W')$ is an open subset of $f \circ \varphi_\alpha^{-1}(\varphi_\alpha(U_\alpha \cap f^{-1}(W_\beta)))$, in particular of $f(R)$ which contains s. As s was arbitrary, it proves that $f(R)$ is open.

4. Let us calculate singular points of C. For $P(x, y, z) = y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3$ we have

$$\frac{\partial P}{\partial x} = -12x^2 + g_2 z^2, \qquad \frac{\partial P}{\partial y} = 2yz, \qquad \frac{\partial P}{\partial z} = y^2 + 2g_2 xz + 3g_3 z^2,$$

   and $[a : b : c]$ is a singular point of C if and only if

$$\begin{cases} -12a^2 + g_2 c^2 = 0, \\ 2bc = 0, \\ b^2 + 2g_2 ac + 3g_3 c^2 = 0. \end{cases}$$

It follows from the second equation that either $b = 0$ or $c = 0$. If $c = 0$ then the first equation implies that $a = 0$ and the third gives $b = 0$, i.e. $[a : b : c]$ is not a point of $\mathbb{P}^2$.

If $b = 0$ then the third equation reads $c(2g_2a + 3g_3c) = 0$. As we have already seen $c = 0$ does not give a point in $\mathbb{P}^2$, hence we have $2g_2a + 3g_3c = 0$. If $g_2 = 0$ then vanishing of g_3 would imply that $[0 : 0 : 1]$ is a singular point of C. In other words, if $g_2 = 0$ and C is non-singular then $g_2^3 - 27g_3^2 \neq 0$.

If $g_2 \neq 0$ we have $a = -\frac{3g_3}{2g_2}c$. Then the first equation gives

$$-12\frac{9g_3^2}{4g_2^2}c^2 + g_2c^2 = \frac{1}{g_2^2}(-27g_3^2 + g_2^3)c^2 = 0.$$

If $g_2^3 - 27g_3^2 \neq 0$ the above equality implies that $c = 0$, i.e. $a = b = c = 0$ and the curve C is non-singular. If, on the other hand, $g_2^3 - 27g_3^2 = 0$ then $[\frac{-3g_3}{2g_2} : 0 : 1]$ is a singular point of C.

13 06.03. Lecture 13. Elliptic curves as complex tori

Recall that to a lattice $\Lambda \subset \mathbb{C}$ we assigned Weierstrass $\wp$-function

$$\wp(z) = z^{-2} + \sum_{\omega \in \Lambda \setminus \{0\}} ((z-\omega)^{-2} - \omega^{-2})$$

and proved that

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

for $g_2 = g_2(\Lambda)$, $g_3 = g_3(\Lambda)$ as in Lemma 12.12.

Definition 13.1. Let C_Λ be the projective curve in $\mathbb{P}^2$ defined by the polynomial

$$Q_\Lambda(x,y,z) = y^2z - 4x^3 + g_2xz^2 + g_3z^3$$

where $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ are defined as in Lemma 12.12.

Lemma 13.2. *The cubic curve C_Λ is non-singular.*

Proof. Let

$$\alpha = \wp(\frac{1}{2}\omega_1),\ \beta = \wp(\frac{1}{2}\omega_2),\ \gamma = \wp(\frac{1}{2}(\omega_1+\omega_2)).$$

It follows from Proposition 12.13 that α, β and γ are distinct. We will show that

$$Q_\Lambda(x,y,z) = y^2z - 4(x-\alpha z)(x-\beta z)(x-\gamma z).$$

Smoothness will follow.

We need to check that α, β and γ are roots of the polynomial $4x^3 - g_2x - g_3$. By Lemma 12.12 it suffices to check that $\wp'(\frac{1}{2}\omega_1)$, $\wp'(\frac{1}{2}\omega_2)$ and $\wp'(\frac{1}{2}(\omega_1+\omega_2))$ are zero.

We know that $\wp'(z)$ is an odd doubly periodic function with periods ω_1 and ω_2. Then

$$\wp'(\frac{1}{2}\omega_1) = -\wp'(-\frac{1}{2}\omega_1) = -\wp'(\frac{1}{2}\omega_1).$$

Similarly for $\frac{1}{2}\omega_2$ and $\frac{1}{2}(\omega_1+\omega_2)$. ☐

Λ is an additive subgroup of $\mathbb{C}$ hence we can form the quotient

$$\mathbb{C}/\Lambda = \{\Lambda + z \mid z \in \mathbb{C}\}.$$

There is a canonical surjective map $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$, $\pi(z) = z+\Lambda$, which induces the quotient topology on $\mathbb{C}/\Lambda$.

Moreover, map π is open, i.e. the image of any open set is open. Indeed, if $U \subset \mathbb{C}$ is open then $\pi^{-1}\pi(U) = \bigcup_{\omega\in\Lambda} \omega + U$ is a union of open sets, hence it is open. By definition, $\pi(U) \subset \mathbb{C}/\Lambda$ is open.

The restriction of π to parallelogram

$$P = \{s\omega_1 + t\omega_2 \mid s,t \in [0,1]\}$$

is surjective, hence $\mathbb{C}/\Lambda$ is an image of a compact set. Thus it is itself compact. Topologically $\mathbb{C}/\Lambda$ is a torus obtained by gluing opposite edges of P.

Lemma 13.3. *There is a well-defined map* $u\colon \mathbb{C}/\Lambda \to C_\Lambda$*,*

$$u(\Lambda + z) = \begin{cases} [\wp(z) : \wp'(z) : 1] & \text{if } z \notin \Lambda, \\ [0 : 1 : 0] & \text{otherwise.} \end{cases}$$

Proof. It follows from Lemma 12.8 that $\wp(z) = \wp(z + \Lambda)$ and $\wp'(z) = \wp'(z + \Lambda)$, thus $\wp$ and $\wp'$ are well-defined on $\mathbb{C}/\Lambda$. Lemma 12.12 implies that point $[\wp(z) : \wp'(z) : 1]$ lies on C_Λ for any $z \in Z$. □

Proposition 13.4. *Mapping* $u\colon \mathbb{C}/\Lambda \to C_\Lambda$ *is a homeomomorphism.*

Proof. We check that u is bijection, and that u and u^{-1} are continuous.

First we prove that u is injective. Suppose that $u(z) = u(w)$. Then $\wp(z) = \wp(w)$, hence by Proposition 12.13 $z \in \Lambda \pm w$. If $z \in \Lambda - w$ then $\wp'(z) = -\wp'(w)$. Because $u(z) = u(w)$ we know that $\wp'(z) = \wp'(w)$, i.e. $\wp'(z) = 0 = \wp'(w)$. As in the proof of Lemma 13.2 it follows that $\wp(w) = \wp(\frac{1}{2}\omega_1)$ or $\wp(w) = \wp(\frac{1}{2}\omega_2)$ or $\wp(w) = \wp(\frac{1}{2}(\omega_1 + \omega_2))$. Proposition 12.13 again implies that $w \in \frac{1}{2}\Lambda$, hence $\Lambda + w = \Lambda - w$, i.e. $z \in \Lambda + w$.

Next we check that u is surjective. Let $[a : b : c]$ be a point in C_Λ. We can assume that $[a : b : c] \neq [0 : 1 : 0]$. It follows from the equation that if $c = 0$ then $a = 0$, so we can assume that $c = 1$. By Proposition 12.13 there is z such that $\wp(z) = a$. Then by Lemma 12.12 and the definition of C_Λ, we have

$$\wp'(z)^2 = b^2$$

so $\wp(z) = \pm b$. Since $\wp$ is even and $\wp'$ is odd it follows that either $u(z)$ or $u(-z)$ is equal to $[a : b : c]$.

Since $\mathbb{C}/\Lambda$ has quotient topology, to prove that u is continuous it is enough to check that $\mathbb{C} \to C_\Lambda$ is continuous. As $\wp(z)$ and $\wp'(z)$ are holomorphic on $\mathbb{C} \setminus \Lambda$, u can possibly be discontinuous on Λ.

We can write $\wp(z) = g(z)/z^2$ and $\wp'(z) = h(z)/z^3$ for functions g and h holomorphic in the neighbourhood of 0 and such that $g(z) \neq 0$, $h(z) \neq 0$. Then in a neighbourhood of 0 $u(z) = [zg(z) : h(z) : z^3]$ which tends to $[0 : 1 : 0]$. Thus u is continuous at 0, hence at every $\omega \in \Lambda$.

We have shown that $u\colon \mathbb{C}/\Lambda \to C_\Lambda$ is a continuous bijection. Since $\mathbb{C}/\Lambda$ is compact and C_Λ is Hausdorff it follows that u is a homeomorphism. □

Next, we want to show that u is holomorphic. First, we need to check that $\mathbb{C}/\Lambda$ is a Riemann surface.

$\mathbb{C}/\Lambda$ has the quotient topology induced by the map $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$. It follows from Lemma 12.6 that there exists $\delta > 0$ such that

$$|n\omega_1 + m\omega_2| \geq \delta.$$

Hence, if $a \in \mathbb{C}$, the restriction on π to the open disc

$$U_a = \{z \in \mathbb{C} \mid |z - a| < \frac{1}{4}\delta\}$$

is a homeomorphism onto $\pi(U_a)$. Indeed, the distance between z and $z + \Lambda$ is at least δ.

Now, we check that the atlas is holomorphic.

If $\pi(U_a) \cap \pi(U_b) \neq \emptyset$, there is unique $\lambda = n\omega_1 + m\omega_2$ such that

$$|a - b + \lambda| < \frac{1}{2}\delta.$$

Then

$$(\pi|_{U_b})^{-1} \circ \pi|_{U_a \cap \pi^{-1}\pi(U_b)} \colon U_a \cap \pi^{-1}\pi(U_b) \to U_b$$

is given by the translation by λ. Therefore the charts

$$\varphi_a = (\pi|_{U_a})^{-1} \colon \pi(U_a) \to U_a$$

form a holomorphic atlas.

$\mathbb{C}/\Lambda$ is Hausdorff because there is a homeomorphism $u\colon \mathbb{C}/\Lambda \to \mathbb{C}_\Lambda$ and $\mathbb{C}_\Lambda$ is Hausdorff.

If S is a Riemann surface then, by the property of quotient topology, map $f : \mathbb{C}/\Lambda \to S$ is holomorphic if and only if $\pi \circ f : \mathbb{C} \to S$ is holomorphic.

Proposition 13.5. *The homeomorphism*

$$u(\Lambda + z) = \begin{cases} [\wp(z) : \wp'(z) : 1] & \text{if } z \notin \Lambda, \\ [0 : 1 : 0] & \text{otherwise.} \end{cases}$$

is holomorphic and so is its inverse.

Proof. Let us fix some holomorphic charts $\varphi_\alpha \colon U_\alpha \to V_\alpha$ on $\mathbb{C}/\Lambda$ and $\psi_\beta \colon W_\beta \to Y_\beta$ on C_Λ such that $w + \Lambda \in U_\alpha$ and $u(w + \Lambda) \in W_\beta$ for some $w \in \mathbb{C}$. We need to check that

$$\psi_\beta \circ u \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap u^{-1}(W_\beta)) \to Y_\beta$$

is holomorphic.

As we have just seen we might take φ_α to be the inverse of the projection map $\pi \colon \mathbb{C} \to \mathbb{C}/\Lambda$ restricted to some sufficiently small disc $V_\alpha \subset \mathbb{C}$. Then $\varphi_\alpha^{-1} = \pi|_{V_\alpha}$.

If $w \notin \Lambda$ then $u(\Lambda + w) = [\wp(z), \wp(z'), 1]$ and as in the proof of the existence of a holomorphic atlas on a complex curve (see Proposition 11.3), we might assume that ψ_β is given by

$$[x : y : z] \mapsto x/z \qquad \text{or} \qquad [x : y : z] \mapsto y/z.$$

Since both $\wp(z)$ and $\wp'(z)$ are holomorphic near w, so is $\psi_\beta \circ u \circ \varphi_\alpha^{-1}$.

If $w \in \Lambda$ then $u(\Lambda + w) = [0 : 1 : 0]$ and as in the proof of Proposition 11.3 we can assume that $\psi_\beta([x : y : z]) = x/y$ because $\frac{\partial Q_\Lambda}{\partial z}(0, 1, 0) \neq 0$, where as before $Q_\Lambda(x, y, z) = y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3$. We thus need to check that

$$\psi_\beta \circ u \circ \varphi_\alpha^{-1}(z) = \begin{cases} \wp(z)/\wp'(z) & \text{for } z \notin \Lambda, \\ 0 & \text{for } z \in \Lambda \end{cases}$$

is holomorphic at 0, hence at any point in Λ.

We have $\wp(z) = \frac{g}{z^2}$ and $\wp'(z) = \frac{h}{z^3}$, where g and h are holomorphic and not equal to zero at zero. Thus, $\wp(z)/\wp'(z) = \frac{zg(z)}{h(z)}$ is holomorphic at zero.

By the inverse function theorem, if $f : U \to V$ is a holomorphic bijection between open subsets of $\mathbb{C}$ then so is f^{-1}. It follows that u^{-1} is holomorphic. □

14 09.03. Lecture 14. Singular curves and their resolution of singularities

We shall consider a singular projective curve $C \subset \mathbb{P}^2$ and find a compact Riemann surface $\widetilde{C}$ with a surjective continuous map $\pi\colon \widetilde{C} \to C$ such that $\pi^{-1}(\mathrm{Sing}(C))$ is a finite set and π is a homeomorphism outside $\mathrm{Sing}(C)$ and its preimage. We call such $\pi\colon \widetilde{C} \to C$ *a resolution of singularities* for C.

Definition 14.1. An ordered pair (f,g) of meromorphic functions defined on an open neighbourhood of $0 \in \mathbb{C}$ is a *pair* if f is non-constant on any neighbourhood of 0 and the mapping

$$t \mapsto (f(t), g(t))$$

is one-to-one near 0.

A *parameter change* is a holomorphic function ρ defined on an open neighbourhood of $0 \in \mathbb{C}$ such that $\rho(0) = 0$, $\rho'(0) \neq 0$. Two pairs (f, g) and $(\widetilde{f}, \widetilde{g})$ are *equivalent*, $(f, g) \sim (\widetilde{f}, \widetilde{g})$ if there is a parameter change ρ such that $\widetilde{f} = f \circ \rho$, $\widetilde{g} = g \circ \rho$. By the inverse function theorem it is an equivalence relation on the set of pairs. The equivalence class of a pair (f, g) is called a *meromorphic element*; we shall use notation $\langle f, g\rangle$ or $\langle f(t), g(t)\rangle$ for it.

We denote by $\mathcal{M}$ the set of meromorphic elements. We shall make $\mathcal{M}$ into a Riemann surface with infinitely many connected components which will provide resolutions for projective curves.

We define open sets. Let (f, g) be a pair and $r > 0$ sufficiently small such that f and g are defined on the disc $D(0, r)$ and $t \mapsto (f(t), g(t))$ is one-to-one on $D(0, r)$. Then $(f(t + t_0), g(t + t_0))$ is a pair for any $t_0 \in D(0, r)$ and we define

$$U(f, g, r) = \{\langle f(t + t_0), g(t + t_0)\rangle \mid t_0 \in D(0, r)\} \subset \mathcal{M}.$$

Lemma 14.2. *There is a topology on $\mathcal{M}$ such that any open set is a union of $U(f_i, g_i, r_i)$.*

Proof. It suffices to show that any finite intersection of $U(f_i, g_i, r_i)$ is again open in the above sense, i.e. that for any

$$\langle f, g\rangle \in U(f_1, g_1, r_1) \cap U(f_2, g_2, r_2),$$

there exists $r > 0$ such that

$$U(f, g, r) \subset U(f_1, g_1, r_1) \cap U(f_2, g_2, r_2).$$

By definition

$$\langle f, g\rangle = \langle f_1(t_1 + t), g_1(t_1 + t)\rangle = \langle f_2(t_2 + t), g_2(t_2 + t)\rangle,$$

i.e. there exist ρ_1, ρ_2 such that

$$f_1(t_1 + \rho_1(t)) = f(t) = f_2(t_2 + \rho_2(t)), \qquad g_1(t_1 + \rho_1(t)) = g(t) = g_2(t_2 + \rho_2(t))$$

for all t is some open disc $D(0,s)$. There exists a smaller disc $D(0,r) \subset D(0,s)$ such that ρ_1 and ρ_2 are holomorphic on $D(0,r)$, ρ_1', ρ_2' do not vanish and

$$\rho_1(D(0,r)) \subset D(0,r_1 - |t_1|), \qquad \rho_2(D(0,r)) \subset D(0,r_2 - |t_2|).$$

For $t_0 \in D(0,r)$ function $\sigma(t) = \rho_1(t_0+t) - \rho_1(t_0)$ is a parameter change (it vanishes at zero and its differential does not), so

$$\langle f(t_0+t), g(t_0+t)\rangle = \langle f_1(t_1+\rho_1(t_0+t)), g_1(t_1+\rho_1(t_0+t))\rangle =$$
$$\langle f_1(t_1+\rho_1(t_0)+\sigma(t)), g_1(t_1+\rho_1(t_0)+\sigma(t))\rangle = \langle f_1(t_1+\rho_1(t_0)+t), g_1(t_1+\rho_1(t_0)+t)\rangle$$

(the last equality follows from the equivalence relation). Since $|t_1+\rho_1(t_0)| < r_1$, we have $\langle f(t_0+t), g(t_0+t)\rangle \in U(f_1,g_1,r_1)$. As t_0 was arbitrary,

$$U(f,g,r) \subset U(f_1,g_1,r_1).$$

Similarly one can show the other inclusion $U(f,g,r) \subset U(f_1,g_1,r_1)$. □

We need to define an atlas.

Definition 14.3. Let $\mathcal{A}$ be the set of all ordered triples (f,g,r) where (f,g) is a pair and $r > 0$ is such that f and g are defined on $D(0,r)$ and the mapping $t \mapsto (f(t),g(t))$ is one-to-one on $D(0,r)$.

Lemma 14.4. *If $(f,g,r) \in \mathcal{A}$ then the map $D(0,r) \to U(f,g,r)$, $t_0 \mapsto \langle f(t_0+t), g(t_0+t)\rangle$ is a homeomorphism.*

Proof. The map is surjective by definition of $U(f,g,r)$. If $\langle f(t_0+t), g(t_0+t)\rangle = \langle f(t_1+t), g(t_1+t)\rangle$ then $(f(t_0),g(t_0)) = (f(t_1),g(t_1))$ and $t_0 = t_1$. It shows that the map is injective. From the definition of the topology on $\mathcal{M}$ it follows that the map is a homeomorphism. □

Definition 14.5. If $\alpha = (f,g,r) \in \mathcal{A}$ let $U_\alpha = U(f,g,r)$, $V_\alpha = D(0,r)$ and $\varphi_\alpha \colon U_\alpha \to V_\alpha$ be the inverse of the homeomorphism of Lemma 14.4.

Proposition 14.6. *$\mathcal{M}$ is a Riemann surface with the holomorphic atlas $\Phi = \{\varphi_\alpha \colon U_\alpha \to V_\alpha \,|\, \alpha \in A\}$.*

Proof. Direct calculation that the composition is holomorphic. The proof that $\mathcal{M}$ is Hausdorff will be given later. □

Functions $\psi, \chi \colon \mathcal{M} \to \mathbb{C} \cup \{\infty\}$,

$$\psi(\langle f,g\rangle) = f(0), \qquad \chi(\langle f,g\rangle) = g(0)$$

are meromorphic because their compositions with inverses of φ_α are given by

$$\psi \circ \varphi_\alpha^{-1}(t_0) = f(t_0), \qquad \chi \circ f_\alpha^{-1}(t_0) = g(t_0).$$

Definition 14.7. If f is a meromorphic function defined on a neighbourhood of a point $a \in \mathbb{C} \cup \{\infty\}$ then the *Riemann surface of* f is the connected component of $\mathcal{M}$ containing $\langle a+t, f(a+t)\rangle$ if $a \in \mathbb{C}$, or $\langle t^{-1}, f(t^{-1})\rangle$, if $a = \infty$. This meromorphic element is called *germ* of f at a and denoted by $[f]_a$.

Let (f, g) be a pair. If f is holomorphic at 0 then $f(t) = \sum_{n\geq 0} c_n t^n$ and we have the multiplicity m of $f(t) - c_0$ at 0, i.e. minimal $m > 0$ such that $c_m \neq 0$. Then $f(t) = c_0 + t^m h(t)$ and $h(0) = c_m \neq 0$. We can find m'th holomorphic root $k(t)$ of $h(t)$ and then

$$\rho(t) = tk(t)$$

is such that $f(t) = c_0 + \rho(t)^m$. Then

$$\langle f, g\rangle = \langle c_0 + t^m, g \circ \rho^{-1}(t)\rangle.$$

If f has a pole at 0, then $f(t) = t^{-m}h(t)$ where $h(t)$ is holomorphic near 0 and $h(0) \neq 0$. We can find holomorphic m'th root of $1/h(t)$ and multiply it by t to get a parameter change ρ such that $f(t) = \rho(t)^{-m}$. Then

$$\langle f, g\rangle = \langle t^{-m}, g \circ \rho^{-1}(t)\rangle.$$

It shows that any element of $\mathcal{M}$ is of the form $\langle a + t^m, g\rangle$ or $\langle t^{-m}, g\rangle$. If $m = 1$ it is a germ of a holomorphic function. If $m > 1$ it is called a *branch point* of the component of $\mathcal{M}$ to which it belongs.

Definition 14.8. Let $P \in \mathbb{C}[x, y, z]$ be a non-constant irreducible homogeneous polynomial of degree d not divisible by z. The Riemann surface S_P of P is the open subset of $\mathcal{M}$ consisting of all those elements $\langle f, g\rangle \in \mathcal{M}$ satisfying

$$P(f(t), g(t), 1) = 0$$

for all t in some neighbourhood of 0. If $C = \{[x : y : z] \mid P(x, y, z) = 0\} \subset \mathbb{P}^2$ we write $\widetilde{C}$ for S_P and define $\pi \colon \widetilde{C} \to C$ by

$$\pi(\langle f, g\rangle) = [f(0) : g(0) : 1].$$

if f and g are holomorphic near 0, and otherwise

$$\pi(\langle f, g\rangle) = [\widetilde{f}(0) : \widetilde{g}(0) : 0]$$

where $\widetilde{f}(t) = t^n f(t)$, $\widetilde{g}(t) = t^n g(t)$ and n is the maximum of the multiplicites of the pole of f and g at 0.

Theorem 14.9. *$\widetilde{C}$ is a compact Riemann surface, map $\pi \colon \widetilde{C} \to C$ is continuous and surjective. If C is non-singular then π is a holomorphic bijection, and in general $\pi^{-1}(Sing(C))$ is finite and*

$$\pi \colon \widetilde{C} \setminus \pi^{-1}(Sing(C)) \to C \setminus Sing(C)$$

is a holomorphic bijection.

We sketch the proofs of some Lemmas needed in the proof.

Lemma 14.10. *π is continuous and its restriction to $\widetilde{C} \setminus \pi^{-1}(Sing(C))$ is holomorphic.*

Proof. π is continuous at any $\langle f_0, g_0\rangle$ with f_0 and g_0 holomorphic at 0 because it is the composition of the continuous projection $\Pi\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ with map $\widetilde{C} \to \mathbb{C}^3 \setminus \{0\}$,

$$\langle f, g\rangle \mapsto (\psi(\langle f, g\rangle), \chi(\langle f, g\rangle), 1).$$

If f_0 or g_0 has a pole at 0 then, if the multiplicity of the pole of f_0 is greater than or equal the multiplicity of the pole of g_0, π is the composition of Π with

$$\langle f, g\rangle \mapsto (1, \frac{\chi(\langle f, g\rangle)}{\psi\langle f, g\rangle}, \frac{1}{\psi\langle f, g\rangle}),$$

hence it is continuous, as both maps are holomorphic.

To show that the map is holomorphic we consider any chart $\varphi_\alpha\colon U_\alpha \to V_\alpha$ on $\mathcal{M}$ and ψ_β on $C \setminus \mathrm{Sing}(C)$. We know φ_α^{-1}; map ψ_β is a quotient of homogeneous coordinates which takes value in $\mathbb{C}$. The composition $\psi_\beta \circ \pi \circ \varphi_\alpha^{-1}$ is then a quotient of f and g and 1, i.e. it is a meromorphic function which takes values in $\mathbb{C}$ (because ψ_β takes values in $\mathbb{C}$). Any such function is holomorphic which finishes the proof. □

Lemma 14.11. *The restriction of π to $\widetilde{C} \setminus \pi^{-1}(Sing(C))$ is a bijection.*

Proof. Let $[a : b : c]$ be a non-singular point of C with $c \neq 0$. Then either $\frac{\partial P}{\partial x}$ or $\frac{\partial P}{\partial y}$ does not vanish at $[a : b : c]$. (If they both vanish, Euler's relation implies that $\frac{\partial P}{\partial z}$ is zero and the point is singular). We can assume that $\frac{\partial P}{\partial y} \neq 0$. The implicit function theorem implies existence of open neighbourhoods U and V of a and b in $\mathbb{C}$ and a holomorphic $h\colon U \to V$ such that $P(x, y, 1) = 0$ with $x \in U$ and $y \in V$ if and only if $y = h(x)$.

If $\pi(\langle f, g\rangle) = [a : b : 1]$ then $g = h(f(t))$ (it follows from the definition of π that f and g are holomorphic at 0). Since (f, g) is a pair and f is holomorphic at zero, $\rho(t) = f(t) - a$ is a parameter change, hence

$$\langle f, g\rangle = \langle a + \rho(t), h(a + \rho(t))\rangle = \langle a + t, h(a + t)\rangle$$

so $[a : b : c]$ has precisely one preimage.

For $c = 0$ we consider polynomial $P(x, 1, z)$ or $P(1, y, z)$ for the implicit function theorem and proceed as above. □

Lemma 14.12. *$\pi\colon \widetilde{C} \to C$ is surjective.*

Proof. We need to prove that the preimage of a singular point of C is non-empty. We can assume that $[0 : 1 : 0] \notin C$, so $\frac{\partial P}{\partial y}$ is not identically zero. Then Bezout's theorem implies that there are finitely many point on C such that $\frac{\partial P}{\partial y}$ vanishes at them.

Let $[a : b : c] \in C$ be singular. First, assume $c = 1$. Then there exists $\varepsilon > 0$ such that if $0 < |a - x| < \varepsilon$ there is no $y \in \mathbb{C}$ such that $[x : y : 1] \in C$ and $\frac{\partial P}{\partial y}(x, y, 1) = 0$.

Recall that we have a branched cover $\varphi\colon C \to \mathbb{P}^1$, $\varphi([x : y : z]) = ([x : z])$. Let $D_+(a, \varepsilon)$ be an open disc with the line from a to $a + \varepsilon$ removed, similarly $D_-(a, \varepsilon)$. As

$D_\pm(a,\varepsilon)$ is simply connected, φ restricts to a homeomorphism of connected component of $\varphi^{-1}(D_\pm(a,\varepsilon))$ with $D_\pm(a,\varepsilon)$. We know that $\varphi\colon C \to \mathbb{P}^1$ is a branch cover of degree $d = \deg C$, so by inverting φ, we get $2d$ holomorphic homemorphisms on their images $f_j^\pm\colon D_\pm(a,\varepsilon) \to C$.

We have $P(x, f_j^\pm(x), 1) = 0$ and $f_j^\pm(x) \neq f_i^\pm(x)$ if $i \neq j$. Since C is compact, maps $f_j^\pm$ are bounded and as $x \to a$ the only possible value they can take is y s.t. $[a : y : 1] \in C$. Point $y = b$ satisfies this property and lies in the closure of $f_j^\pm D_\pm(0,\varepsilon)$. Shrinking the discs if necessary we can assume that b is the only such point, so $f(x) \to b$ as $x \to a$.

We have defined $2d$ functions, the cover $\varphi\colon C \to \mathbb{P}^1$ has d sheets. Renumbering, we can assume that f_j^+ agrees with f_j^- on the lower half of $D_+(a,\varepsilon) \cap D_-(a,\varepsilon)$. It agrees with $f_{\sigma(j)}^-$ on the upper half. We can further assume that there exists m such that

$$\sigma(i) = \begin{cases} i+1 & \text{for } i = 1, \ldots, m-1, \\ 1 & \text{for } i = m \end{cases}$$

Then we can glue functions $f_1^\pm, \ldots, f_m^\pm$ to a function

$$g\colon \{t \in \mathbb{C} \,|\, 0 < |t| < \varepsilon^{\frac{1}{m}}\} \to C, \quad g(t) = \begin{cases} f_j^+(a + t^m) & \text{if } (2j-2)\pi/m < \arg(t) < 2j\pi/m \\ f_j^-(a + t^m) & \text{if } (2j-1)\pi/m < \arg(t) < (2j+1)\pi/m. \end{cases}$$

Since g is bounded and has b as a limit when $t \to 0$, it can be extended to a holomorphic function $g\colon D(0, \varepsilon^{\frac{1}{m}}) \to C$. We have $P(a + t^m, g(t), 1) = 0$ and the mapping $t \mapsto (a + t^m, g(t))$ is injective since $f_j^\pm(x) \neq f_i^\pm(x)$. Thus, $\langle a + t^m, g(t)\rangle \in \mathcal{M}$ and we have $\pi(\langle a + t^m, g(t)\rangle = [a : b : c]$.

If $c = 0$ then $a = 1$ because $[0 : 1 : 0] \notin C$.There exists $\varepsilon > 0$ such that if $|x| \geq \frac{1}{\varepsilon}$ there is no y such that $[x : y : 1] \in C$ and $\frac{\partial P}{\partial y}(x, y, 1) = 0$. Then we take "cut" discs $D_\pm(\infty, \varepsilon)$ and glue to find an element $\langle t^{-m}, g(t)\rangle \in \mathcal{M}$ such that $\pi(\langle t^{-m}, g(t)\rangle) = [1 : b : 0]$. □

Remark 14.13. The permutation σ of the above proof can be decomposed into disjoint cycles $\sigma = \sigma_1 \ldots \sigma_l$, where σ_i is a cycle of length m_i. Since σ was a permutation of d discs, $m_1 + \ldots + m_l = d$. Gluing as above, we can construct holomorphic functions g_j such that

$$\psi^{-1}(a) = \{\langle a + t^{m_j}, g_j(t)\rangle \,|\, 1 \preceq j \preceq l\}$$

for the map $\psi\colon \widetilde{C} \xrightarrow{\pi} C \xrightarrow{\varphi} \mathbb{P}^1$. The multiplicity of 0 of $\psi - a$ at $\langle a + t^{m_j}, g_j(t)\rangle$ is m_j, hence ψ takes value $a \in \mathbb{P}^1$ precisely d times, counting with multiplicities.

Note that we have found $m_1 \ldots, m_l$ such that $m_1 + \ldots + m_l = d$ and constructed $g_1 \ldots, g_l$ such that $P(a + t^{m_i}, g_i(t), 1) = 0$. In other words, if x is close to a, for any m_j'th root $(x-a)^{\frac{1}{m_j}}$ of $(x - a)$, $y = g_j((x-a)^{\frac{1}{m_j}})$ is a solution to the equation $P(x, y, 1) = 0$.

Finally, if $|x - a|$ is very small positive, points $[x : y_j : 1]$ lie in different sheets of the covering so d numbers

$$y_j = g_j(e^{2\pi i s/m_j}(x-a)^{\frac{1}{m_j}})$$

for $1 \leq i \leq l$, $1 \leq s \leq m_j$ are distinct. As $\deg P = d$, it follows that

$$P(x, y, 1) = K \prod_{1 \leq j \leq l} \prod_{1 \leq s \leq m_j} (y - g_j(e^{2\pi i s/m_j}(x-a)^{\frac{1}{m_j}}))$$

for some $K \in \mathbb{C}^*$.

We omit proofs that $\widetilde{C}$ is compact and C and $\widetilde{C}$ are connected.

We sketch the proof of

Lemma 14.14. *$\mathcal{M}$ is Hausdorff.*

Proof. Let $\langle f, g\rangle$ and $\langle \widetilde{f}, \widetilde{g}\rangle$ be points in $\mathcal{M}$ which do not have disjoint open neighbourhoods. We want to show that they are equal. As we have seen, we can assume that $f(t) = a + t^m$ or $f(t) = t^{-m}$ and $\widetilde{f}(t) = b + t^n$ or $\widetilde{f}(t) = b^{-n}$.

For k large enough $U(f, g, 1/k)$ and $U(\widetilde{f}, \widetilde{g}, 1/k)$ are open neighbourhoods of $\langle f, g\rangle$ and $\langle \widetilde{f}, \widetilde{g}\rangle$ hence by assumption they have a common element, i.e.

$$\langle f(s_k + t), g(s_k + t)\rangle = \langle \widetilde{f}(t_k + t), \widetilde{g}(t_k + t)\rangle,$$

for $s_k, t_k \in \mathcal{D}(0, 1/k)$. In particular, $f(s_k) = \widetilde{f}(t_k)$ and $g(s_k) = \widetilde{g}(t_k)$. When k goes to infinity, s_k and t_k tend to zero, so $f(0) = \widetilde{f}(0)$. It follows that either $f(t) = a + t^m$ and $\widetilde{f}(t) = a + t^n$ or $f(t) = t^{-m}$ and $\widetilde{f}(t) = t^{-n}$. Moreover, $s_k^n = t_k^m$, so either both are zero or both are non-zero.

If for some k, $s_k = 0 = t_k$ then the two meromorphic elements are equal and the proof finishes. So we assume that $s_k \neq 0$ and $t_k \neq 0$ for all k.

Let σ_k be any any complex n'th root of s_k. Then

$$\left(\frac{(\sigma_k)^m}{t_k}\right)^n = \frac{(s_k)^m}{(t_k)^n} = 1$$

so $\frac{(\sigma_k)^m}{t_k}$ is a complex n'th root of 1 for each k. Since there are only n complex n'th roots of unity, there exists ω such that $\frac{(\sigma_k)^m}{t_k} = \omega$ for infinitely many k. Then

$$g(t^n) = \widetilde{g}(\omega^{-1}t^m)$$

for $t = t_k$ and these values of k. The uniqueness theorem of complex analysis implies that $g(t^n) = \widetilde{g}(\omega^{-1}t^m)$ in some neighbourhood of 0. We also have $f(t^n) = \widetilde{f}(\omega^{-1}t^m)$ as both are equal to $a + t^{mn}$ or t^{-mn}.

Mappings $t \mapsto (f(t), g(t))$ and $t \mapsto (\widetilde{f}(t), \widetilde{g}(t))$ are one-to-one in a neighbourhood of zero, hence so are $t \mapsto (f(t^n), g(t^n))$ and $t \mapsto (\widetilde{f}(\omega^{-1}t^m), \widetilde{g}(\omega^{-1}t^m))$. We have shown that these mappings are the same so $n = m$ and $g(t^n) = \widetilde{g}(\omega^{-1}t^n)$, so $g(t) = \widetilde{g}(\omega^{-1}t)$ for small t. Since also $f(t) = \widetilde{f}(\omega^{-1}t)$, we have

$$(f(t), g(t)) = (\widetilde{f}(\rho(t)), \widetilde{g}(\rho(t))),$$

for $\rho(t) = \omega^{-1}t$. □

09.03. Homework IV

1. Let Λ be a lattice in $\mathbb{C}$. Show that $g_2(\Lambda)^3 - 27g_3(\Lambda)^2 \neq 0$.

2. Find the first three terms of the Puiseux expansion about $[0 : 0 : 1]$ of the curve defined by $P(x, y, z) = x^3 + y^3 + 3xyz$.

3. Calculate genus of the curve defined by $P(x, y, z) = x^3 + y^3 + 3xyz$.

Solutions to Homework IV

1. We know that the cubic curve C_Λ defined by polynomial $y^2z - 4x^3 + g_2(\Lambda)xz^2 + g_3(\Lambda)z^3$ is non-singular. In particular, its affine part $D \subset \mathbb{P}^2$ defined by polynomial $P(x,y) = y^2 - 4x^3 + g_2(\Lambda)x + g_3(\Lambda)$ is non-singular.

   Assume that $(a,b) \in D$ is a singular point. Then $\frac{\partial P}{\partial y}(a,b) = 2b = 0$, i.e. $b = 0$. Thus, a is a root of the polynomial $P(x,0) = -4x^3 + g_2(\Lambda)x + g_3(\Lambda)$ and of $\frac{\partial P}{\partial x}(0,x) = -12x^2 + g_2(\Lambda) = \frac{\partial}{\partial x}P(x,0)$. Such $a \in \mathbb{C}$ exists if and only if $P(x,0)$ has multiple roots. In other words, if D is non-singular, then $P(x,0)$ has distinct roots $\lambda_1, \lambda_2, \lambda_3$, i.e. $P(x,0) = -4(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$. We have

$$\begin{aligned}&-4(x-\lambda_1)(x-\lambda_2)(x-\lambda_3) = -4(x^2 - (\lambda_1+\lambda_2)x + \lambda_1\lambda_2)(x-\lambda_3)\\ &= -4x^3 + 4(\lambda_1+\lambda_2+\lambda_3)x^2 - 4(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x + 4\lambda_1\lambda_2\lambda_3.\end{aligned}$$

   Hence

$$\begin{aligned}&\lambda_1+\lambda_2+\lambda_3 = 0,\\ &-4(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) = -4(\lambda_1\lambda_2 - (\lambda_1+\lambda_2)^2) = 4((\lambda_1+\lambda_2)^2 - \lambda_1\lambda_2) = g_2(\Lambda),\\ &4\lambda_1\lambda_2\lambda_3 = -4\lambda_1\lambda_2(\lambda_1+\lambda_2) = g_3(\Lambda).\end{aligned}$$

   Polynomial $P(x,0)$ has distinct roots if and only if

$$\begin{aligned}&0 \neq (\lambda_1-\lambda_2)^2(\lambda_2-\lambda_3)^2(\lambda_1-\lambda_3)^2 = (\lambda_1-\lambda_2)^2(\lambda_1+2\lambda_2)^2(2\lambda_1+\lambda_2)^2\\ &= (\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2)^2(4\lambda_1^4 + 4\lambda_2^4 + 20\lambda_1^3\lambda_2 + 33\lambda_1^2\lambda_2^2 + 20\lambda_1\lambda_2^3)\\ &= 4\lambda_1^6 + 4\lambda_2^6 + 12\lambda_1^5\lambda_2 + 12\lambda_1\lambda_2^5 - 3\lambda_1^4\lambda_2^2 - 3\lambda_1^2\lambda_2^4 - 26\lambda_1^3\lambda_2^3.\end{aligned}$$

   On the other hand, we have

$$\begin{aligned}&\frac{1}{16}(g_2(\Lambda)^3 - 27g_3(\Lambda)^2) = 4((\lambda_1+\lambda_2)^2 - \lambda_1\lambda_2)^3 - 27\lambda_1^2\lambda_2^2(\lambda_1+\lambda_2)^2\\ &= 4(\lambda_1+\lambda_2)^6 - 12\lambda_1\lambda_2(\lambda_1+\lambda_2)^4 - 15\lambda_1^2\lambda_2^2(\lambda_1+\lambda_2)^2 - 4\lambda_1^3\lambda_2^3\\ &= 4\lambda_1^6 + 4\lambda_2^6 + 12\lambda_1^5\lambda_2 + 12\lambda_1\lambda_2^5 - 3\lambda_1^4\lambda_2^2 - 3\lambda_1^2\lambda_2^4 - 26\lambda_1^3\lambda_2^3.\end{aligned}$$

   Thus if the curve D is non-singular, then $g_2(\Lambda)^3 - 27g_3(\Lambda)^2 \neq 0$.

2. We have $P(x,y,1) = x^3 + y^3 + 3xy$. The Newton polygon is a line segment joining $(0,3)$ and $(1,1)$. Its equation is $\alpha + \frac{1}{2}\beta = \frac{3}{2}$. Then

$$P(x,y,1) = \sum_{\alpha+\frac{1}{2}\beta=\frac{3}{2}} c_{\alpha\beta}x^\alpha y^\beta + x^3.$$

   Function

$$f_0(t) = \sum_{\alpha+\frac{1}{2}\beta=\frac{3}{2}} c_{\alpha\beta}t^\beta = t^3 + 3t = t(t+i\sqrt{3})(t-i\sqrt{3})$$

   has roots 0 and $\pm i\sqrt{3}$.

Let us consider the case $\pm i\sqrt{3}$ first. The first substitution is $x = x_1^2$, $y = x_1(\pm i\sqrt{3} + y_1)$ and

$$P(x_1, y_1, 1) = x_1^6 + x_1^3(\pm i\sqrt{3} + y_1)^3 + 3x_1^3(\pm i\sqrt{3} + y_1).$$

Thus

$$P_1(x_1, y_1) = x_1^3 + y_1^3 \pm 3i\sqrt{3}y_1^2 - 9y_1 - 3(\pm i\sqrt{3}) \pm 3i\sqrt{3} + 3y_1 = x_1^3 + y_1^3 - 6y_1 \pm 3i\sqrt{3}y_1^2.$$

Polynomial $P_1(x_1, y_1)$ has as the Newton polygon the line segment joining $(0, 1)$ and $(3, 0)$ with equation $\alpha + 3\beta = 3$. Then

$$P_1(x_1, y_1) = \sum_{\alpha+3\beta=3} d_{\alpha\beta} x_1^\alpha y_1^\beta + y_1^3 \pm 3i\sqrt{3}y_1^2$$

and

$$f_1(t) = \sum_{\alpha+3\beta=3} d_{\alpha\beta} t^\beta = 1 - 6t$$

has root $\frac{1}{6}$.
We get substitutions

$$x_1 = x_2, \quad y_1 = x_2^3(\frac{1}{6} + y_2)$$

hence

$$P_1(x_2, y_2) = x_2^3 + x_2^9(\frac{1}{6} + y_2)^3 - 6x_2^3(\frac{1}{6} + y_2) \pm 3i\sqrt{3}x_2^6(\frac{1}{6} + y_2)^2$$

and

$$P_2(x_2, y_2) = 1 + x_2^6(y_2^3 + \frac{1}{2}y_2^2 + \frac{1}{12}y_2 + \frac{1}{216}) - 1 - 6y_2 \pm 3i\sqrt{3}x_2^3(y_2^2 + \frac{1}{3}y_2 + \frac{1}{36})$$
$$= \sum_{\alpha+3\beta\geq 3} e_{\alpha\beta} x_2^\alpha y_2^\beta.$$

Polynomial

$$\sum_{\alpha+3\beta=3} e_{\alpha\beta} t^\beta = -6t \pm \frac{i\sqrt{3}}{12}$$

has two roots $(\pm\frac{i\sqrt{3}}{72})$. Then

$$x_2 = x_3, \quad y_2 = x_3^3((\pm\frac{i\sqrt{3}}{72}) + y_3).$$

Then, the expansion is

$$y = x^{\frac{1}{2}}(\pm i\sqrt{3} + x^{\frac{3}{2}}(\frac{1}{6} + x^{\frac{3}{2}}(\pm\frac{i\sqrt{3}}{72} + \ldots))) = \pm i\sqrt{3}x^{\frac{1}{2}} + \frac{1}{6}x^2 \pm \frac{i\sqrt{3}}{72}x^{\frac{7}{2}} + \ldots$$

If, on the other hand we take the root 0 of $f_0(t)$ then the first substitution is $x = x_1^2$, $y = x_1 y_1$ and

$$\begin{aligned} P(x_1, y_1, 1) &= x_1^6 + x_1^3 y_1^3 + 3x_1^3 y_1 \\ P_1(x_1, y_1) &= x_1^3 + y_1^3 + 3y_1 = \sum_{\alpha+3\beta\geq 3} g_{\alpha\beta} x^\alpha y^\beta. \end{aligned}$$

Function

$$f_1(t) = \sum_{\alpha+3\beta=3} g_{\alpha\beta} t^\beta = 3t + 1$$

has one root $t = -\frac{1}{3}$. We get

$$x_1 = x_2, \qquad y_1 = x_2^3(-\frac{1}{3} + y_2),$$

$$\begin{aligned} P_1(x_2, y_2) &= x_2^3 + x_2^9(y_2 - \frac{1}{3})^3 + 3x_2^3(y_2 - \frac{1}{3}) \\ P_2(x_2, y_2) &= 1 + x_2^6(y_2^3 - y_2^2 + \frac{1}{3}y_2 - \frac{1}{27}) + 3y_2 - 1 \\ &= x_2^6 y_2^3 - x_2^6 y_2^2 + \frac{1}{3}x_2^6 y_2 - \frac{1}{27}x_2^6 + 3y_2 \\ &= \sum_{\alpha+6\beta\geq 6} h_{\alpha\beta} x^\alpha y^\beta. \end{aligned}$$

Function

$$f_2(t) = \sum_{\alpha+6\beta=6} h_{\alpha\beta} t^\beta = 3t - \frac{1}{27}$$

has root $\frac{1}{81}$. Then $x_2 = x_3$, $y_2 = x_3^6(\frac{1}{81} + y_2 3)$

The Puiseux expansion is:

$$y = x^{\frac{1}{2}}(x^{\frac{3}{2}}(-\frac{1}{3} + x^{\frac{6}{2}}(\frac{1}{81} + \ldots))) = 0 \cdot x^{\frac{1}{2}} - \frac{1}{3}x^2 + \frac{1}{81}x^5 + \ldots$$

3. A point $[a : b : c]$ is a singular point of a curve $\{x^3 + y^3 + 3xyz = 0\}$ if

$$\begin{cases} 3a^2 + 3bc = 0, \\ 3b^2 + 3ac = 0, \\ ab = 0. \end{cases}$$

It follows from the last equation that either $a = 0$ or $b = 0$. If $a = 0$ then the second equation implies that $b = 0$. If $b = 0$ then the first equation implies that $a = 0$. Hence $p = [0 : 0 : 1]$ is the only singular point of C.

It follows from Noether formula that

$$g(C) = \frac{(3-1)(3-2)}{2} - \delta(p).$$

Point $[0:1:0]$ does not lie on C. To check whether it lies on tangent lines to inflection points of C we calculate the matrix of second derivatives

$$\begin{pmatrix} 6x & 3z & 3y \\ 3z & 6y & 3x \\ 3y & 3x & 0 \end{pmatrix}$$

Its determinant is $54(xyz - y^3 - x^3)$. It follows that inflection points of C satisfy $x^3 + y^3 = 0$ and $xyz = 0$. There are three inflection points $[1 : -1 : 0]$, $[\varepsilon_3 : -1 : 0]$ and $[\varepsilon_3^2 : -1 : 0]$. Since $\frac{\partial P}{\partial y}(x, y, z) = 3y^2+3xz$ does not vanish at the above inflection points, $[0 : 1 : 0]$ does not lie on any line tangent to the inflection point of C. It follows that

$$\delta(p) = \frac{1}{2}(I_p(P, \frac{\partial P}{\partial y}) - \nu_\varphi(p) + |\pi^{-1}(p)|).$$

We know that $|\pi^{-1}(p)|$ is the number of essentially distinct Puiseux expansions. From the previous exercise we learn that there are 3 Puiseux expansions:

$$y_1(x) = i\sqrt{3}x^{\frac{1}{2}} + \frac{1}{6}x^2 + \frac{i\sqrt{3}}{72}x^{\frac{7}{2}} + \dots,$$
$$y_2(x) = -i\sqrt{3}x^{\frac{1}{2}} + \frac{1}{6}x^2 - \frac{i\sqrt{3}}{72}x^{\frac{7}{2}} + \dots, \quad y_3(x) = 0 \cdot x^{\frac{1}{2}} - \frac{1}{3}x^2 + \frac{1}{81}x^5 + \dots$$

If we consider

$$g_1(t) = i\sqrt{3}t + \frac{1}{6}t^4 + \frac{i\sqrt{3}}{72}t^7 + \dots$$

then $y_1(x) = g_1(x^{\frac{1}{2}})$ and $y_2(x) = g_1(-x^{\frac{1}{2}})$. Thus, these two expansions are not essentially distinct. It follows that $|\pi^{-1}(p)| = 2$.

We have $P(0, y, 1) = y^3$ so $\nu_\varphi(p) = 3$.

Finally, $\frac{\partial P}{\partial y} = 3y^2 + 3xz$ and

$$\begin{aligned}
&I_p(x^3 + y^3 + 3xyz, y^2 + xz) = I_p(z^4(x^3 + y^3 + 3xyz) - x^2z^3(y^2 + xz), y^2 + xz) \\
&= I_p(y^3z^4 + 3xyz^5 - x^2y^2z^3, y^2 + xz) = I_p(z^3(y^3z + 3xyz^2 - x^2y^2), y^2 + xz) \\
&= I_p(y(y^2z + 3xz^2 - x^2y), y^2 + xz) = I_p(y, y^2 + xz) + I_p(y^2z + 3xz^2 - x^2y, y^2 + xz) \\
&= I_p(y, xz) + I_p(y^2z^3 + 3xz^4 - x^2yz^2 - 3y^2z^3 - 3xz^4, y^2 + xz) \\
&= 1 + I_p(x^2yz^2 + 2y^2z^3, y^2 + xz) = 1 + I_p(x^2y + 2y^2z, y^2 + xz) \\
&= 2 + I_p(x^2 + 2yz, y^2 + xz) = 2 + I_p(x^2z^2 + 2yz^3 - xy^2z - x^2z^2, y^2 + xz) \\
&= 2 + I_p(2yz^3 - xy^2z, y^2 + xz) = 3 + I_p(2z^2 - xy, y^2 + xz) = 3.
\end{aligned}$$

It follows that

$$\delta(p) = \frac{1}{2}(3 - 3 + 2) = 1, \quad g(C) = 1 - 1 = 0.$$

15 13.03. Lecture 15. Newton Polygons and Puiseux expansions

Today we investigate what a singular curve of degree d looks like in a neighbourhood of a singular point. We assume that C is given by $P \in \mathbb{C}[x, y, z]$ and that $[0 : 0 : 1] \in C$ is singular.

Newton's idea was to think about equation $P(x, y, 1) = 0$ as an implicit equation for y as a function of x near 0. If $\frac{\partial P}{\partial y}(0, 0, 1) \neq 0$ the implicit function theorem tells us that y is locally a holomorphic function in x, hence can be expanded as a power series in x. When the derivative vanishes we need to consider fractional powers of x.

Let us suppose first that $P(x, y, 1)$ is a quasi-homogeneous polynomial:

$$P(x, y, 1) = \sum_{\alpha+\mu\beta=\nu} c_{\alpha\beta} x^\alpha y^\beta.$$

Clearly, $P(x, y, 1)$ is quasi-homogeneous if all (α, β) such that $c_{\alpha\beta} \neq 0$ lie on one line in $\mathbb{R}^2$. Then putting $y = tx^\mu$, we get

$$P(x, tx^\mu, 1) = \sum_{\alpha+\mu\beta=\nu} c_{\alpha\beta} x^\alpha t^\beta x^{\mu\beta} = x^\nu (\sum_{\alpha+\mu\beta=\nu} c_{\alpha\beta} t^\beta) = x^\nu f(t).$$

If t_0 is such that $f(t_0) = 0$ then $y = t_0 x^\mu$ is the solution to the equation $P(x, y, 1) = 0$.

In general, let

$$P(x, y, 1) = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha y^\beta$$

and define the *carrier* of P as

$$\Delta(P) = \{(\alpha, \beta) \in \mathbb{Z} \mid c_{\alpha\beta} \neq 0\}.$$

Let

$$x + \mu y = \nu$$

be a line which contains at least two points of $\Delta(P)$ and such that $\alpha + \mu\beta \geq \nu$ for all $(\alpha, \beta) \in \Delta(P)$. If $y = tx^\mu$ then

$$P(x, tx^\mu, 1) = x^\nu f(t) + \sum_{\alpha+\mu\beta>\nu} c_{\alpha\beta} t^\beta x^{\alpha+\mu\beta}.$$

For a root t_0 of $f(t)$ we can think of $y = t_0 x^\mu$ as an approximate solution to $P(x, y, 1) = 0$.

To show that this way we can construct a solution we first define the *Newton polygon* of P:

Definition 15.1. If $p, q \in \mathbb{R}^2$ let $[p, q] = \{tp + (1 - t)q \mid t \in [0, 1]\}$ be the line segment from p to q. Consider the convex subset of $\mathbb{R}^2$ consisting of those $(x, y) \in \mathbb{R}^2$ such that

$$x \geq a \text{ and } y \geq b$$

for some $(a, b) \in [\delta_1, \delta_2]$, $\delta_1, \delta_2 \in \Delta(P)$. Its boundary consists of a vertical half-line and a horizontal half-line joined by a union of finitely many straight line segments. This union is the *Newton polygon* of P.

Because $P(0,0,1) = 0$, $(0,0) \notin \Delta(P)$. One can always choose coordinates such that $P(x,y,z)$ is not divisible by x. If it is true then there is some $(0,\beta) \in \Delta(P)$.

If $\Delta(P)$ is a single point, it is $(0,\beta_0)$ and all terms of $P(x,y,1)$ are divisible by y^{β_0}, so

$$P(x,y,1) = y^{\beta_0} Q(x,y).$$

and $Q(0,0) \neq 0$. In this case the only solutions to $P(x,y,1) = 0$ in the neighbourhood of 0 are given by $y = 0$.

If $\Delta(P)$ has more than one point, the steepest segment of the Newton polygon is the starting line of the procedure described above. Let $(0,\beta_0)$ be the upper endpoint of this segment and let $-\frac{1}{\mu_0}$ be its slope. Then μ_0 is a positive rational,

$$\mu_0 = \frac{p_0}{q_0}.$$

We can write

$$P(x,y,1) = \sum_{\alpha+\mu_0\beta\geq\nu_0} c_{\alpha\beta} x^\alpha y^\beta$$

where $\nu_0 = \mu_0\beta_0$. Since there is at least one point $(\alpha,\beta) \in \Delta(P)$ other than $(0,\beta_0)$ which lies on this segment, polynomial

$$f_0(t) = \sum_{\alpha+\mu_0\beta=\nu_0} c_{\alpha\beta} t^\beta$$

has a non-zero root, say t_0. Then

$$y_0 = t_0 x_0^\mu$$

gives us the first approximate solution.

Next we make the substitution

$$x = (x_1)^{q_0}, \qquad y = x^\mu(t_0 + y_1) = x_1^{p_0}(t_0 + y_1)$$

to get

$$P(x_1^{q_0}, x_1^{p_0}(t_0+y_1), 1) = \sum_{\alpha+\mu_0\beta\geq\nu_0} c_{\alpha\beta} x_1^{q_0(\alpha+\mu_0\beta)}(t_0+y_1)^\beta = x_1^{q_0\nu_0} P_1(x_1,y_1).$$

Then

$$P_1(x_1,y_1) = \sum_{q_0\alpha+p_0\beta\geq q_0\nu_0} c_{\alpha\beta} x_1^{q_0\alpha+p_0\beta-q_0\nu_0}(t_0+y_1)^\beta$$

is a polynomial not divisible by x_1.

Remark 15.2. From the construction of Newton's polygon we learn that y^{β_0} is the smallest power of y appearing in $P(x,y,1)$. Let $y_1^{\beta_1}$ be the smallest power of y_1 appearing in $P_1(x_1,y_1)$. Then

$$P_1(0,y_1) = \sum_{q_0\alpha+\rho_0\beta=q_0\nu_0} (t_0+y_1)^\beta$$

so β_1 is the smallest integer such that $c_{\alpha,\beta} \neq 0$ and $\alpha + \mu_0\beta = \nu_0$, for some $\alpha \geq 0$. Either $\beta_1 < \beta_0$ or $P_1(0,y_1)$ is a constant multiple of $y_1^{\beta_0}$. If this is the case then the degree $\beta_0 - 1$ term of $c_{0\beta_0}(t_0+y_1)^{\beta_0}$ must cancel with term of $c_{\alpha\beta_0-1}(t_0+y_1)^{\beta_0-1}$. In particular, $\beta_0 - 1$ must lie on the line $\alpha + \mu_0\beta = \nu_0$. It follows that its slope μ_0 needs to be integer, so $q_0 = 1$.

We now repeat the whole process, replacing $P(x, y, 1)$ by $P(x_1, y_1)$ and continue indefinitely. We obtain a sequence of positive rationals $\mu_0 = \frac{p_0}{q_0}, \mu_1 = \frac{p_1}{q_1} \dots$ and complex numbers $t_0, t_1, \dots$ and successive "approximate solutions" $(x, y), (x_1, y_1), \dots$ to the equation $P(x, y, 1) = 0$. We will have

$$x = x_1^{\frac{1}{q_1}},\ x_1 = x_2^{\frac{1}{q_2}}, \quad y = x^{\mu_0}(t_0 + y_1),\ y_1 = x_1^{\mu_1}(t_1 + y_2), \dots.$$

We would like to conclude that

$$y = t_0 x^{\mu_0} + t_1 x_1^{\mu_1} x^{\mu_0} + t_2 x_2^{\mu_2} x_1^{\mu_1} x^{\mu_0} + \dots = t_0 x^{\mu_0} + t_1 x^{\mu_0 + \mu_1/q_0} + t_2 x^{\mu_0 + \mu_1/q_1 + \mu_2/q_0 q_1} + \dots$$

is a genuine solution near $(0, 0)$. This series is called a *Puiseux expansion* for the curve $C = \{[x : y : z] \,|\, P(x, y, z) = 0\}$ near $[0 : 0 : 1]$.

By the above remark, $q_i = 1$ unless $\beta_{i-1} > \beta_i$. Since $\beta_0 \geq \beta_1 \geq \beta_2 \dots$ is a sequence of positive integers, $q_i = 1$ for all but finitely many i. The product n of all q_i is well-defined and the Puiseux expansion may be expressed as a formal power series in $x^{\frac{1}{n}}$.

Theorem 15.3. *Any Puiseux expansion $y = \sum_{r \geq 1} a_r x^{r/n}$ for the curve C near $[0 : 0 : 1]$ is a power series in $x^{1/n}$ which converges for x sufficiently close to 0 and satisfies*

$$P(x, \sum_{r \geq 1} a_r x^{r/n}, 1) = 0.$$

Proof. From the last lecture, we know that there are $m_1 + \dots + m_l = d$ and holomorphic $g_1 \dots, g_l$ such that

$$P(x, y, 1) = K \prod_{1 \leq j \leq l} \prod_{1 \leq s \leq m_j} (y - g_j(e^{2\pi i s/m_j}(x - a)^{\frac{1}{m_j}})).$$

We can expand each $g_j(t) = \sum_{r \geq 0} a_r^{(j)} t^r$ as a convergent power series near 0. If N is the lowest common multiple of $m_1, \dots, m_l$ and n then series

$$g_j(e^{2\pi i s/m_j} x^{1/m_j}) = \sum_{r \geq 0} a_r^{(j)} e^{2\pi i r s/m_j} x^{r/m_j}$$

and the Puiseux expansion can be regarded as elements of the ring $\mathbb{C}\{x^{1/N}\}$ of formal power series. It is an integral domain, so if $Q \in \mathbb{C}\{x^{1/N}\}[y]$ satisfies $Q(c) = 0$ for some $c \in \mathbb{C}\{x^{1/N}\}$ and $Q(y) = k(y - c_1) \dots (y - c_d)$ then $c = c_j$ for some j. Therefore, it suffices to check that as a formal power series

$$P(x, \sum_{r \geq 1} a_r x^{r/n}, 1) = 0$$

as then the Puiseux expansion must coincide with one of the $\sum_{r \geq 0} a_r^{(j)} e^{2\pi i r s/m_j} x^{r/m_j}$, hence converge.

The exponent of the smallest power of $x^{1/N}$ in $P(x, \sum_{r=1}^{M} a_r x^{r/n}, 1)$ is at least $p_0 \beta_0 + \dots + p_M \beta_M$. It tends to zero as M tends to infinity, since each p_j and β_j are positive integers. This tell us that every coefficient in the formal power series $P(x, \sum_{r \geq 1} a_r x^{r/n}, 1)$ is 0, so the power series is 0. $\square$

Recall that we have $\pi\colon \widetilde{C} \to C$ and that

$$\pi^{-1}([0:0:1]) = \{\langle t^{m_j}, g_j(t)\rangle \mid 1 \leq j \leq l\}.$$

Any Puiseux expansion of C is of the form

$$y = g_j(e^{2\pi i s/m_j} x^{1/m_j})$$

for $j \in [1, l]$, $s \in [1, m_j]$. We regard two Puiseux expansions essentially different if they come from different functions g_j. Then the number of points of $\pi^{-1}([0:0:1])$ is the number of essentially different Puiseux expansions of C near $[0:0:1]$.

16 16.03. Lecture 16. The degree–genus formula for singular curves

We know that any irreducible projective curve C has a resolution of singularity $\pi\colon \widetilde{C} \to C$, that the map π is a homeomorphism when restricted to $\pi^{-1}(C \setminus \mathrm{Sing}(C))$ and that, for a singular $p \in C$, $\pi^{-1}(p)$ is the number of essentially different Puiseux expansions of C in the neighbourhood of p. The curve $\widetilde{C}$ is smooth, hence its genus g is well-defined. We shall call it the genus of C and relate to the degree of C. We shall assign to each singular $p \in C$ a positive integer $\delta(p)$ and prove *Noether's formula*

$$g = \frac{1}{2}(d-1)(d-2) - \sum_{p \in \mathrm{Sing}(C)} \delta(p).$$

As before, assume that $[0:1:0] \notin C$ and define $\varphi\colon C \to \mathbb{P}^1$ via $\varphi([x:y:z]) = [x:z]$. We regard $\psi\colon \widetilde{C} \xrightarrow{\pi} C \xrightarrow{\varphi} \mathbb{P}^1$ as a branched cover of $\mathbb{P}^1$ and define

$$R = \pi^{-1}\{[a:b:c] \in C \mid \frac{\partial P}{\partial y}(a,b,c) = 0\}$$

to be the set of *ramification points* of ψ and its image $\psi(R)$ the *branch locus* of ψ.

Proposition 16.1. *Given any triangulation (V,E,F) of $\mathbb{P}^1$ such that $\psi(R)$ is contained in the set of vertices V there is a triangulation $(\widetilde{V},\widetilde{E},\widetilde{F})$ of $\widetilde{C}$ such that*

$$\widetilde{V} = \pi^{-1}(V), \qquad |\widetilde{E}| = d|\widetilde{E}|, \qquad |\widetilde{F}| = d|F|.$$

Proof. Since $\pi^{-1}(\mathrm{Sing}(C)) \subset R$ and $\pi|_{\widetilde{C}\setminus\pi^{-1}(\mathrm{Sing}(C))}$ is a homeomorphism, the proof is a straightforward modification of the proof in the smooth case, see Proposition 10.6. □

If $p = [a:b:c]$ we define $\nu_\varphi(p)$ to be the multiplicity of the polynomial $P(a,y,c)$ at $y = b$.

Lemma 16.2. $|\widetilde{V}| = d|V| - \sum_{p\in\pi(R)}(\nu_\varphi(p)-1) + \sum_{p\in Sing(C)}(|\pi^{-1}(p)|-1)$.

Recall that we proved

Lemma 8.4 *The inverse image $\varphi^{-1}([a:c])$ of any $[a:c] \in \mathbb{P}^1$ under φ contains exactly*

$$d - \sum_{p\in\varphi^{-1}([a:c])} (\nu_\varphi(p)-1)$$

points. In particular $\varphi^{-1}([a:c])$ contains d points if and only of $\varphi^{-1}([a:c])$ contains no ramification points of φ.

Proof of Lemma 16.2. The inverse image of any $q \in \mathbb{P}^1$ under $\varphi\colon C \to \mathbb{P}^1$ contains exactly $d - \sum_{p\in\varphi^{-1}(q)}(\nu_\varphi(p)-1)$.

If $p \notin \pi(R)$ then $\nu_\varphi(p) = 1$ and $\pi(R) \subset \varphi^{-1}(V)$ so

$$|\varphi^{-1}V| = d|V| - \sum_{p\in\pi(R)} (\nu_\varphi(p)-1).$$

The set $\varphi^{-1}(V)$ contains all singular points of C, hence

$$|\widetilde{V}| = |\pi^{-1}\varphi^{-1}V| = d|V| - \sum_{p\in\pi(R)} (\nu_\varphi(p) - 1) + \sum_{p\in\mathrm{Sing}(C)} (|\pi^{-1}(p)| - 1).$$

□

Recall that we proved that for a non-singular $[a : b : c] \in C$, we have $\nu_\varphi[a : b : c] \geq 2$ if and only if $P(x, y, z) = P_y(x, y, z) = 0$ and $\nu_\varphi[a : b : c] > 2$ if and only if $P(a, b, c) = P_y(a, b, c) = P_{yy}(a, b, c) = 0$ if and only if $[a : b : c]$ is the inflection point on C and the tangent line to C at $[a : b : c]$ contains $[0 : 1 : 0]$.

Lemma 16.3. *Suppose that $[0 : 1 : 0]$ does not lie on C or the tangent line to C at any of the finitely many $p \in C \setminus Sing(C)$ which are inflection points. Then if $p \in \pi(R)$ and $p \notin Sing(C)$ we have*

$$\nu_\varphi(p) = 2, \qquad\qquad I_p(P, \frac{\partial P}{\partial y}) = 1.$$

Proof. Since $[0 : 1 : 0] \notin C$, the coefficient $P(0, 1, 0)$ of y^d in $P(x, y, z)$ is non-zero, hence $P_y(x, y, z)$ is a polynomial of degree $d - 1$ which is not identically zero. Since P is irreducible, P_y and P cannot have a common factor, hence C and $D = \{P_y = 0\}$ intersect in finitely many points.

If $p \in \pi(R)$ then by definition of R, $p \in C \cap D$ and $\nu_\varphi(p) \geq 2$. Let us assume further that $p \in C$ is a non-singular point. It follows from the description of the points with $\nu_\varphi(q) > 2$ that $\nu_\varphi(p) \leq 2$, hence $\nu_\varphi(p) = 2$. To prove that the intersection index is equal to 1 we need to check that tangent lines are distinct (we know that $p \in D$ is non-singular because $P_{yy}(a, b, c) \neq 0$). If they would coincide then

$$[P_x(a, b, c) : P_y(a, b, c) : P_z(a, b, c)] = [P_{xy}(a, b, c) : P_{yy}(a, b, c), P_{yz}(a, b, c)]$$

in particular $P_{yy}(a, b, c) = 0$ which contradicts the assumption on tangent lines. □

Corollary 16.4. *If coordinates are chosen as above then*

$$\chi(\widetilde{C}) = d(3 - d) + \sum_{p\in Sing(C)} (I_p(P, \frac{\partial P}{\partial y}) - \nu_\varphi(p) + |\pi^{-1}(p)|).$$

Proof. By definition

$$\chi(\widetilde{C}) = |\widetilde{V}| - |\widetilde{E}| + |\widetilde{F}| = d(|V| - |E| + |F|) - \sum_{p\in\pi(R)} (\nu_\varphi(p) - 1) + \sum_{p\in\mathrm{Sing}(C)} (|\pi^{-1}(p)| - 1).$$

The sum $|V| - |E| + |F| = \chi(\mathbb{P}^1) = 2$.

By Lemma 16.3

$$\sum_{p\in\pi(R)\setminus\mathrm{Sing}(C)} (\nu_\varphi(p) - 1) = \sum_{p\in\pi(R)\setminus\mathrm{Sing}(C)} I_p(P, \frac{\partial P}{\partial y}).$$

By construction, $\mathrm{Sing}(C) \subset \pi(R)$. Then, since the degree of C is d and $D = \{\frac{\partial P}{\partial y}\}$ is $d-1$, Bezout's theorem implies that

$$\sum_{p\in\pi(R)\setminus\mathrm{Sing}(C)} I_p(P, \frac{\partial P}{\partial y}) = d(d-1) - \sum_{p\in\mathrm{Sing}(C)} I_p(P, \frac{\partial P}{\partial y}).$$

Then we get

$$\begin{aligned}\chi(\widetilde{C}) &= 2d - \sum_{p\in\pi(R)\setminus\mathrm{Sing}(C)} I_p(P, \frac{\partial P}{\partial y}) + \sum_{p\in\mathrm{Sing}(C)} (|\pi^{-1}(p)| - 1 - \nu_\varphi(p) + 1)\\ &= 2d - d(d-1) + \sum_{p\in\mathrm{Sing}(C)} (I_p(P, \frac{\partial P}{\partial y}) + |\pi^{-1}(p)| - \nu_\varphi(p)).\end{aligned}$$

□

Definition 16.5. Let p be a singular point of the irreducible curve

$$C = \{[x : y : z] \in \mathbb{P}^2 \,|\, P(x, y, z) = 0\}$$

and suppose that coordinates have been chosen so that $[0 : 1 : 0]$ does not lie on C or on the tangent line to C at any of the inflection points. We define

$$\delta(p) = \frac{1}{2}(I_p(P, \frac{\partial P}{\partial y}) - \nu_\varphi(p) + |\pi^{-1}(p)|).$$

Theorem 16.6. *The genus g of an irreducible projective curve C of degree d in $\mathbb{P}^2$ is*

$$g = \frac{1}{2}(d-1)(d-2) - \sum_{p\in Sing(C)} \delta(p).$$

Proof. Follows from Corollary 16.4 and $\chi(\widetilde{C}) = 2 - 2g$. □

It remains to check that $\delta(p)$ does not depend on the choice of coordinates.

Extending the results about intersection multiplicity to holomorphic curves, one can check that

Lemma 16.7. *We have*

$$I_p(P, \frac{\partial P}{\partial y}) = \sum_{\langle f,g\rangle\in\pi^{-1}(p)} \mu(\frac{\partial P}{\partial y}(f, g, 1)), \quad \nu_\varphi(p) = \sum_{\langle f,g\rangle\in\pi^{-1}} \mu(f - f(0)),$$

where $\mu(\frac{\partial P}{\partial y}(f, g, 1))$ is the multiplicity of zero or minus multiplicity of the pole of $\frac{\partial P}{\partial y}(f(t), g(t), 1)$ at $t = 0$. In the second formula, we omit $f(0)$ if f has a pole at 0.

Then we have

Lemma 16.8. *Suppose that neither* $[0:1:0]$ *or* $[\alpha:\beta:\gamma]$ *lie on* C *or the tangent line to any inflection point of* $C \setminus Sing(C)$*. Suppose that* $p \in Sing(C)$ *and let* $\nu_\varphi^{[\alpha:\beta:\gamma]}(p)$ *be the smallest positive integer such that*

$$(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z})^m P$$

does not vanish at p*. Then*

$$I_p(P, \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z}) - \nu_\varphi^{[\alpha:\beta:\gamma]}(p) = I_p(P, \frac{\partial P}{\partial y}) - \nu_\varphi(p).$$

Proof. Note that $\nu_\varphi(p)$ was defined as the multiplicity of 0 of the polynomial $P(a, y, c)$, i.e. minimal m such that $\frac{d^m}{dy^m}P(a, y, c) \neq 0$. It follows that if $[\alpha:\beta:\gamma] = [0:1:0]$ there is nothing to prove.

Otherwise, we can find a projective transformation $[x:y:z] \mapsto [x':y':z']$ which fixes $[0:1:0]$, such that, up to scalar, $\frac{\partial P}{\partial y}(x', y', z') = \frac{\partial P}{\partial y'}(x', y', z')$ and which maps $[\alpha:\beta:\gamma]$ to $[1:0:0]$. Indeed, the first two conditions imply that the matrix is of the form $\begin{pmatrix} a & 0 & b \\ c & d & e \\ f & 0 & g \end{pmatrix}$. Such a transformation does not change the intersection multiplicity, hence without loss of generality we can assume that $[\alpha:\beta:\gamma] = [1:0:0]$.

For an element $\langle f, g\rangle \in \widetilde{C}$, we have $P(f(t), g(t), 1) = 0$ for all t near 0. Differentiating this equation gives

$$f'(t)\frac{\partial P}{\partial x}(f(t), g(t), 1) + g'(t)\frac{\partial P}{\partial y}(f(t), g(t), 1) = 0.$$

We conclude that

$$\mu(f') + \mu(\frac{\partial P}{\partial x}(f, g, 1)) = \mu(g') + \mu(\frac{\partial P}{\partial y}(f, g, 1)).$$

Since $\mu(f') = \mu(f - f(0)) - 1$ and $\mu(g') = \mu(g - g(0)) - 1$, summing the above equality over all $\langle f, g\rangle \in \pi^{-1}(p)$, we get that

$$\nu_\varphi^{[0:1:0]}(p) + I_p(P, \frac{\partial P}{\partial x}) = \nu_\varphi^{[1:0:0]} + I_p(P, \frac{\partial P}{\partial y}).$$

□

17.03. Workshop IV

1. Calculate the first two terms of the Puiseux expansion about $[0:0:1]$ of the curve defined by $P(x,y,z) = y^4z^3 + 2x^3y^2z^2 + 4x^5yz + x^6z + x^7$.

2. Calculate the genus of the curve defined by $P(x,y,z) = y^2z - x^3$.

Solutions to workshop IV

1. The carrier of polynomial

$$P(x,y,1) = y^4 + 2x^3y^2 + 4x^5y + x^6 + x^7$$

is

$$\Delta(P) = \{(0,4),\ (3,2),\ (5,1),\ (6,0),\ (7,0)\}.$$

The Newton polygon is the line segment joining $(0,4)$ and $(6,0)$. Its equation is $\alpha + \frac{3}{2}\beta = 6$. Then

$$P(x,y,1) = \sum_{\alpha+\frac{3}{2}\beta \geq 6} c_{\alpha\beta} x^\alpha y^\beta,$$
$$f_0(t) = \sum_{\alpha+\frac{3}{2}\beta = 6} c_{\alpha\beta} t^\beta = t^4 + 2t^2 + 1 = (t^2+1)^2 = (t-i)^2(t+i)^2.$$

Polynomial $f_0(t)$ has roots $\pm i$, hence the first substitution is

$$x = x_1^2,\ y = x_1^3(\pm i + y_1).$$

We have

$$\begin{aligned}
&P(x_1,y_1,1) = x_1^{12}(\pm i + y_1)^4 + 2x_1^{12}(\pm i + y_1)^2 + 4x_1^{13}(\pm i + y_1) + x_1^{12} + x_1^{14},\\
&P_1(x_1,y_1) = (\pm i + y_1)^4 + 2(\pm i + y_1)^2 + 4x_1(\pm i + y_1) + 1 + x_1^2\\
&= y_1^4 \pm 4iy_1^3 - 6y_1^2 - (\pm 4iy_1) + 1 + 2y_1^2 \pm 4iy_1 - 2 \pm 4ix_1 + 4x_1y_1 + 1 + x_1^2\\
&= y_1^4 \pm 4iy_1^3 - 4y_1^2 \pm 4ix_1 + 4x_1y_1 + x_1^2.
\end{aligned}$$

The carrier of polynomial P_2 is

$$\Delta(P_2) = \{(0,4),\ (0,3),\ (0,2),\ (1,0),\ (1,1),\ (2,0)\}.$$

Newton polygon is the line segment joining $(0,2)$ and $(1,0)$. Its equation is $\alpha + \frac{1}{2}\beta = 1$. Then

$$P_1(x_1,y_1) = \sum_{\alpha+\frac{1}{2}\beta \geq 1} d_{\alpha\beta} x^\alpha y^\beta$$
$$f_1(t) = \sum_{\alpha+\frac{1}{2}\beta = 1} d_{\alpha\beta} t^\beta = -4t^2 \pm i = 4(t - \frac{1}{2}\sqrt{\pm i})(t + \frac{1}{2}\sqrt{\pm i}).$$

Polynomial $f_1(t)$ has two roots $\pm\frac{1}{2}\sqrt{\pm i}$. We have

$$x_1 = x_2^{\frac{1}{2}}, \quad y_1 = x_2(\pm\frac{1}{2}\sqrt{\pm i} + y_2).$$

We get Puiseux expansion

$$y = x^{\frac{3}{2}}(\pm i + x^{\frac{1}{4}}(\pm\frac{1}{2}\sqrt{\pm i} + \ldots)) = \pm ix^{\frac{3}{2}} \pm \frac{1}{2}\sqrt{\pm i}x^{\frac{7}{4}} + \ldots$$

2. Let $C = \{y^2z - x^3 = 0\}$ be a cubic curve. By Noether formula

$$g(C) = \frac{1}{2}(3-1)(3-2) - \sum_{p \in \mathrm{Sing}(C)} \delta(p).$$

We need to find singular points of C. If $P(x, y, z) = y^2z - x^3$ then

$$\frac{\partial P}{\partial x} = 3x^2, \qquad \frac{\partial P}{\partial y} = 2yz, \qquad \frac{\partial P}{\partial z} = y^2.$$

It follows that if $[a : b : c]$ is a singular point then $a = b = 0$. Moreover $[0 : 0 : 1]$ is a singular point of C.

Let us find inflection points of C. The matrix of second derivatives

$$\begin{pmatrix} 6x & 0 & 0 \\ 0 & 2z & 2y \\ 0 & 2y & 0 \end{pmatrix}$$

has determinant $-24xy^2$. Hence, $[a : b : c]$ is an inflection point if

$$\begin{cases} a^3 = b^2c \\ ab^2 = 0 \end{cases}$$

$$\begin{cases} 0 = b^2c \\ a = 0 \end{cases} \qquad \begin{cases} a = 0 \\ b = 0 \end{cases}$$

$$\begin{cases} b = 0 \\ a = 0 \end{cases} \qquad \begin{cases} c = 0 \\ a = 0 \end{cases} \qquad \begin{cases} b = 0 \\ a = 0 \end{cases}$$

The solutions are $[0 : 0 : 1]$ and $[0 : 1 : 0]$. Since $[0 : 0 : 1]$ is the singular point of C, $[0 : 1 : 0]$ is the only inflection point.

Tangent line to C at $[0 : 1 : 0]$ is

$$z = 0.$$

Then $[1 : 0 : 1]$ is a point which does not lie on C and on the tangent line to the inflection point of C.

Then

$$\delta(p) = \frac{1}{2}(I_p(P, \frac{\partial P}{\partial(x+z)}) - \nu_{\varphi}^{[1:0:1]}(p) + |\pi^{-1}(p)|),$$

where $\nu_{\varphi}^{[1:0:1]}(p)$ is the smallest positive integer such that $(\frac{\partial}{\partial x}P + \frac{\partial}{\partial z}P)^m$ does not vanish at p.

First, let us calculate $|\pi^{-1}(p)|$, i.e. number of essentially different Puiseux expansions. We have

$$P(x, y, 1) = y^2 - x^3 = \sum_{\alpha + \frac{3}{2}\beta = 3} c_{\alpha\beta} x^\alpha y^\beta.$$

Polynomial

$$f_0(t) = \sum_{\alpha+\frac{3}{2}\beta=3} = t^2 - 1 = (t-1)(t+1)$$

has two roots ± 1, hence

$$x = x_1^2, \quad y = x_1^3(\pm 1 + y_1)$$

Note that $P(x_1^2, \pm x_1^3) = 0$, hence $P(x, y, 1)$ has finite Puiseux expansions $y = \pm x^{\frac{3}{2}}$. They are not essentially different, hence

$$|\pi^{-1}(p)| = 1.$$

Further

$$(\frac{\partial}{\partial x}P + \frac{\partial}{\partial z}P)^2 = \frac{\partial^2}{\partial x^2}P + 2\frac{\partial^2}{\partial x \partial z}P + \frac{\partial^2}{\partial z^2}P = 6x$$

vanishes at p while

$$(\frac{\partial}{\partial x}P + \frac{\partial}{\partial z}P)^3 = \frac{\partial^3}{\partial x^3}P + 3\frac{\partial^3}{\partial x^2 \partial z}P + 3\frac{\partial^3}{\partial x \partial z^2}P + \frac{\partial^3}{\partial z^3}P = 6$$

does note, hence

$$\nu_{\varphi}^{[1:0:1]}(p) = 3.$$

Finally,

$$\frac{\partial}{\partial x}P + \frac{\partial}{\partial z}P = 3x^2 + y^2$$

and

$$\begin{aligned}
&I_p(3x^2 + y^2, y^2 z - x^3) = I_p(3x^2 + y^2, x(3x^2 + y^2) + 3y^2 z - 3x^3) \\
&= I_p(3x^2 + y^2, y^2(x + 3z)) = 2I_p(3x^2 + y^2, y) + I_p(3x^2 + y^2, x + 3z) \\
&= 4I_p(x, y) + 0 = 4
\end{aligned}$$

It follows that

$$\begin{aligned}
\delta(p) &= \frac{1}{2}(4 - 3 + 1) = 1, \\
g(C) &= 1 - 1 = 0.
\end{aligned}$$

17 20.03. Lecture 17. Holomorphic differentials

We have associated to a lattice Λ in $\mathbb{C}$ a non-singular cubic curve $C_\Lambda \subset \mathbb{P}^2$. Now, we would like to understand how to a non-singular cubic curve assign a lattice in $\mathbb{C}$. For this we need to know how to integrate a holomorphic differential along a piecewise smooth path in a Riemann surface.

Definition 17.1. A *piecewise-smooth path* in a Riemann surface S is a continuous map $\gamma\colon [a,b] \to S$ such that if $\varphi\colon U \to V$ is a holomorphic chart and $[c,d] \subset \gamma^{-1}(U)$ then $\varphi \circ \gamma\colon [c,d] \to V$ is a piecewise-smooth path in the open subset V of $\mathbb{C}$.

We say that γ is *closed* if $\gamma(a) = \gamma(b)$.

Recall that a meromorphic function on an open $W \subset \mathbb{C}$ can be interpreted as a holomorphic function $W \to \mathbb{P}^1$ of Riemann surfaces. It motivates

Definition 17.2. A *meromorphic function* on a Riemann surface S is a function $f\colon S \to \mathbb{P}^1$ which is holomorphic in the sense of Riemann surfaces and not identically ∞ on any connected component of S.

If S is a compact Riemann surface then every holomorphic function $S \to \mathbb{C}$ is constant but there are lots of meromorphic functions (which is not easy to prove in general). Examples are Weiertrass $\wp$-function on a complex torus, and rational functions on non-singular projective curves. By a *rational function* on $C\backslash \mathrm{Sing}(C)$, where $C = \{P(x,y,z) = 0\}$ we mean a meromorphic function

$$[x:y:z] \mapsto \frac{S(x,y,z)}{T(x,y,z)}$$

where S and T are homogenous polynomials of the same degree ans T does not vanish identically at C.

Does it make sense to differentiate a meromorphic function on a Riemann surface?

If

$$\Phi = \{\varphi_\alpha\colon U_\alpha \to V_\alpha \mid \alpha \in A\}$$

is a holomorphic atlas on S then a meromorphic function $g\colon S \to \mathbb{P}^1$ is determined by the collection of meromorphic functions

$$\{g \circ \varphi_\alpha^{-1}\colon V_\alpha \to \mathbb{P}^1 \mid \alpha \in A\}.$$

Conversely, a collection of meromorphic functions

$$\{G_\alpha\colon V_\alpha \to \mathbb{P}^1 \mid \alpha \in A\}$$

defines a meromorphic function on S if and only if

$$G_\alpha(\varphi_\alpha(u)) = G_\beta(\varphi_\beta(u)),\ \forall u \in U_\alpha \cap U_\beta.$$

We can differentiate $g \circ \varphi_\alpha^{-1}\colon V_\alpha \to \mathbb{P}^1$ to get meromorphic functions $(g \circ \varphi_\alpha^{-1})'\colon V_\alpha \to \mathbb{P}^1$ but they do not define a meromorphic function on S since $(g \circ \varphi_\alpha^{-1})'$ and $(g \circ \varphi_\beta)'$ do not necessarily agree on $U_\alpha \cap U_\beta$. Instead, the chain rule tells us that

$$(g \circ \varphi_\alpha^{-1})'(\varphi_\alpha(u)) = ((g \circ \varphi_\beta^{-1})(\varphi_\beta \circ \varphi_\alpha^{-1}))'(\varphi_\alpha(u)) = (g \circ \varphi_\beta^{-1})'(\varphi_\beta(u))(\varphi_\beta \circ \varphi_\alpha^{-1})'(\varphi_\alpha(u)).$$

So if we differentiate g_α we do not get a meromorphic function on S but an abstract object called a *meromorphic differential*, denoted by dg which is glued from $(g \circ \varphi_\alpha^{-1})'$. We can then multiply a meromorphic differential dg by a meromorphic function f. We will say that $f\,dg$ and $\widetilde{f}\,d\widetilde{g}$ are equal if the defining functions $(f \circ \varphi_\alpha^{-1})(g \circ \varphi_\alpha^{-1})'$ and $(\widetilde{f} \circ \varphi_\alpha^{-1})(\widetilde{g} \circ \varphi_\alpha^{-1})'$ agree on every V_α. More precisely, we have

Definition 17.3. A *meromorphic differential* on a Riemann surface S is an equivalence class of pairs (f, g) of meromorphic functions on S such that

$$(f, g) \sim (\widetilde{f}, \widetilde{g})$$

if and only if for every holomorphic chart $\varphi\colon U \to V$ on S and every $z \in V$, we have

$$f \circ \varphi^{-1}(z)(g \circ \varphi^{-1}(z))' = \widetilde{f} \circ \varphi^{-1}(z)(\widetilde{g} \circ \varphi^{-1}(z))'.$$

Abstractly, we also have

Definition 17.4. Let $\{\varphi_\alpha\colon U_\alpha \to V_\alpha \,|\, \alpha \in A\}$ be a holomorphic atlas on a Riemann surface S. A meromorphic differential η on S is given by a collection

$$\{\eta_\alpha\colon V_\alpha \to \mathbb{P}^1 \,|\, \alpha \in A\}$$

of meromorphic function such that if $u \in U_\alpha \cap U_\beta$ then

$$\eta_\alpha(\varphi_\alpha(u)) = \eta_\beta(\varphi_\beta(u))(\varphi_\beta \circ \varphi_\alpha^{-1})'(\varphi_\alpha(u)).$$

Remark 17.5. If η and ζ are meromorphic differentials and ζ is not identically zero on any connected component of S then η/ζ defines a meromorphic function on S glued from η_α/ζ_α. It follows that $\eta = f\zeta$. Thus, to show that every abstract meromorphic differential is of the form $f\,dg$ it suffices to show that there is at least one non-constant meromorphic function on every Riemann surface. This is beyond the scope of this lecture but we have seen examples on non-singular curves and complex tori.

Definition 17.6. The meromorphic differential $f\,dg$ *has a pole* at a point $p \in S$ is the meromorphic function $(\varphi \circ \varphi^{-1})(g \circ \varphi^{-1})'$ has a pole at $\varphi(p)$ when $\varphi\colon U \to V$ is a holomorphic chart on an open neighbourhood U of $p \in S$. We call $f\,dg$ *holomorphic differential* if it has no poles.

Definition 17.7. If $f\,dg$ is a holomorphic differential on S then the *integral of* $f\,dg$ along a piecewise-smooth path $\gamma\colon [a, b] \to S$ is

$$\int_\gamma f\,dg = \int_a^b f \circ \gamma(t)(g \circ \gamma)'(t)dt.$$

We need to check that if $(f, g) \sim (\widetilde{f}, \widetilde{g})$ then $\int_\gamma f\,dg = \int_\gamma \widetilde{f}\,d\widetilde{g}$, i.e. that

$$\int_a^b f \circ \gamma(t)(g \circ \gamma)'(t)dt = \int_a^b \widetilde{f} \circ \gamma(t)(\widetilde{g} \circ \gamma)'(t)dt.$$

We can find $a = a_0 < a_1 < \ldots < a_p = b$ and $\alpha_1, \ldots \alpha_p \in A$ such that $\gamma([a_{i-1}, a_i]) \subset U_{\alpha_i}$. Then

$$\int_\gamma f\,dg = \sum_{i=1}^{p} \int_{a_{i-1}}^{a_i} f \circ \gamma(t)(g \circ \gamma)'(t)dt =$$
$$\sum_{i=1}^{p} \int_{a_{i-1}}^{a_i} (f \circ \varphi_{\alpha_i}^{-1})(\varphi_{\alpha_i}\gamma)(t)(g \circ \varphi_{\alpha_i}^{-1})' \circ (\varphi_{\alpha_i}\gamma)(t)(\varphi_{\alpha_i} \circ \gamma)'(t)dt =$$
$$\sum_{i=1}^{p} \int_{a_{i-1}}^{a_i} (\widetilde{f} \circ \varphi_{\alpha_i}^{-1})(\varphi_{\alpha_i}\gamma)(t)(\widetilde{g} \circ \varphi_{\alpha_i}^{-1})' \circ (\varphi_{\alpha_i}\gamma)(t)(\varphi_{\alpha_i} \circ \gamma)'(t)dt =$$
$$\sum_{i=1}^{p} \int_{a_{i-1}}^{a_i} \widetilde{f} \circ \gamma(t)(\widetilde{g} \circ \gamma)'(t)dt = \int_\gamma \widetilde{f}\,d\widetilde{g}.$$

If the Riemann surface is $\mathbb{C}$ then $\int_\gamma f\,dg = \int_\gamma f(z)g'(z)dz$ is the integral of $f(z)g'(z)$ along γ in the usual sense of complex analysis.

If $g\colon \mathbb{S} \to C$ is a complex-valued holomorphic mapping on any Riemann surface S then $\int_\gamma dg = g(\gamma(b)) - g(\gamma(a))$.

Definition 17.8. If $\psi\colon S \to R$ is a holomorphic mapping between Riemann surfaces S and R an if $f\,dg$ is a holomorphic differential on R then we define a holomorphic differential $\psi^*(f\,dg)$ on S by

$$\psi^*(f\,dg) = (f \circ \psi)\,d(g \circ \psi).$$

If $\gamma\colon [a, b] \to S$ is a piecewise-smooth path in S then

$$\int_\gamma \psi^*(f\,dg) = \int_a^b f \circ \psi \circ \gamma(t)(g \circ \psi \circ \gamma)'(t) = \int_{\psi\circ\gamma} f\,dg.$$

Let $C \subset \mathbb{P}^2$ be an irreducible projective curve defined by $P \in \mathbb{C}[x, y, z]$. An *abelian integral* is an integral of the form $\int_\gamma f\,dg$ where f and g are rational functions on $C \setminus \mathrm{Sing}(C)$ and γ is a piecewise-smooth path not passing through any poles of $f\,dg$. We usually assume that C is not the line at infinity $\{z = 0\}$ and we can take g to be $g([x : y : z]) = x/z$. If we work in inhomogeneous coordinates $[x : y : 1]$, we write dx for dg. In affine coordinates f becomes a rational function $R(x, y)$ in the usual sense and

$$\int_\gamma f\,dg = \int_\gamma R(x, y)dx,$$

where y is regarded as a multivalued function of x via the equation $P(x, y, 1) = 0$ which defines C in affine coordinates.

Given a lattice $\Lambda \in \mathbb{C}$ we have defined a biholomorphism

$$u\colon \mathbb{C}/\Lambda \to C_\Lambda,$$

where

$$C_\Lambda = \{y^2 z = 4x^3 - g_2(\Lambda)xz^2 - g_3(\Lambda)z^3\}.$$

There is a meromorphic differential on C_Λ given in inhomogeneous coordinates $[x : y : 1]$ by $y^{-1}\,dx$. Let

$$\eta = u^*(y^{-1}\,dx).$$

Then η is a meromorphic differential on $\mathbb{C}/\Lambda$. If $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ is the canonical projection then

$$\pi^*\eta = \pi^* u^*(y^{-1}\,dx) = (u \circ \pi)^*(y^{-1}dx) = (\wp')^{-1} d\wp = (\wp')^{-1}\wp'\,dz = dz.$$

where $z\colon \mathbb{C} \to \mathbb{C}$ denotes the identity function. Since π is locally a holomorphic bijection with a holomorphic inverse and dz is a holomorphic differential on $\mathbb{C}$, it follows that η has no poles. Since u is a holomorphic bijection with a holomorphic inverse it follows that $y^{-1}\,dx$ has no poles.

Now, choose any $\lambda \in \Lambda$ and define $\widetilde{\gamma}\colon [0,1] \to \mathbb{C}$ by $\widetilde{\gamma}(t) = t\lambda$. If $\gamma := \pi \circ \widetilde{\gamma}\colon [0,1] \to \mathbb{C}_\Lambda$ then $\gamma(0) = \Lambda + 0 = \Lambda + \lambda = \gamma(1)$. By definition

$$\int_\gamma \eta = \int_{\widetilde{\gamma}} \pi^*\eta = \int_{\widetilde{\gamma}} dz = \widetilde{\gamma}(1) - \widetilde{\gamma}(0) = \lambda.$$

On the other hand, if $\gamma\colon [0,1] \to \mathbb{C}/\Lambda$ is a piecewise-smooth closed path then we can find a continuous path $\widetilde{\gamma}\colon [0,1] \to \mathbb{C}$ such that $\gamma = \pi \circ \widetilde{\gamma}$. It is locally smooth since π is a holomorphic bijection with holomorphic inverse. Moreover, $\pi \circ \widetilde{\gamma}(a) = \pi \circ \widetilde{\gamma}(b)$, hence

$$\int_\gamma \eta = \int_{\widetilde{\gamma}} \pi^*\eta = \int_{\widetilde{\gamma}} dz = \widetilde{\gamma}(b) - \widetilde{\gamma}(a) \in \Lambda.$$

Thus, we have proved

Proposition 17.9. $\Lambda = \{\int_\gamma \eta \mid \gamma$ *is a piecewise-smooth path in* $\mathbb{C}/\Lambda\}$.

Since $u\colon \mathbb{C}/\Lambda \to C_\Lambda$ is a bijection with a holomorphic inverse and $\eta = u^*(y^{-1}dx)$, we get

Corollary 17.10. $\Lambda = \{\int_\gamma y^{-1}\,dx \mid \gamma$ *is a piecewise-smooth path in* $C_\Lambda\}$.

This means that we can recover the lattice from $C_\Lambda \subset \mathbb{P}^2$. We can also describe u^{-1} in terms of integrals of the differential $y^{-1}\,dx$ on C_Λ.

Proposition 17.11. *The inverse of the holomorphic bijection* $u\colon \mathbb{C}/\Lambda \to C_\Lambda$ *is given by*

$$u^{-1}(p) = \Lambda + \int_{[0:1:0]}^{p} y^{-1}\,dx$$

where the integral is over any piecewise-smooth path γ *in* C_Λ *from* $[0:1:0]$ *to* p.

Note that the definition of $u^{-1}(p)$ makes sense because any two paths from $[0:1:0]$ to p differ by a closed path and a differential along a closed path gives an element in the lattice Λ.

Proof. Suppose that $u(\Lambda + z) = p$ and that $\gamma\colon [a,b] \to C_\Lambda$ is any piecewise-smooth path from $[0:1:0]$ to p. Then $u^{-1} \circ \gamma$ is a path in $\mathbb{C}/\Lambda$ from $\Lambda + 0$ to $\Lambda + z$ and hence

$$\int_\gamma y^{-1}\, dx = \int_{u^{-1}\circ\gamma} u^*(y^{-1}dx) = \int_{u^{-1}\circ\gamma} \eta = \int_{\widetilde{\gamma}} \pi^*\eta = \widetilde{\gamma}(b) - \widetilde{\gamma}(a).$$

Then $\Lambda + \widetilde{\gamma}(b) = u^{-1}(p) = \Lambda + z$ and $\Lambda + \widetilde{\gamma}(a) = u^{-1}([0:1:0]) = \Lambda + 0$ so

$$\Lambda + \int_\gamma y^{-1} dx = \Lambda + \widetilde{\gamma}(b) - \widetilde{\gamma}(a) = \Lambda + z.$$

□

18 23.03. Lecture 18. Abel's theorem

As an application of Bezout theorem we proved that any projective line intersects a smooth cubic in three points, counting with multiplicity. This allowed us to define an abelian group structure on a smooth cubic by the rule that three points add to zero if and only if they lie on an intersection of the cubic with a projective line. The neutral element could be chosen as any (out of nine) inflection point of C.

Weierstrass $\wp$-function allowed us to view a non-singular cubic C_Λ as $\mathbb{C}/\Lambda$, for some lattice $\Lambda \subset \mathbb{C}$. (In fact, one can show that any smooth cubic $\{y^2 z = x(x-z)(x-\lambda z)\} \subset \mathbb{P}^2$ is biholomorphic to $\mathbb{C}_\Lambda$, for some lattice Λ.) Addition in $\mathbb{C}$ endows $\mathbb{C}/\Lambda$ with a structure of an abelian group. Today, we show that these two group structures coincide. In particular, we give the proof of associativity that was missing in Lecture 7.

Recall that a line in $\mathbb{P}^2$ meets a non-singular projective cubic curve C in $\mathbb{P}^2$ either

(a) in three distinct points p, q, r each with multiplicity one, or

(b) in two distinct points p with multiplicity one and q with multiplicity two (i.e. L is tangent to C at q but not at p and q is not an inflection point of C), or

(c) in one point p with multiplicity three (L is tangent to C at p and p is an inflection point of C).

That means that we have

$$\begin{aligned}&p + q + r = 0,\\ &p + 2q = 0,\\ &p + p + p = 0.\end{aligned}$$

Since we choose one inflection point to be the neutral element 0, it follows from the last formula that the remaining inflection points are of order 3.

There are precisely nine points of order 1 or 3 in $\mathbb{C}/\Lambda$:

$$\Lambda + \frac{j}{3}\omega_1 + \frac{k}{3}\omega_2.$$

They can be arranged in a 3×3 array such that the three entries in a row, column or on a diagonal add up to $\Lambda + 0$ (recall Homework II).

In order to state Abel's theorem, recall that C_Λ is defined by polynomial

$$Q_\Lambda(x, y, z) = y^2 z - 4x^3 + g_2(\Lambda) x z^2 + g_3(\Lambda) z^3,$$

map $u\colon \mathbb{C}/\Lambda \to C_\Lambda$ is defined as

$$u(\Lambda + z) = \begin{cases} [\wp(z) : \wp'(z) : 1] & \text{if } z \notin \Lambda,\\ [0 : 1 : 0] & \text{if } z \in \Lambda,\end{cases}$$

and its inverse

$$u^{-1}(p) = \int_{[0:1:0]}^{p} y^{-1} dx.$$

Theorem 18.1 (Abel's theorem for tori)**.** *If $t, v, w \in \mathbb{C}$ then*

$$t + v + w \in \Lambda$$

if and only if there is a line L in $\mathbb{P}^2$ whose intersection with C_Λ consists of the points $u(\Lambda+t)$, $u(\Lambda+v)$, $u(\Lambda+w)$ (allowing for multiplicities).

Equivalently, if $p, q, r \in C_\Lambda$ then

$$\Lambda + \int_{[0:1:0]}^{p} y^{-1}dx + \int_{[0:1:0]}^{q} y^{-1}dx + \int_{[0:1:0]}^{r} y^{-1}dx = \Lambda + 0$$

if and only if p, q, r are the points of intersection of C_Λ with a line in $\mathbb{P}^2$.

Proof. First, we show that if L is a line intersecting C in p, q, r (allowing for multiplicities) then

$$\Lambda + \int_{[0:1:0]}^{p} y^{-1}dx + \int_{[0:1:0]}^{q} y^{-1}dx + \int_{[0:1:0]}^{r} y^{-1}dx = \Lambda + 0.$$

Case 1 Suppose that L is tangent line $z = 0$ to C_Λ at the point of inflection $[0:1:0]$. Then $p = q = r = [0:1:0]$.

Case 2 Suppose that $L = \{cy = bz\}$. Then L meets C_Λ in three points

$$p_1(b,c) = [a_1 : b : c], \qquad p_2(b,c) = [a_2 : b : c], \qquad p_3(b,c) = [a_3 : b : c]$$

where a_1, a_2, a_3 are roots of $Q_\Lambda(x, b, c)$.

Define

$$\mu\colon \mathbb{P}^1 \to \mathbb{C}/\Lambda, \quad \mu([b:c]) = \Lambda + \int_{[0:1:0]}^{p_1(b,c)} y^{-1}dx + \int_{[0:1:0]}^{p_2(b,c)} y^{-1}dx + \int_{[0:1:0]}^{p_3(b,c)} y^{-1}dx.$$

We shall prove later that μ is holomorphic and that any holomorphic map from $\mathbb{P}^1 \to \mathbb{C}$ is constant.

Since $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ is a covering projection, we can lift μ to a map $\widetilde{\mu}\colon \mathbb{P}^1 \to \mathbb{C}$. Since π is holomorphic and locally an isomorphism, $\widetilde{\mu}$ is holomorphic, hence constant. It follows that $\mu(b,c) = \mu(1,0) = \Lambda + 0$.

Case 3 Suppose that L is any line. Then

$$L = \{sx + t(cy - bz) = 0\}.$$

Fix b, c and let $L \cap C_\Lambda = \{q_1(s,t), q_2(s,t), q_3(s,t)\}$. Define $\nu\colon \mathbb{P}^1 \to \mathbb{C}/\Lambda$ by

$$\nu([s:t]) = \Lambda + \int_{[0:1:0]}^{q_1(s:t)} y^{-1}dx + \int_{[0:1:0]}^{q_2(s,t)} y^{-1}dx + \int_{[0:1:0]}^{q_3(s,t)} y^{-1}dx.$$

As in Case 2 ν is holomorphic, hence constant, i.e. $\nu([s:t]) = \nu([0:1]) = \Lambda + 0$.

It follows from the definition of u^{-1} that if t, v, w are complex numbers and $u(\Lambda + t)$, $u(\Lambda + v)$, $u(\Lambda + w)$ are the intersection points of C_Λ with a line in $\mathbb{P}^2$ then $t + v + w \in \Lambda$.

Let now $t, v, w \in \mathbb{C}$ be such that $t + v + w \in \Lambda$. Let

$$p = u(\Lambda + t), \qquad q = u(\Lambda + v), \qquad r = u(\Lambda + w)$$

If $p \neq q$ let L be a line in $\mathbb{P}^2$ through p and q. If $p = q$ let L be the line tangent to C_Λ at p. In both cases L meets C_Λ in another point $\widetilde{r}$. Then

$$u^{-1}(p) + u^{-1}(q) + u^{-1}(\widetilde{r}) = \Lambda + 0 = \Lambda + t + v + w = u^{-1}(p) + u^{-1}(q) + u^{-1}(r).$$

It follows that $r = \widetilde{r}$ which finishes the proof. □

In the proof we have used

Lemma 18.2. *Map* $\mu \colon \mathbb{P}^1 \to \mathbb{C}/\Lambda$,

$$\mu([b : c]) = \Lambda + \int_{[0:1:0]}^{p_1(b,c)} y^{-1} dx + \int_{[0:1:0]}^{p_2(b,c)} y^{-1} dx + \int_{[0:1:0]}^{p_3(b,c)} y^{-1} dx$$

is holomorphic.

Proof. For all but finitely many $b \in \mathbb{C}$ the partial derivative $\frac{\partial Q_\Lambda}{\partial x}(x, y, z)$ is non-zero at $(a, b, 1)$ when $Q_\Lambda(a, b, 1) = 0$. For such a, b the polynomial $Q(x, b, 1)$ has three distinct roots a_1, a_2, a_3 and $p_i(b, 1) = [a_i, b, 1]$.

By the implicit function theorem applied to $Q_\Lambda(x, y, 1)$ there are open neighbourhoods U and V_1, V_2, V_3 of b and a_1, a_2, a_3 and holomorphic functions $g_i \colon U \to V_i$ such that if $x \in V_i$ and $y \in U$ then

$$Q_\Lambda(x, y, 1) = 0 \Leftrightarrow x = g_i(y).$$

Hence, there are holomorphic maps $U \to C_\Lambda$ given by

$$\psi_i(w) = [g_i(w) : w : 1].$$

We may choose V_1, V_2 and V_3 to be disjoint. This means that for $w \in U$, $g_1(w)$, $g_2(w)$ and $g_3(w)$ are distinct roots of the polynomial $Q_\Lambda(x, w, 1)$, so

$$p_i(w, 1) = [g_i(w) : w : 1] = \psi_i(w).$$

Thus, if γ is a path in U from b to w then $\psi_i \circ \gamma$ is a path in V_i from $p_i(b, 1)$ to $p_i(w, 1)$ and

$$\int_{\psi_i \circ \gamma} y^{-1} dx = \int_\gamma \frac{g_i'(w)}{w} \, dw.$$

Then

$$\mu[w : 1] = \mu[b : 1] + \sum_{1 \le i \le 3} \int_b^w \frac{g_i'(y)}{y} \, dy$$

We can find Puiseux expansions to see that $g_i'(0) = 0$. It follows functions $\frac{g_i'(y)}{y}$ are holmorphic on U, so their integrals from b to w are holomorphic functions of w near b. It follows that μ is holomorphic in a neighbourhood of $[b : 1]$.

We have shown that μ is holomorphic except possibly at finitely many points of $\mathbb{P}^1$. Since a bounded continuous function $U \to \mathbb{C}$ whose restriction to $U \setminus \{a\}$ is holomorphic, is holomorphic on U, we conclude that μ is holomorphic. □

Lemma 18.3. *Any holomorphic map* $f\colon \mathbb{P}^1 \to \mathbb{C}$ *is constant.*

Proof. We can regard f as a map $\mathbb{P}^1 \to \mathbb{P}^1$ which never takes the value $\{\infty\}$. We proved in Lemma 11.11 that any non-constant holomorphic map $\mathbb{P}^1 \to \mathbb{P}^1$ is rational, hence takes value ∞ at least one. □

19 27.03. Lecture 19. The Riemann-Roch theorem

We shall relate genus g of a curve to meromorphic differentials on the curve.

Definition 19.1. A *divisor* D on C is a formal sum

$$D = \sum_{p \in C} n_p p$$

such that $n_p \in \mathbb{Z}$ for every $p \in C$ and $n_p = 0$ for all but finitely many $p \in C$. The *degree* of D is then

$$\deg(D) = \sum_{p \in C} n_p$$

If $n_p = 0$ for $p \notin \{p_1, \ldots, p_l\}$ then we also write

$$D = m_1 p_1 + \ldots + m_k p_k.$$

We add and subtract divisors by adding and subtracting n_p's. This way divisors on C form a $\mathbb{C}$-vector space $\mathrm{Div}(C)$ and the degree defines a group homomorphism $\deg\colon \mathrm{Div}(C) \to \mathbb{Z}$.

If $n_p \geq 0$ for all p, we write $D \geq 0$ and say that D is *effective*. We write $D \geq D'$ if $D - D' \geq 0$.

Let f be a meromorphic function on C which is not identically zero. If $p \in C$ we can choose a holomorphic chart $\varphi_\alpha\colon U_\alpha \to V_\alpha$ on C such that $p \in U_\alpha$. Then $f \circ \varphi_\alpha^{-1}$ is a meromorphic function on $V_\alpha \subset \mathbb{C}$. If m is a positive integer we say that f has a pole or a zero of multiplicity m at p if $f \circ \varphi_\alpha^{-1}$ has a pole or a zero of multiplicity m at $\varphi_\alpha(p)$. This is independent of the choice of a chart.

Similarly, if $\omega = f\,dg$ is a meromorphic differential on C which is not identically zero we say that ω has a pole or a zero of multiplicity m at p if the meromorphic function $(f \circ \varphi_\alpha^{-1})(g \circ \varphi_\alpha^{-1})'$ which represents ω has a pole or zero of multiplicity m at $\varphi_\alpha(p)$.

Definition 19.2. The *divisor of a meromorphic function* f which is not identically zero is

$$(f) = \sum n_p\, p$$

where $n_p = -m$ if f has a pole of multiplicity m at p, $n_p = m$ if f has a zero of multiplicity m at p and $n_p = 0$ otherwise. Note that

$$(fg) = (f) + (g), \qquad \left(\frac{f}{g}\right) = (f) - (g).$$

A divisor of a meromorphic function is called *principal.* Two divisors are *linearly equivalent*,

$$D \sim D'$$

if $D - D'$ is principal.

If ω is a meromorphic differential we can analogously define (ω). The divisor of a meromorphic differential is called a *canonical divisor* and is denoted κ. We have noted that if η is another meromorphic differential then $\eta = f\omega$, hence

$$(\eta) = (f) + (\omega) \sim (\omega),$$

i.e. the linear equivalence class of canonical divisor is well-defined.

Proposition 19.3. *A principal divisor on C has degree zero; that is a meromorphic function on C which is not identically zero has the same number of zeroes and poles, counted with multiplicities.*

Proof. Let g be a meromorphic function. We consider meromorphic differential $\frac{dg}{g}$ which has poles precisely at the points $q_1, \dots, q_t$ where g has zeroes or poles. The same remains true if we compose g with some holomorphic chart.

We can choose coordinates in $\mathbb{P}^2$ such that $[0 : 1 : 0] \notin C$ and $0, \varphi(q_1), \dots, \varphi(q_t)$ and ∞ are distinct and are not branch points of the map $\varphi\colon C \to \mathbb{P}^1$ defined by $[x : y : z] \mapsto [x : z]$. We can choose a triangulation (V, E, F) of $\mathbb{P}^1$ such that every branch point of φ belongs to V and $\varphi(q_1), \dots, \varphi(q_t)$ and 0 are in the interior of a face f_0 while ∞ is in the interior of a different face $f_\infty \in F$. There is a triangulation $(\widetilde{V}, \widetilde{E}, \widetilde{F})$ of C given by

$$\begin{aligned}
\widetilde{V} &= \varphi^{-1}(V), \\
\widetilde{E} &= \{\widetilde{e}\colon [0, 1] \to C \text{ continuous, s.t. } \varphi \circ \widetilde{e} \in E\}, \\
\widetilde{F} &= \{\widetilde{f}\colon \Delta \to C \text{ continuous, s.t. } \varphi \circ \widetilde{f} \in F\}.
\end{aligned}$$

By subdividing the triangulation if necessary we can assume that each face has at most one branch point among its vertices. This means that if $\widetilde{f} \in \widetilde{F}$ and $f = \psi \circ \widetilde{f} \in F$ then

$$\varphi\colon \widetilde{f}(\Delta) \to f(\Delta)$$

is a homeomorphism whose restriction to $\widetilde{f}(\Delta - \{(0, 0), (0, 1), (1, 0)\})$ is the restriction of a holomorphic chart if $f \neq f_\infty$. If $f = f_\infty$ we need to compose φ with the mapping $z \to \frac{1}{z}$. The boundary $\widetilde{\gamma}$ of $\widetilde{f}(\Delta)$ in C is the image of $f \circ \sigma_i$, it is a closed piecewise-smooth path (recall that we chose an orientation on the boundary of Δ). Then

$$\sum_{\widetilde{f} \in \widetilde{F}} \int_{\widetilde{\gamma}} \frac{dg}{g} = 0$$

because if σ_i is the edge of some face f then σ_i^{-1} is the edge of another face in the triangulation.

On the other hand $\int_{\widetilde{\gamma}} \frac{dg}{g} = \int_\gamma \frac{(g \circ \varphi^{-1})'(z)}{(g \circ \varphi^{-1})(z)} dz$. The meromorphic function $\frac{(g \circ \varphi^{-1})'(z)}{(g \circ \varphi^{-1})(z)}$ has a pole at a point a inside γ with residue ρ if and only if $g \circ \varphi^{-1}$ has either a zero at a with multiplicity ρ or a pole at a with multiplicity $-\rho$. Since $\varphi|_{\widetilde{f}(\Delta^o)}$ is a chart, the statement is equivalent to g having a zero or a pole at $\varphi^{-1}(a)$. It follows from Cauchy's theorem that

$$\int_\gamma \frac{(g \circ \varphi^{-1})'(z)}{(g \circ \varphi^{-1})(z)} dz = \pm(Z(\widetilde{f}) - P(\widetilde{f}))$$

and the sign depends on whether γ is positively or negatively oriented in $\mathbb{C}$. Since we assumed that all poles of dg/g lie in $\varphi^{-1}(f_0(\Delta^0))$, the sign is consistent, hence

$$\sum_{\widetilde{f} \in \widetilde{F}, \varphi \circ \widetilde{f} = f_0} \int_{\widetilde{g}} \frac{dg}{g} = \pm(Z - P).$$

As the remaining faces of $\widetilde{F}$ contain no poles we get that

$$0 = \sum_{\widetilde{f} \in \widetilde{F}} \int_{\widetilde{\gamma}} \frac{dg}{g} = \pm(Z - P).$$

□

It remains to prove that we can choose a triangulation of $\mathbb{P}^2$ such that 0 and $\varphi(q_i)$ lie in an interior of one face.

Lemma 19.4. *Let $\{p_1, \dots, p_r, q_1, \dots, q_s\}$ be a set of $r + s$ points in $\mathbb{C}$ with $r \geq 3$. Then there is a triangulation (V, E, F) of $\mathbb{P}^1$ such that $V = \{p_1, \dots, p_r\}$ and the set $\{q_1, \dots, q_s\}$ is contained in the interior of a face. We may also assume that ∞ is in the interior of a different face.*

Proof. The proof is by induction on r. First suppose $r = 3$.

We can assume that 0 does not belong to the set $\{p_1, \dots, p_r, q_1 \dots, q_s\}$ and that arguments of the complex numbers $p_1, p_2, p_3, q_1, \dots, q_s$ are distinct. We can choose a real number R such that

$$R > \max\{|p_1|, |p_2|, |p_3|, |q_1|, \dots, |q_s|\}$$

and we can reorder p_1, p_2, p_3 so that

$$\arg(p_1) < \arg(p_2) < \arg(p_3).$$

For $\varepsilon > 0$ sufficiently small we can find a piecewise smooth path from p_1 to $R\exp(\varepsilon + \arg(p_1))$, then along the arc of the circle of radius R to $R\exp(-\varepsilon + \arg(p_2))$ and then to p_2. Three paths like that give a triangulation of $\mathbb{P}^1$ with two faces, the interior, containing $q_1, \dots, q_s$, and the exterior, containing ∞.

Now assume the inductive hypothesis that $r > 3$ and that we have a triangulation of $\mathbb{P}^1$ with vertices $p_1, \dots, p_{r-1}$. If p_r lies in the interior of the face containing $q_1, \dots, q_s$ or on its boundary we can choose a subdivision of that face so that all $q_1, \dots, q_s$ still lie in one face.

In the remaining cases we proceed as in the proof that we have a triangulation of $\mathbb{P}^1$ with prescribed vertices, i.e. we choose arbitrary subdivisions. □

Corollary 19.5. *Two linearly equivalent divisors on C have the same degree. In particular, the degree of a canonical divisor is well-defined.*

Proposition 19.6. *If κ is a canonical divisor on C then*

$$\deg \kappa = 2g - 2.$$

Proof. Since any two meromorphic differentials are linearly equivalent, it is enough to show that there is some meromorphic differential ω on C such that $\deg(\omega) = 2g - 2$.

Let $P(x, y, z)$ be a homogeneous polynomial defining C. We may assume that $[0 : 1 : 0] \notin C$ so that the coefficient $P(0, 1, 0)$ of y^d in P is non-zero. Then $\frac{\partial P}{\partial y}$ is not identically zero. Since C is irreducible, $\{\frac{\partial P}{\partial y} = 0\}$ does not have common components with C and by Bezout's theorem there are finitely many points of C at which $\frac{\partial P}{\partial y}$ vanishes. Since $[0 : 1 : 0] \notin C$, applying a projective transformation of the form

$$x \mapsto x, \qquad y \mapsto y, \qquad z \mapsto \alpha x + z$$

we can assume that if $[a : b : c] \in C$ and $\frac{\partial P}{\partial y} = 0$ then $c \neq 0$.

Let now $\omega = d(x/z)$. Near points $[a : b : c] \in C$ with $c \neq 0$ and $\frac{\partial P}{\partial y}(a, b, c) \neq 0$ we can take x/z as a local holomorphic chart (because the implicit function theorem allow us to represent y as a holomorphic function of x). It follows that ω has no poles and no zeroes at such points.

At a point $[a : b : 0] \in C$ we have $a \neq 0$ (because $[0 : 1 : 0] \notin C$) and $\frac{\partial P}{\partial y} \neq 0$ (by our choice of coordinates). Then $v = z/x$ is a local holomorphic chart and

$$\omega = d(1/v) = -v^{-2}\, dv$$

has a pole of multiplicity 2. By assumption, $\frac{\partial P}{\partial y}(a, b, 0) \neq 0$ whenever $[a : b : 0] \in C$ so line $\{z = 0\}$ is not tangent to C at any of the intersection points. It follows that the intersection multiplicity at each point of intersection is equal to one, hence by Bezout theorem, there are precisely d points of intersection. These points contribute to $-2d$ of degree of ω.

It remains to consider points with $\frac{\partial P}{\partial y}(a, b, c) = 0$. These are precisely ramification points of map $\varphi\colon C \to \mathbb{P}^1$, $\varphi([x : y : z]) = [x : z]$. At these points c is non-zero, so $\frac{\partial P}{\partial x}(a, b, c) \neq 0$. (If it was then Euler's relation would give that $c\frac{\partial P}{\partial z}(a, b, c) = 0$, hence $\frac{\partial P}{\partial z}(a, b, c) = 0$ which cannot happen because C is non-singular). Therefore,

$$u = y/z$$

is a local chart and locally x/z is a holomorphic function $f(u)$ of u satisfying

$$P(f(u), u, 1) = 0.$$

Differentiating we get

$$P_x f' + P_y = 0$$
$$P_{xx}(f')^2 + P_{xy}f' + P_x f^{(2)} + P_{xy}f' + P_{yy} = P_{xx}(f')^2 + 2P_{xy}f' + P_x f^{(2)} + P_{yy} = 0$$

If $f'(u_0) = 0$ then

$$f^{(2)} = -\frac{P_{yy}}{P_x}.$$

In general, if $f^{(k)}(u_0) = 0$, for $k = 1, \ldots, m-1$ then

$$f^{(m)}(u_0) = -\frac{\frac{\partial^m}{\partial y^m} P(u_0, f(u_0), 1)}{\frac{\partial P}{\partial x}(u_0, f(u_0), 1)}.$$

It follows that the smallest positive integer m such that $f^{(m)}(u_0) \neq 0$ is equal to the smallest positive integer m such that

$$\frac{\partial^m P}{\partial y^m}(u_0, f(u_0), 1) \neq 0.$$

Since

$$\omega = d(f(u)) = f'(u)\, du$$

the multiplicity of zero of ω at a ramification point of φ is precisely one less than the ramification index at this point. We can assume that the coordinates were chosen is such a way that there are precisely $d(d-1)$ ramification points, each of index 2. Hence, we get $d(d-1)$ to the degree of ω.

To sum up

$$\deg \omega = d^2 - d - 2d = d(d-3).$$

By degree-genus formula,

$$\deg \omega = 2g - 2.$$

□

To state Riemann-Roch theorem we need one more definition

Definition 19.7. Let $D = \sum_{p \in C} n_p\, p$ be a divisor on C. Then $\mathcal{L}(D)$ is a complex vector space

$$\mathcal{L}(D) = \{(f) \mid (f) + D \geq 0\} \cup 0.$$

In other words a meromorphic function f on C belongs to $\mathcal{L}(D)$ if it is holomorphic outside of the points for which $n_p > 0$ and the order of the pole at such a point is at most n_p. Moreover, f has zero of order at least $-n_p$ at every $p \in C$ such that $n_p < 0$.

We define

$$l(D) = \dim \mathcal{L}(D).$$

Theorem 19.8 (Riemann-Roch)**.** *If D is any divisor on a non-singular projective curve C of genus g in $\mathbb{P}^2$ and κ is a canonical divisor on C then*

$$l(D) - l(\kappa - D) = \deg(D) + 1 - g.$$

20 30.03. Lecture 20. The proof of the Riemann-Roch theorem

On the last lecture we defined divisors, linear equivalence, degree of divisor and a vector space $\mathcal{L}(D)$. We calculated degree of the canonical divisor κ, i.e. divisor of any meromorphic differential on C, to be $2g-2$. We formulated the theorem

Theorem 19.8 (Riemann-Roch). *If D is any divisor on a non-singular projective curve C of genus g in $\mathbb{P}^2$ and κ is a canonical divisor on C then*

$$l(D) - l(\kappa - D) = \deg(D) + 1 - g.$$

We have also proved that degree of a principal divisor (f) is 0. As a corollary, we get

Corollary 20.1. *If $\deg(D) < 0$ then $l(D) = 0$.*

Proof. If f is meromorphic function which is not identically zero such that

$$(f) + D \geq 0$$

then $\deg D = \deg((f) + D) \geq 0$. □

Lemma 20.2. *If $D \sim D'$ then $l(D) = l(D')$.*

Proof. If $D = D' + (g)$ then $f \mapsto fg$ defines an isomorphism between $\mathcal{L}(D)$ and $\mathcal{L}(D')$. □

Before proving Riemann-Roch theorem, we discuss some of its corollaries

Corollary 20.3. *The genus of a non-singular projective curve C in $\mathbb{P}^2$ equals the dimension $l(\kappa)$.*

Proof. Riemann-Roch theorem for $D = 0$ tell us that

$$l(0) - l(\kappa) = 1 - g.$$

$l(0)$ is the dimension of the vector space of holmorphic functions. We shall see soon that any holomorphic function is constant, hence $l(0) = 1$. (More precisely, we will show that $0 \leq l(0) - l(-p) \leq 1$. Together with $l(-p) = 0$ and the fact that there is a one dimensional space of constant functions we get $l(0) = 1$.) □

Let now

$$H = \sum_{p \in C} I_p(C, L)p$$

where $I_p(C, L)$ is the intersection multiplicity at p of C with a line L in $\mathbb{P}^2$, $L = R(x, y, z) = 0$.

By Bezout's theorem, H is a divisor of degree d on C. Then

$$\deg(\kappa - mH) = \deg \kappa - md$$

becomes negative if m is large enough. Thus, for $m >> 0$, $l(\kappa - mH) = 0$. On the other hand, if $Q(x, y, z)$ is any homogeneous polynomial of degree m then

$$f = \frac{Q(x, y, z)}{R^m(x, y, z)}$$

is a meromorphic function such that $(f)+mH \geq 0$, i.e. $f \in \mathcal{L}(mH)$. Any two polynomials Q and Q' define the same function on C if and only if their difference is divisible by P. Hence, if $\mathbb{C}_k[x, y, z]$ is the vector space of homogenous polynomials of degree k then

$$\begin{aligned} &l(mH) \geq \dim \mathbb{C}_m[x, y, z]/P(x, y, z)\mathbb{C}_{m-d}[x, y, z] \\ &= \frac{1}{2}(m+1)(m+2) - \frac{1}{2}(m-d+1)(m-d+2) = md + \frac{1}{2}d(3-d) = md + 1 - g. \end{aligned}$$

We have shown that

$$l(mH) - l(\kappa - mH) \geq \deg(mH) + 1 - g,$$

when m is large enough. Riemann-Roch theorem tells us that there is equality, i.e. that any meromorphic function is of the form Q/R^m and in particular it is rational.

The same argument shows that if there are many lines $L_1, \ldots, L_m$ in $\mathbb{P}^2$ and corresponding divisors H_j then if m is large enough every meromorphic function on C satisfying

$$(f) + H_1 + \ldots + H_m \geq 0$$

is rational. On the other hand, every meromorphic function f satisfies

$$(f) + H_1 + \ldots + H_m \geq 0$$

for some lines $L_1, \ldots, L_m$ (we just need to take lines through the poles). It proves

Theorem 20.4. *All meromorphic functions on a non-singular projective curve in $\mathbb{P}^2$ are rational.*

In order to prove Riemann-Roch, we need three lemmas

Lemma 20.5. *Given any divisor D on C and any positive integer m_0 there exists $m \geq m_0$ and points $p_1 \ldots, p_k$ of C (not necessarily distinct) such that*

$$D + p_1 + \ldots + p_k \sim mH.$$

Proof. By adding points to D we can assume that $D \geq 0$. For each of $p \in C$ such that $n_p > 0$ we can choose a line through p whose points of intersections with C (allowing for multiple intersections) are

$$q_1^{(p)} = p, q_2^{(p)}, \ldots, q_d^{(p)}.$$

Since the ratio of two linear homogeneous polynomials in x, y, z defines a meromorphic function on C,

$$q_1^{(p)} + \ldots + q_d^{(p)} \sim H.$$

Then

$$mH \sim \sum_{n_p > 0} n_p\, p + \sum_{2 \leq i \leq d} n_p q_i^{(p)} = D + \sum p_j.$$

□

Lemma 20.6. *Let ω be a meromorphic differential on C with precisely one pole. Then this pole is not a simple pole (i.e. its multiplicity is at least 2).*

Proof. Let us assume that $\omega = g\,dh$ has one simple pole at q. We choose $\varphi\colon C \to \mathbb{P}^1$ a branch cover such that 0, $\varphi(q)$ and ∞ are distinct and are not branch points of φ. As on the previous lecture, we chose a triangulation of $\mathbb{P}^1$ such that every branch point is a vertex, 0 and $\varphi(q)$ belong to the same face f_0 and ∞ to another face f_∞. It induces a triangulation of C and, considering subdivision if necessary, we can assume that we have a triangulation $(\widetilde{V}, \widetilde{E}, \widetilde{F})$ of C such that each face has at most one branch point among its vertices. It means that $\varphi|_{\widetilde{f}(\Delta)}$ is a holomorphic chart (we might have to compose it with $z \mapsto 1/z$ if $\varphi \circ \widetilde{f} = f_\infty$.

The boundary $\widetilde{\gamma}$ of every face in $\widetilde{F}$ is a closed path. Then

$$\int_{\widetilde{\gamma}} \omega = \int_{\gamma} (g \circ \varphi^{-1})(h \circ \varphi^{-1})'(z) dz$$

where $\gamma = \varphi \circ \widetilde{\gamma}$. If $q \in \widetilde{f}(\Delta^0)$ then $f = f_0$ and $(g \circ \varphi^{-1})(h \circ \varphi^{-1})'$ has a simple pole inside γ. Since a residue of a simple pole is always non-zero, we get by Cauchy's theorem $\int_{\widetilde{\gamma}} \omega \neq 0$. By assumption q was the only pole of ω, hence the integral along boundary of any other face in $\widetilde{F}$ is zero. Thus, we get

$$\sum_{\widetilde{f} \in \widetilde{F}} \int_{\widetilde{\gamma}} \omega \neq 0.$$

On the other hand, in the above sum each edge σ_i occurs twice, once as σ_i, once as σ_i^{-1}, hence the integrals cancel each other. It follows that

$$\sum_{\widetilde{f} \in \widetilde{F}} \int_{\widetilde{\gamma}} \omega = 0.$$

The contradiction shows that our assumption that ω has one simple pole cannot be satisfied. ☐

Lemma 20.7. *If D is a divisor on C, κ is a canonical divisor and p is any point of C then*

$$0 \leq l(D+p) - l(\kappa - D - p) - l(D) + l(\kappa - D) \leq 1.$$

Proof. Suppose that $D = \sum_{q \in C} n_q q$. Then $\mathcal{L}(D) = \mathcal{L}(D+p)$ if there is no meromorphic function f such that

$$(f) + D + p \geq 0$$

and f has a pole at p of multiplicity exactly $n_p + 1$ (if $n_p \geq 0$) or a zero at p of multiplicity exactly $-n_p - 1$ (if $n_p < 0$). Otherwise, $\mathcal{L}(D)$ is a linear subspace of $\mathcal{L}(D+p)$ of codimension one (because if we subtract λf, for some $\lambda \neq 0$, we get an element of $\mathcal{L}(D)$). In particular,

$$0 \leq l(\mathcal{D}+p) - l(\mathcal{D}) \leq 1, \quad 0 \leq l(\kappa - D) - l(\kappa - D - p) \leq 1.$$

We need to show that we cannot have

$$l(\mathcal{D}-p)-l(\mathcal{D})=1=l(\kappa-D)-l(\kappa-D-p).$$

If we had there would be functions f and g with

$$(f)+D+p\geq 0,\quad (g)+\kappa-D\geq 0$$

with "equalities" at the point p. Then $fg\omega$ is a meromorphic differential

$$(fg\omega)=(f)+(g)+\kappa\geq -D-p+D=-p,$$

i.e. a differential with a pole of order exactly one at p. By the previous Lemma, we know that such a meromorphic differential does not exist. Hence, either $l(D+p)-l(D)=0$ or $l(\kappa-D)-l(\kappa-D-p)=0$. □

Proof of Riemann-Roch theorem. First, we check that $l(D)-l(\kappa-D)\geq \deg D-g+1$.

We saw that there exists m_0 such that if $m\geq m_0$ then

$$l(mH)-l(\kappa-mH)\geq \deg(mH)-g+1.$$

We also know that $D+p_1+\ldots+p_k\sim mH$. Then

$$\deg(mH)=\deg(D)+k,\quad l(mH)-l(\kappa-mH)=l(D+p_1+\ldots+p_k)-l(\kappa-D-p_1-\ldots-p_k).$$

By the above Lemma

$$l(D+p_1+\ldots+p_k)-l(\kappa-D-p_1-\ldots-p_k)-l(D)+l(\kappa-D)\leq k.$$

Then

$$\begin{aligned}l(D)-l(\kappa-D)&\geq l(D+p_1+\ldots+p_k)-l(\kappa-D-p_1-\ldots-p_k)-k\\&=l(mH)-l(\kappa-mH)-k\\&\geq \deg(mH)-g+1-k=\deg(D)-g+1.\end{aligned}$$

The inequality applied to $\kappa-D$ gives

$$l(\kappa-D)-l(D)\geq \deg(\kappa-D)-g+1=2g-2-\deg(D)-g+1=-\deg(D)+g-1.$$

Hence, we have equality. □

Riemann-Rich theorem gives us an alternative proof of the associativity of the abelian group structure on a smooth cubic curve in $\mathbb{P}^2$:

Let p,q,r be points of C and let

$$a=p+q,\qquad b=a+r=(p+q)+r,\qquad c=q+r,\qquad d=p+c=p+(q+r).$$

Since p, q and $-a$ are collinear, there is a homogeneous linear polynomial which vanishes at p, q and $-a$. Similarly, there is a linear polynomial which vanishes at a, $-a$ and p_0. The ratio of these polynomials defines a rational function φ with zeros at p and q and poles at a and p_0. By the same argument there is a rational function ψ with zeros at a and r and poles at b and p_0. Then $\varphi\psi$ is a meromorphic function with zeros at p, q, r and poles at b and p_0 (with multiplicity two). Similarly, there is a meromorphic function with zeros at p, q, r and poles at d and p_0 (with multiplicity two). If $b \neq d$ the ratio of these functions is a meromorphic function with single zero at d and single pole at b.

By the degree-genus formula, $g = 1$, so $\deg(\kappa) = 0$ and $\kappa - b < 0$, i.e. $l(\kappa - b) = 0$. Then Riemann-Roch theorem gives

$$l(b) = \deg(b) + 1 - g = 1$$

so the only meromorphic functions with at most simple poles at b are constant. It follows that $b = d$.

31.03. Workshop V

1. Show that given any nine points in $\mathbb{P}^2$ there is a cubic which contains them.

2. Let S be a compact connected Riemann surface of genus zero. Assuming that the Riemann-Roch theorem applies to S show that if $D = p$, for any $p \in S$, then $l(D) = 2$. Deduce that there exists a meromorphic function f on S with a simple pole at p and no other poles. (In fact f is a holomorphic bijection $S \to \mathbb{P}^1$).

3. Show that a point $p \neq [0 : 1 : 0]$ of the cubic curve C_Λ associated with a lattice $\Lambda \subset \mathbb{C}$ has order two if and only if the tangent to C_Λ at p passes through $[0 : 1 : 0]$. Show that the points of order two form a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4. Use Abel's theorem to show that if $u, v, w \in \mathbb{C} \setminus \Lambda$ and u, v, w are distinct modulo Λ then $u + v + w \in \Lambda$ is and only if

$$0 = \det \begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix}.$$

Solution to Workshop V

1. A general equation of a cubic is

$$P(x,y,z) = Ax^3+By^3+Cz^3+Dx^2y+Ex^2z+Fxy^2+Gy^2z+Hxz^2+Iyz^2+Jxyz = 0$$

Let $p_1 = [a_1 : b_1 : c_1], \ldots, p_9 = [a_9 : b_9 : c_9 : d_9]$ be arbitrary points in $\mathbb{P}^2$. The cubic $C = \{P(x,y,z) = 0\}$ contains them if and only if $P(a_i, b_i, c_i) = 0$ for $i = 1, \ldots, 9$. Thus, we get 9 linear equations on coefficients $A, \ldots, J$:

$$\left\{ \ P(a_i, b_i, c_i) = 0. \right.$$

By rank-nullity theorem the space of solutions is at least one dimensional, hence there exists a non-zero solution $(A_0, \ldots, J_0)$. Then

$$\{A_0x^3+B_0y^3+C_0z^3+D_0x^2y+E_0x^2z+F_0xy^2+G_0y^2z+H_0xz^2+I_0yz^2+J_0xyz = 0\}$$

is a cubic which contains $p_1, \ldots, p_9$.

2. We know that $\deg(\kappa) = 2g - 2 = -2$. It follows that $\deg(\kappa - p) = -3 < 0$ so $l(\kappa - p) = 0$. Then Riemann-Roch theorem gives

$$l(p) - l(\kappa - p) = l(p) = \deg(p) - g + 1 = 1 - 0 + 1 = 2.$$

Constant function f_0 has a trivial divisor, hence $(f_0) + p > 0$. Since $\mathcal{L}(0) \subsetneqq \mathcal{L}(p)$ there exists f which is an element of $\mathcal{L}(p)$ but not of $\mathcal{L}(0)$. It follows that (f) is not effective, i.e. f has some poles. Since $(f) + p \geq 0$, f can have only one pole of multiplicity one at p. Hence, f is the required meromorphic function.

3. By Abel's theorem three point q, r and s on C_Λ add up to zero if and only if they are collinear. We have $2q = -r$ where r is the second intersection point of the line tangent to C at q with C. It follows that $2p = 0 = -0$ if and only if the tangent line to C at p contains $0 \in C_\Lambda$, i.e. the point $[0 : 1 : 0]$.

We know that C_Λ is given by $Q(x,y,z) = y^2z - 4x^3 - g_2xz^2 + g_3z^3$. A point $[0 : 1 : 0]$ lies on the line tangent to C_Λ at $p = [a : b : c]$ if and only if

$$0 \cdot \frac{\partial Q}{\partial x}(a,b,c) + 1 \cdot \frac{\partial Q}{\partial y}(a,b,c) + 0 \cdot \frac{\partial Q}{\partial z}(a,b,c) = 0,$$

i.e. if and only if $0 \cdot \frac{\partial Q}{\partial y}(a,b,c) = 2bc = 0$.

It follows that points of order two lie on the intersection of C_Λ with lines $\{z = 0\}$ and $\{y = 0\}$. If $z = 0$ then $x = 0$ so the neutral element $[0 : 1 : 0]$ is the only point of intersection $C_\Lambda \cap \{z = 0\}$. If $y = 0$ then $4x^3 + g_2xz^2 - g_3z^3 = 4(x - \mu_1z)(x - \mu_2z)(x - \mu_3z) = 0$. Since curve C_Λ is non-singular, all μ_i are pairwise distinct. It follows that $C_\Lambda \cap \{z = 0\} = \{p_1, p_2, p_3\}$.

By assumption $p_i = -p_i$. Moreover, as p_1, p_2, p_3 are collinear, $p_1 + p_2 + p_3 = 0$. It follows that $p_3 = -p_3 = p_1 + p_2$. Then a map $p_1 \mapsto (1,0)$, $p_2 \mapsto (0,1)$, $p_3 \mapsto (1,1)$, $0 \mapsto (0,0)$ is an isomorphism of the subgroup of C_Λ with points of order two and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4. The sum $u+v+w$ lies in Λ if and only if $u+v+w=0$ in $\mathbb{C}/\Lambda$. Riemann surface $\mathbb{C}/\Lambda$ is biholomorphic to C_Λ and the map is given by $z \mapsto u(z) = [\wp(z) : \wp'(z) : 1]$, for $z \notin \Lambda$. Thus $u+v+w \in \Lambda$ if and only if $u(u)+u(v)+u(w)=0$ in C_Λ, i.e. if and only if $u(u)$, $u(v)$ and $u(w)$ are collinear. By assumption $u(u)$, $u(v)$ and $u(w)$ are distinct. There exists a line $\alpha x+\beta y+\gamma z=0$ which contains all three points if and only if

$$\begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0$$

for some $(\alpha, \beta, \gamma) \neq (0,0,0)$. A vector (α,β,γ) like this exists if and only if $\det \begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} = 0$ which finishes the proof.

21 03.04. Lecture 21. Revision

Student should know

- What is an affine and projective curve.
- What is a tangent line to a smooth point of a projective curve and an arbitrary point of an affine curve.
- What is multiplicity of a point on a curve.
- What is projective space.
- That a projective curve is compact and Hausdorff
- How to formulate Bezout theorem and its weak version
- What is intersection multiplicity and how to calculate it
- That a non-singular projective curve is irreducible
- When intersection multiplicity is one and when it is greater than one
- What is a Hessian of a polynomial and an inflection point of a curve
- How many inflection points can a curve of degree d have
- What is a general form of a non-singular cubic in $\mathbb{P}^2$
- How a line can intersect a non-singular cubic
- What is a group law on a non-singular cubic in $\mathbb{P}^2$
- What is a ramification point and a branch locus
- How many ramification points can there be
- What is a covering projection
- What is a triangulation
- What is Euler characteristic
- The degree–genus formula for smooth curves
- What is a Riemann surface
- That every $C \setminus \mathrm{Sing}(C)$ is a Riemann surface
- That a lattice in $\mathbb{C}$ gives a cubic curve
- That an elliptic curve is biholomorphic to a complex tori

- That every curve has a resolution of singularities
- That the resolution of singularities is constructed by holomorphic elements
- How does a projective curve look like near singular point
- What is Newton polygon
- What is the algorithm for calculating Puiseux expansions
- When Puiseux expansions are essentially different
- That there is a degree–genus formula for singular curves
- How to calculate genus of a singular curve
- What is a holomorphic differential and its integral
- That Λ is the lattice of integrals of η along closed paths
- What is Abel's theorem
- What is a divisor, a principal divisor and divisor of a holomorphic differential
- What is degree of a divisor
- What is the formulation of Riemann-Roch theorem

06.04. Lecture 22. Mock Exam

Student should be able to

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Exam

1. (a) Find singular points their multiplicity and tangent lines of the affine curve: $y^3 = x^3 - x^2$.

   (b) Find singular points and inflections points of projective curve C defined by polynomial $P(x, y, z) = y^2z - x^2(x + z)$. Find tangent lines to C at smooth inflection points.

   (c) Find the intersection points of C with tangent lines at smooth inflection points.

   [Similar to Workshop and Homework I and IV (for finding inflection points)]

2. (a) Use Bezout theorem to show that irreducible cubic curve cannot have more than one singular point.

   (b) Show that given any five points on $\mathbb{P}^2$ there is at least one conic containing them.

   (c) Use Bezout theorem to deduce that a projective curve of degree 4 in $\mathbb{P}^2$ with 4 singular points is reducible. [Hint: show that any conic containing the 4 singular points and any other point of C must have a component common with C].

   [Similar to Workshop II]

3. (a) Show that the projective curve $C = \{y^2z = x^3\}$ has a unique singular point p.

   (b) Show that a line through the singular point and point $[t : s : 0]$ with $s \neq 0$ on the line $\{z = 0\}$ intersects C at the singular point and $[s^2t : s^3 : t^3]$.

   (c) Show that the map $f \colon \mathbb{P}^1 \to C$, $f[s : t] = [s^2t, s^3, t^3]$ is a homeomorphism. Deduce that the degree-genus formula cannot be applied to singular curves. [Hint: recall that any continuous bijection from a compact space to a Hausdorff space is always a homeomorphism]

   [Similar to Workshop III]

4. Recall that if $p = [0 : 0 : 1]$ is a singular point of a curve $C = \{P(x, y, z) = 0\} \subset \mathbb{P}^2$ such that $[0 : 1 : 0]$ doesn't lie on C or the tangent line to C at any of non-singular inflection points of C then we define $\delta(p) = \frac{1}{2}(I_p(P, \frac{\partial P}{\partial y}) - \nu_\varphi(p) + |\pi^{-1}(p)|)$, where $\nu_\varphi(p)$ is the minimal m such that $(\frac{\partial}{\partial y}P)^m$ does not vanish at p and $\pi \colon \widetilde{C} \to C$ is the resolution of singularities.

   (a) Calculate the first three terms of Puiseux expansion of C around the singular point.

   (b) Calculate the intersection multiplicity of C with the curve $-4x^2 - 2xz = 0$ at the singular point of C.

   (c) Calculate the genus of the curve defined by $P(x, y, z) = y^2z - x^2(x + z)$.

   [Similar to Workshop/Homework IV]

Solutions

1. (a) Let $C \subset \mathbb{C}^2$ be a curve defined by polynomial $P(x,y) = y^3 - x^3 - x^2$. Then

$$\frac{\partial P}{\partial x} = -3x^2 - 2x = -x(3x-2), \quad \frac{\partial P}{\partial y} = 3y^2.$$

A point $[a:b]$ is a singular point of C if $3b^2 = 0$ and $a(3a-2) = 0$. It follows that $b = 0$ and either $a = 0$ or $a = \frac{2}{3}$. It remains to check whether points $(0,0)$ and $(\frac{2}{3}, 0)$ lie on C. We have:

$$P(0,0) = 0, \qquad P(\frac{2}{3}, 0) = -8/27 - 4/9,$$

hence $p = (0,0)$ is the only singular point of C.
To calculate its multiplicity, we look at further differentials of P:

$$\frac{\partial^2 P}{\partial x^2} = -6x - 2, \qquad \frac{\partial^2 P}{\partial x \partial y} = 0, \qquad \frac{\partial^2 P}{\partial y^2} = 6y.$$

Since $\frac{\partial P}{\partial x}(0,0) \neq 0$, point $(0,0)$ is a singular point of multiplicity two. Tangent lines are linear factors of the equation

$$\frac{1}{2!}\frac{\partial^2 P}{\partial x^2}(0,0)x^2 + \frac{1}{1!1!}\frac{\partial^2 P}{\partial x \partial y}(0,0)xy + \frac{1}{2!}\frac{\partial^2 P}{\partial y^2}(0,0)y^2 = -x^2.$$

It follows that curve C has tangent line $x = 0$ at the singular point $(0,0)$.

(b) Let now $C \subset \mathbb{P}^2$ be the projective curve defined by polynomial $P(x,y,z) = y^2 z - x^2(x+z)$. To calculate singular points, we first calculate derivatives

$$\frac{\partial P}{\partial x} = -3x^2 - 2xz, \qquad \frac{\partial P}{\partial y} = 2yz, \qquad \frac{\partial P}{\partial z} = y^2 - x^2.$$

Thus $[a:b:c]$ is a singular point if and only if

$$\begin{cases} a(-3a-2c) = 0 \\ bc = 0 \\ a^2 - b^2 = 0. \end{cases}$$

It follows from the second equation that either $b = 0$ or $c = 0$. In the first case $b = 0$, the third equation implies $a = 0$. Then the first equation holds, hence $[0:0:1]$ is a singular point of C. If $c = 0$ then the first equation implies $a = 0$. Then the third equation holds if $b = 0$ which contradicts the assumption that $[a:b:c]$ is a point in $\mathbb{P}^2$. It follows that $[0:0:1]$ is the only singular point of C.
To find inflection points of C we look at the matrix of second derivatives

$$\begin{pmatrix} -6x-2z & 0 & -2x \\ 0 & 2z & 2y \\ -2x & 2y & 0 \end{pmatrix}$$

It determinant is $-8x^2z+4y^2(6x+2z)=8y^2z+24xy^2-8x^2z$.
A point $[a:b:c]$ is an inflection point of C if

$$\begin{cases} b^2c=a^3+a^2c \\ b^2c=-3ab^2+a^2c \end{cases}$$
$$\begin{cases} b^2c=a^3+a^2c \\ 0=a^3+3ab^2=a(a^2+3b^2) \end{cases}$$
$$\begin{cases} b^2c=0 \\ 0=a \end{cases} \qquad \begin{cases} b^2c=a^3+a^2c \\ a^2=-3b^2 \end{cases}$$
$$\begin{cases} b=0 \\ a=0 \end{cases} \qquad \begin{cases} c=0 \\ a=0 \end{cases} \qquad \begin{cases} -\frac{1}{3}a^2c=a^3+a^2c \\ a^2=-3b^2 \end{cases}$$
$$\begin{cases} b=0 \\ a=0 \end{cases} \qquad \begin{cases} c=0 \\ a=0 \end{cases} \qquad \begin{cases} 0=a^2(a+\frac{4}{3}c) \\ b^2=-\frac{1}{3}a^2 \end{cases}$$
$$\begin{cases} b=0 \\ a=0 \end{cases} \qquad \begin{cases} c=0 \\ a=0 \end{cases} \qquad \begin{cases} a=0 \\ b=0 \end{cases} \qquad \begin{cases} c=-\frac{3}{4}a \\ b^2=-\frac{1}{3}a^2 \end{cases}$$

It follows that the inflection points are:

$$[0:0:1], \qquad [0:1:0], \qquad [1:\frac{i\sqrt{3}}{3}:-\frac{3}{4}], \qquad [-1:\frac{i\sqrt{3}}{3}:\frac{3}{4}].$$

Point $[0:0:1]$ is singular. Tangent lines at smooth inflection points are:

$$\begin{aligned} &z=0, \\ &-\frac{3}{2}x-\frac{2i\sqrt{3}}{4}y-\frac{4}{3}z=0, \\ &-\frac{3}{2}x+\frac{2i\sqrt{3}}{4}y-\frac{4}{3}z=0. \end{aligned}$$

(c) Line $z=0$ intersects C in points $[a:b:c]$ such that

$$\begin{cases} c=0, \\ -a^3=0 \end{cases}$$

It follows that $[0:1:0]$ is the only intersection point.
Line $-\frac{3}{2}x-\frac{i\sqrt{3}}{2}y-\frac{4}{3}z=0$ intersects C in points $[a:b:c]$ such that

$$\begin{cases} -\frac{3}{2}a-\frac{i\sqrt{3}}{2}b-\frac{4}{3}c=0, \\ b^2c-a^2(a+c)=0 \end{cases}$$
$$\begin{cases} \frac{i\sqrt{3}}{2}b=-\frac{3}{2}a-\frac{4}{3}c, \\ b^2c-a^2(a+c)=0 \end{cases}$$
$$\begin{cases} \frac{i\sqrt{3}}{2}b=-\frac{3}{2}a-\frac{4}{3}c, \\ -\frac{64}{27}c^3-\frac{16}{3}ac^2-3a^2c-a^3-a^2c=-(a^3+4a^2c+\frac{16}{3}ac^2+\frac{64}{27}c^3)=-(a+\frac{4}{3}c)^3=0 \end{cases}$$

It follows that $[1 : \frac{i\sqrt{3}}{3} : -\frac{3}{4}]$ is the only intersection point.

Finally, Line $-\frac{3}{2}x + \frac{i\sqrt{3}}{2}y - \frac{4}{3}z = 0$ intersects C in points $[a : b : c]$ such that

$$\begin{cases} -\frac{3}{2}a + \frac{i\sqrt{3}}{2}b - \frac{4}{3}c = 0, \\ b^2c - a^2(a+c) = 0 \end{cases}$$

$$\begin{cases} \frac{i\sqrt{3}}{2}b = \frac{3}{2}a + \frac{4}{3}c \Rightarrow b^2 = -\frac{64}{27}c^2 - \frac{16}{3}ac - 3a^2, \\ b^2c - a^2(a+c) = 0 \end{cases}$$

$$\begin{cases} \frac{i\sqrt{3}}{2}b = \frac{3}{2}a + \frac{4}{3}c, \\ -\frac{64}{27}c^3 - \frac{16}{3}ac^2 - 3a^2c - a^3 - a^2c = -(a^3 + 4a^2c + \frac{16}{3}ac^2 + \frac{64}{27}c^3) = -(a + \frac{4}{3}c)^3 = 0 \end{cases}$$

It follows that $[1 : -\frac{i\sqrt{3}}{3} : -\frac{3}{4}]$ is the only intersection point.

2. (a) Let $C \subset \mathbb{P}^2$ be a cubic and assume that p_0, p_1 are singular points of C. Let L be a line passing through p_0 and p_1. Since $p_i \in C$ is singular, $I_{p_1}(C, L) \geq 2$. It follows that $\sum_{p \in C \cap L} I_p(C, L) \geq 4$.

   By Bezout theorem, if C and L had no common components then $\sum_{p \in C \cap L} I_p(C, L) = 3$. It follows that C and L have a common component.

   As L is irreducible, it is the common component of C. It shows that C is reducible.

   (b) Let $p_0 = [a_0 : b_0 : c_0], \dots, p_4 = [a_4 : b_4 : c_4]$ be arbitrary points in $\mathbb{P}^2$. A conic Q is defined by a polynomial

$$P(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz.$$

   Two polynomials define the same conic if they differ by a multiplication by scalar, hence we can assume that $A + B + C + D + E + F = 1$. Conic Q contains points p_i if and only if $P(a_i, b_i, c_i) = 0$. Thus, we get a system of six linear equations in $A, \dots, F$.

$$\begin{cases} A + B + C + D + E + F = 1, \\ P(a_i, b_i, c_i) = 0 \end{cases}$$

   Such a system always has a solution $(A_0, \dots, F_0)$ and the first equation guarantees that it is not zero. Then, the conic

$$Q = \{[x : y : z] \,|\, A_0x^2 + B_0y^2 + C_0z^2 + D_0xy + E_0xz + F_0yz\}$$

   contains $p_0, \dots, p_4$.

   (c) Let now C be a projective curve of degree 4 and assume that $p_1, \dots, p_4$ are singular points of C. Let $p_0 \in C$ be any point of $C \setminus \{p_1, \dots, p_4\}$. By the above argument, there exists a conic Q which contains $p_0, \dots, p_4$. Then

$$\sum_{p \in C \cap Q} I_p(C, Q) \geq \sum_{i=1}^{4} I_{p_i}(C, Q) + I_{p_0}(C, Q) \geq \sum_{i=1}^{4} 2 + 1 = 9$$

because if $p_i \in C$ is a singular point and $p \in Q$ then $I_p(C, Q) > 1$. On the other hand, degree of C is 4 and degree of Q if 2. If C and Q had no common component, then by Bezout's theorem

$$\sum_{p \in C \cap Q} I_p(C, Q) = 2 \cdot 4 = 8.$$

The contradiction implies that C and Q have common component, in particular C is reducible.

3. (a) Let $P(x, y, z) = x^3 - y^2 z$. A point $[a : b : c]$ is a singular point of curve $C = \{[x : y : z] \in \mathbb{P}^2 \mid P(x, y, z) = 0\}$ if and only if

$$\frac{\partial P}{\partial x} = 3x^2, \qquad \frac{\partial P}{\partial y} = -2yz, \qquad \frac{\partial P}{\partial z} = -y^2$$

vanish at $[a : b : c]$, i.e. if

$$\begin{cases} 3a^2 = 0, \\ 2bc = 0, \\ b^2 = 0 \end{cases}$$

The first and the last equation imply that $a = 0 = b$. Then c is arbitrary, hence $[0 : 0 : 1]$ is the unique singular point of C.

(b) A line through $[0 : 0 : 1]$ and $[t : s : 0]$ is given by polynomial

$$L_{s,t} = \{-sx + ty = 0\}.$$

If $s \neq 0$, point $[a : b : c]$ lies on the intersection of C with $L_{s,t}$ if and only if

$$\begin{cases} a = \frac{t}{s} b, \\ b^2 c = \frac{t^3}{s^3} b^3 \end{cases}$$

$$\begin{cases} a = \frac{t}{s} b, \\ b^2 (c - \frac{t^3}{s^3} b) = 0 \end{cases}$$

$$\begin{cases} a = 0, \\ b = 0 \end{cases} \qquad\qquad \begin{cases} a = \frac{t}{s} b, \\ c = \frac{t^3}{s^3} b \end{cases}$$

It follows that $[\frac{t}{s} : 1 : \frac{t^3}{s^3}] = [ts^2 : s^3 : t^3]$ is the intersection point of L_{st} with C which is different than p.

(c) Since map $\mathbb{C}^2 \to \mathbb{C}^3$, $(s, t) \mapsto (s^2 t, s^3, t^3)$ is continuous, so is the map $\mathbb{P}^1 \to \mathbb{P}^2$, $[s : t] \mapsto [s^2 t : s^3 : t^3]$. As $(s^2 t)^3 = (s^3)^2 t^3$, its image is contained in C, hence

$$\varphi \colon \mathbb{P}^1 \to C,\ f([s : t]) = [s^2 t : s^3 : t^3]$$

is continuous.

Projective line $\mathbb{P}^1$ is compact and any complex projective curve is Hausdorff, hence φ is a homeomorphism if it is a bijection. Let us construct its inverse. Since $\frac{s}{t} = \frac{s^3}{s^2t}$, we put

$$\psi\colon C \to \mathbb{P}^1,\ \psi([x:y:z]) = \begin{cases} [y:x] & \text{if } y \neq 0, \\ [0:1] & \text{if } y = 0. \end{cases}$$

Then

$$\psi \circ \varphi([s:t]) = \psi([s^2t : s^3 : t^3]) = \begin{cases} [s^3 : s^2t] = [s:t] & \text{if } s \neq 0, \\ [0:1] & \text{if } s = 0. \end{cases}$$

i.e. $\psi\varphi = \mathrm{Id}$.
Let now $[x:y:z] \in C$. If $y = 0$ then $x^3 = 0$, so $[x:y:z] = [0:0:1]$. Then

$$\psi \circ \varphi[0:0:1] = \psi([1:0]) = [0:0:1].$$

If $y \neq 0$ then

$$\psi \circ \varphi[x:y:z] = \psi([y:x]) = [y^2x : y^3 : x^3] = [y^2x : y^3 : y^2z] = [x:y:z],$$

i.e. $\varphi \circ \psi = \mathrm{Id}$. As discussed above, it follows that φ is a homeomorphism. In particular, the genus of C is the genus of $\mathbb{P}^1$, which is zero.
On the other hand, C is of degree three, hence

$$\frac{1}{2}(d-1)(d-2) = 1.$$

For smooth curves the degree-genus formula reads $g(C) = \frac{1}{2}(d-1)(d-2)$. The example of $\{x^3 = y^2z\}$ shows that the degree-genus formula does not hold for singular curves.

4. (a) Curve C is defined by polynomial $P(x,y,z) = y^2z - x^2(x+z)$. We know from the first exercise that C has one singular point $p = [0:0:1]$. Then

$$P(x,y,1) = y^2 - x^3 - x^2 = \sum_{\alpha+\beta\geq 2} c_{\alpha\beta}x^\alpha y^\beta$$

and

$$f_0(t) = \sum_{\alpha+\beta=2} c_{\alpha\beta}t^\beta = t^2 - 1 = (t-1)(t+1)$$

has two roots ± 1.
The first substitutions are

$$x = x_1, \qquad y = x_1(\pm 1 + y_1)$$

$$P(x_1, y_1, 1) = x_1^2(\pm 1 + y_1)^2 - x_1^3 - x_1^2,$$
$$P_1(x_1, y_1) = y_1^2 \pm 2y_1 + 1 - x_1 - 1 = y_1^2 \pm 2y_1 - x_1 = \sum_{\alpha+\beta\geq 1} d_{\alpha\beta}x^\alpha y^\beta.$$

Polynomial

$$f_1(t) = \sum_{\alpha+\beta=1} d_{\alpha\beta} t^{\beta} = \pm 2t - 1$$

has two roots $\pm\frac{1}{2}$.

The second substitutions are

$$x_1 = x_2, \qquad\qquad y_1 = x_2(\pm\frac{1}{2} + y_2),$$

$$P_1(x_2, y_2) = x_2^2(\pm\frac{1}{2} + y_2)^2 \pm 2x_2(\pm\frac{1}{2} + y_2) - x_2,$$
$$P_2(x_2, y_2) = x_2(y_2^2 \pm y_2 + \frac{1}{4}) + 1 \pm y_2 - 1 = x_2 y_2^2 \pm x_2 y_2 + \frac{1}{4}x_2 \pm y_2 = \sum_{\alpha+\beta\geq 1} e_{\alpha\beta} x^{\alpha} y^{\beta}.$$

Polynomial

$$f_2(t) = \sum_{\alpha+\beta=1} e_{\alpha\beta} t^{\beta} = \pm t + \frac{1}{4}$$

has root $-(\pm\frac{1}{4})$. Then

$$x_2 = x_3, \quad y_2 = x_3(-(\pm\frac{1}{4} + y_3))$$

and the Puiseux expansions are

$$y_1(x) = x(1 + x(\frac{1}{2} + x(-\frac{1}{4} + \ldots))) = x + \frac{1}{2}x^2 - \frac{1}{4}x^3 + \ldots$$
$$y_2(x) = x(-1 + x(-\frac{1}{2} + x(+\frac{1}{4} + \ldots))) = -x - \frac{1}{2}x^2 + \frac{1}{4}x^3 + \ldots$$

(b)

$$I_p(y^2z - x^3 - x^2z, 2x^2 + xz) = I_p(2y^2z - 2x^3 - 2x^2z + x(2x^2 + xz), 2x^2 + xz)$$
$$= I_p(2y^2z, 2x^2 + xz) = 2I_p(y, 2x^2 + xz) + I_p(z, 2x^2 + xz)$$
$$= 2I_p(y, x(x + z)) + 0 = 2I_p(y, x) + 2I_p(y, z) = 2 + 0 = 2.$$

(c) Curve C has one singular point, hence Noether formula reads

$$g(C) = \frac{1}{2}(3-1)(3-2) - \delta(p).$$

To calculate $\delta(p)$ we first calculate $|\pi^{-1}(p)|$, i.e. the number of essentially different Puiseux expansions of C around p. From part (a) we know that there are two:

$$y_1(x) = x(1 + x(\frac{1}{2} + x(-\frac{1}{4} + \ldots))) = x + \frac{1}{2}x^2 - \frac{1}{4}x^3 + \ldots$$
$$y_2(x) = x(-1 + x(-\frac{1}{2} + x(+\frac{1}{4} + \ldots))) = -x - \frac{1}{2}x^2 + \frac{1}{4}x^3 + \ldots$$

If $g(t) = t + \frac{1}{2}t^2 - \frac{1}{4}t^3 + \ldots$, then $g(-\varepsilon x)$ is not the second Puiseux expansion, for any root of unity ε. Indeed, we would necessarily have $\varepsilon^2 = 1$ and $g(-x) \neq y_2(x)$. It follows that the two expansions are essentially different, hence

$$|\pi^{-1}(p)| = 2.$$

We learn from the first exercise that point $[0:1:0]$ lies on the tangent line to C at the inflection point $[0:1:0]$. However, $[1:0:1]$ is a point which does not lie on C and on tangent line to any smooth inflection point of C.
We calculate $\nu_\varphi^{[1:0:1]}(p)$. We have

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial z})^2 P = \frac{\partial^2}{\partial x^2}P + 2\frac{\partial^2}{\partial x \partial y}P + \frac{\partial^2}{\partial y}P = -6x - 2z - 2x$$

and it does not vanish at z. Hence

$$\nu_\varphi^{[1:0:1]}(p) = 2.$$

Finally $(\frac{\partial}{\partial x} + \frac{\partial}{\partial z})P = -3x^2 - 2xz - x^2 = -4x^2 - 2xz$. By part (b),

$$I_p(y^2z - x^3 - x^2z, 2x^2 + xz) = 2.$$

Then

$$\delta(p) = \frac{1}{2}(2 - 2 + 2) = 1$$
$$g(C) = 0.$$