LIE GROUPS AND ALGEBRAS

Exercises whose solutions were already discussed are marked with ♠.

1. Basic definitions

Exercise 1.1. ♠ Show that if $G = \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{End}(\mathbb{R}^n)$ so that each tangent space is canonically identified with $\mathrm{End}(\mathbb{R}^n)$ then $(Lg)_* v = gv$, where the product in the right hand side is the usual product of matrices. Also, the adjoint action is given by $\mathrm{Ad}\, g(v) = gvg^{-1}$.

The next 4 exercises are about the group $\mathrm{SU}(2)$ and its adjoint representation

Exercise 1.2. ♠ Define a bilinear form on $\mathrm{su}(2) = \{x \in M_{2\times 2}(\mathbb{C}) \,|\, x + \overline{x}^* = 0, \mathrm{tr}(x) = 0\}$ by $(a, b) = \frac{1}{2}\,\mathrm{tr}(a\overline{b}^T)$. Show that this form is symmetric, positive definite, and invariant under the adjoint action of $\mathrm{SU}(2)$.

Exercise 1.3. ♠ Define a basis in $\mathrm{su}(2)$ by

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Show that the map $\varphi\colon \mathrm{SU}(2) \to \mathrm{GL}(3, \mathbb{R})$, $g \mapsto$ matrix of $\mathrm{Ad}\, g$ in the basis σ_1, σ_2, σ_3 gives a morphism of Lie groups $\mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R})$.

Exercise 1.4. ♠ Let $\varphi\colon SU(2) \to \mathrm{SO}(3, \mathbb{R})$ be as in the previous exercise. Compute the map of tangent spaces $\varphi_*\colon \mathrm{su}(2) \to \mathrm{so}(3, \mathbb{R})$ and show that φ_* is an isomorphism.

Exercise 1.5. ♠ Prove that φ as above establishes an isomorphism $\mathrm{SU}(2)/\mathbb{Z}_2 \simeq \mathrm{SO}(3, \mathbb{R})$ and thus, since $\mathrm{SU}(2) \simeq S^3$, $\mathrm{SO}(3, \mathbb{R}) \simeq \mathbb{RP}^3$.

Exercise 1.6. ♠ Using fiber sequences for $\mathrm{SU}(n-1) \to \mathrm{SU}(n)$ and $U(n-1) \to U(n)$ show that $U(n)$ and $\mathrm{SU}(n)$ are connected, $\mathrm{SU}(n)$ is simply connected and $\pi_1 U(n) = \mathbb{Z}$.

Exercise 1.7. ♠ Show that $\pi_0(\mathrm{SO}(n+1, \mathbb{R})) \simeq \pi_0(\mathrm{SO}(n, \mathbb{R}))$, for $n \geq 2$ and $\pi_1(\mathrm{SO}(n+1, \mathbb{R})) \simeq \pi_1(\mathrm{SO}(n, \mathbb{R}))$, for $n \geq 3$. Calculate these groups.

Exercise 1.8. ♠ Using the Gram-Schmidt orthogonalization process show that $\mathrm{GL}(n, \mathbb{R})/O(n, \mathbb{R})$ is diffeomorphic to the space of upper triangular matrices with positive entries on the diagonal. Deduce from it that $\mathrm{GL}(n, \mathbb{R})$ is homotopic to $O(n, \mathbb{R})$.

Exercise 1.9. ♠ Let L_n be the set of all Lagrangian ($\dim V = n$, $\omega(x, y) = 0$, for all $x, y \in V$) subspaces in $\mathbb{R}^{2n}$ with symplectic form $\omega(x, y) = \sum_{i=1}^n x_i y_{i+n} - y_i x_{i+n}$. Show that L_n has a structure of a smooth manifold and calculate its dimension.

Exercise 1.10. Let $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ be the algebra of quaternions defined by $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$, $i^2 = j^2 = k^2 = -1$.

(1) Let $\mathrm{End}(\mathbb{H}^n)$ be the algebra of endomorphisms of $\mathbb{H}^n$ considered as a right module over $\mathbb{H}$. Show that $\mathrm{End}(\mathbb{H}^n)$ is naturally identified with the algebra $M_{n\times n}(\mathbb{H})$.

(2) Define a $\mathbb{H}$-valued form $(-,-)$ on $\mathbb{H}^n$ via

$$(h, h') = \sum_i \overline{h}_i h'_i,$$

where $\overline{a + bi + cj + dk} = a - bi - cj - dk$. Let

$$U(n, \mathbb{H}) = \{A \in \mathrm{End}(\mathbb{H}^n) \mid (Ah, Ah') = (h, h')\}$$

be the group of 'unitary quaternionic transformations'. Show that it is indeed a group and that a matrix $A \in M_{n\times n}(\mathbb{H})$ is in $U(n, \mathbb{H})$ if and only if $A^*A = \mathrm{Id}$, where $(A^*)_{ab} = \overline{A_{ba}}$.

(3) Define a map $\mathbb{C}^{2n} \xrightarrow{\simeq} \mathbb{H}^n$ by

$$(z_1, \dots, z_{2n}) \mapsto (z_1 + jz_{n+1}, \dots, z_n + jz_{2n}).$$

Show that it is an isomorphism of complex vector spaces and that this isomorphism identifies

$$\mathrm{End}(\mathbb{H}^n) = \{A \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2n}) \mid \overline{A} = J^{-1}AJ\},$$

where $J = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$.

(4) Translate the quaternionic bi-linear form to a form on $\mathbb{C}^{2n}$ and deduce that $U(n, \mathbb{H})$ is identified with $\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{SU}(2n)$.

Exercise 1.11. ♠

(1) Show that $\mathrm{Sp}(1) \simeq \mathrm{SU}(2) \simeq S^3$.
(2) Using the previous exercise construct a fiber sequence relating $\mathrm{Sp}(n)$ and $\mathrm{Sp}(n-1)$.
(3) Calculate $\pi_0(\mathrm{Sp}(n))$ and $\pi_1(\mathrm{Sp}(n))$.

2. Lie groups and Lie algebras

Exercise 2.1. ♠ Consider the group $SL(2, \mathbb{R})$. Show that element $X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ in not in the image of the exponential map.

Exercise 2.2. ♠ Let $f\colon \mathfrak{g} \to G$ be any smooth map such that $f(0) = 1$, $f_*(0) = \mathrm{Id}$; we can view such a map as a local coordinate system near $1 \in G$. Show that the group law written in this coordinate system has the form $f(x)f(y) = f(x + y + B(x, y) + \dots)$, for some bi-linear map $B\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and that $B(x, y) - B(y, x) = [x, y]$.

Exercise 2.3. ♠ Show that, if we denote, for $x \in \mathfrak{g}$ by ξ_x the left-invariant vector field on G such that $\xi_x(1) = x$, then $[\xi_x, \xi_y] = -\xi_{[x,y]}$.

Exercise 2.4. ♠

(1) Show that $\mathbb{R}^3$ with the commutator given by the cross-product is a Lie algebra. Show that this Lie algebra is isomorphic to $\mathrm{so}(3, \mathbb{R})$.
(2) Letr $\varphi\colon \mathrm{so}(3, \mathbb{R}) \to \mathbb{R}^3$ be the isomorphism as above. Prove that under this action the standard action of $\mathrm{so}(3, \mathbb{R})$ on $\mathbb{R}^3$ is identified with the action of $\mathbb{R}^3$ given by the cross-product:

$$a \cdot v = \phi(a)v.$$

Exercise 2.5. ♠ Let P_n be the space of polynomials with real coefficients of degree $\leq n$ in variable x. The Lie group $G = \mathbb{R}$ acts on P_n by translations of the argument $\rho(t)(x) = x + t$, $t \in G$. Calculate the corresponding action of the Lie algebra $\mathfrak{g} = \mathbb{R}$.

Exercise 2.6. ♠ Let G be the Lie group of all maps $A\colon \mathbb{R} \to \mathbb{R}$ having the form $A(x) = ax + b$, $a \neq 0$. Without embedding G into $GL(2, \mathbb{R})$ describe explicitly the corresponding Lie algebra.

Exercise 2.7. ♠ Let $SL(2, \mathbb{C})$ act on $\mathbb{CP}^1$ in the usual way

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x : y) = (ax + by : cx + dy).$$

This defines an action of $(2, \mathbb{C})$ by vector fields on $\mathbb{CP}^1$. Write explicitly vector fields corresponding to $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ in terms of the coordinate $t = x/y$ on the open cell $\mathbb{C} \subset \mathbb{CP}^1$.

Exercise 2.8. ♠ Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$ the Lie algebra of automorphisms of $\mathfrak{g}$ and $\mathrm{Der}(\mathfrak{g})$ its Lie algebra.

(1) Show that $g \mapsto \mathrm{Ad}\, g$ gives a morphism of Lie groups $G \to \mathrm{Aut}(\mathfrak{g})$, similarly $x \mapsto \mathrm{ad} x$ is a morphism of Lie algebras $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$.
(2) Show that $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\mathrm{Der}(\mathfrak{g})$.

Exercise 2.9. ♠ Let $\{H_\alpha\}_{\alpha \in A}$ be a family of closed Lie subgroups in G, with Lie algebras $\mathfrak{h}_\alpha$. Let $H = \bigcap_\alpha H_\alpha$. Show that H is a closed Lie subgroup with Lie algebra $\bigcap_\alpha \mathfrak{h}_\alpha$.

Exercise 2.10. Let J_x, J_y, J_z be the basis in $\mathrm{so}(3, \mathbb{R})$. The standard action of $SO(3, \mathbb{R})$ on $\mathbb{R}^3$ defines an action of $\mathrm{so}(3, \mathbb{R})$ by vector fields on $\mathbb{R}^3$. Abusing the language, we will use the same notation J_x, J_y, J_z for the corresponding vector fields on $\mathbb{R}^3$. Let $\Delta = J_x^2 + J_y^2 + J_z^2$.

(1) Write down J_x, J_y, J_z and Δ in terms of $x, y, z, \partial_x, \partial_y$ and ∂_z.
(2) Show that Δ is well-defined as a differential operator on a sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$.
(3) Show that $\partial_x^2 + \partial_y^2 + \partial_z^2$ can be written as $\frac{1}{r^2}\Delta + \Delta_r$, where Δ_r is a differential operator written in terms of $r = \sqrt{x^2 + y^2 + z^2}$ and $d\partial_r = x\partial_x + y\partial_y + z\partial_z$.

Exercise 2.11. ♠

(1) Let $\mathfrak{g}$ be a 3-dimensional real Lie algebra with basis x, y, z and $[x, y] = z$, $[z, x] = [z, y] = 0$. Show that in the corresponding Lie group $\exp(tx)\exp(sy) = \exp(tsz)\exp(sy)\exp(tx)$ and construct explicitly a Lie group corresponding to $\mathfrak{g}$.
(2) Generalize the previous part to the Lie algebra $\mathfrak{g} = V \oplus \mathbb{R}z$, where V is a real vector space with non-degenerate skew-symmetric form ω and $[v_1, v_2] = \omega(v_1, v_2)z$, $[z, v] = 0$.

Exercise 2.12. ♠ Let G be a complex connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{k} \subset \mathfrak{g}$ be a real form of $\mathfrak{g}$.

(1) Define the $\mathbb{R}$-linear map $\theta\colon \mathfrak{g} \to \mathfrak{g}$ via $\theta(x + iy) = x - iy$, where $x, y \in \mathfrak{k}$. Show that θ is an automorphism of $\mathfrak{g}$ considered as a real Lie algebra, and that it can be uniquely lifted to an automorphism $\theta\colon G \to G$ of the group G considered as a real Lie group.
(2) Let $K = G^\theta$ be the fixed points of θ. Show that K is a real Lie group with Lie algebra $\mathfrak{k}$.

Exercise 2.13. Let $\mathrm{Sp}(n)$ be the unitary quaternionic group. Show that $\mathfrak{sp}(n)_\mathbb{C} = \mathfrak{sp}(n, \mathbb{C})$.

Exercise 2.14. ♠ Let G be a complex connected Lie group.

(1) Show that $g \mapsto \mathrm{Ad}\, g$ is an analytic map $G \to \mathfrak{gl}(\mathfrak{g})$.
(2) Assume that G is compact. Show that then $\mathrm{Ad}\, g = \mathrm{Id}$, for any $g \in G$.
(3) Show that any connected compact Lie group G is abelian.
(4) Show that if G is a connected complex compact Lie group then the exponential map gives an isomorphism of Lie groups

$$\mathfrak{g}/L \simeq G,$$

for some lattice L, i.e. a free abelian group of rank equal to $2\dim\mathfrak{g}$.

3. Lie algebras

Exercise 3.1. ♠ Find the center of the Lie algebra $\mathfrak{sl}(2, F)$ over any field F.

Exercise 3.2. ♠ Show that if $\varphi\colon L_1 \to L_2$ is a homomorphism of Lie algebras then the kernel of φ is an ideal in L_1 and the image is a subalgebra of L_2.

Exercise 3.3. ♠ Show that a Lie bracket on a Lie algebra L is associative if and only if, for all $a, b \in L$, $[a, b]$ lies in the center of L.

Exercise 3.4. Prove that $\mathfrak{sl}(2, \mathbb{C})$ has no non-trivial ideals.

Exercise 3.5. Let L be a complex Lie algebra with basis x, y, z and $[x, y] = z$, $[y, z] = x$, $[z, x] = y$.

(1) Show that L is isomorphic to the Lie subalgebra of $gl(3, \mathbb{C})$ consisting of all 3×3 anti-symmetric matrices.
(2) Find an explicit isomorphism $\mathfrak{sl}(2, \mathbb{C}) \simeq L$.

Exercise 3.6. Let $\mathfrak{g}$, $\mathfrak{h}$ be Lie algebras and $\psi\colon \mathfrak{g} \to \mathrm{der}\mathfrak{h}$ a morphism of Lie algebras. Show that $\mathfrak{g} \oplus \mathfrak{h}$ with the bracket defined as

$$[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], [h_1, h_2] + \psi_{g_1}(h_2) - \psi_{g_2}(h_1))$$

is a Lie algebra. It is the semi-direct product of $\mathfrak{g}$ and $\mathfrak{h}$.

Exercise 3.7. ♠ Let V be an n-dimensional vector space and $\mathfrak{g} = \mathfrak{gl}(V)$. Suppose that $x \in \mathfrak{g}$ is diagonalisable with eigenvalues $\lambda_1, \dots, \lambda_n$. Show that $\mathrm{ad}x$ is diagonalisable with eigenvalues $\lambda_i - \lambda_j$ for $0 \leq i, j \leq n$.

Exercise 3.8. Let I, J be ideals in a Lie algebra $\mathfrak{g}$. Show that $I+J = \{x+y \mid x \in I, y \in J\}$ and $[I, J] = \mathrm{Span}\{[x, y] \mid x \in I, y \in J\}$ are ideals in $\mathfrak{g}$.

4. Representations of Lie groups and Lie algebras

Exercise 4.1. ♠ Let $V = \mathbb{C}^2$ be the standard 2-dimensional representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and let S^kV be the k-th symmetric power of V.

(1) Write explicitly the action of e, f and $h \in \mathfrak{sl}(2, \mathbb{C})$ in the basis x^iy^{k-i}.
(2) Show that S^2V is isomorphic to the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$.
(3) Which of these representations lift to representations of $\mathrm{SO}(3, \mathbb{R})$?

Exercise 4.2. ♠ Show that $\Lambda^n\mathbb{C}^n \simeq \mathbb{C}$ as a representation of $\mathfrak{sl}(n, \mathbb{C})$. Does it also work for $\mathfrak{gl}(n, \mathbb{C})$?

Exercise 4.3. ♠ Let V be a representation of $\mathfrak{sl}(2, \mathbb{C})$ and $C \in \mathrm{End}(V)$ be defined by:

$$C = \rho(e)\rho(f) + \rho(f)\rho(e) + \frac{1}{2}\rho(h)^2.$$

(1) Show that C commutes with the action of $\mathfrak{sl}(2,\mathbb{C})$; for any $x\in\mathfrak{sl}(2,\mathbb{C})$, $[\rho(x),C]=0$.
(2) Show that if $V=V_k$ is an irreducible representation with highest weight k, then C is a scalar operator, $C=c_k\,\mathrm{Id}$. Compute c_k.
(3) Recall that $\mathfrak{so}(3,\mathbb{C})\simeq\mathfrak{sl}(2,\mathbb{C})$. Show that this isomorphism identifies C with a multiple of $\rho(J_x)^2+\rho(J_y)^2+\rho(J_z)^2$.

Exercise 4.4. ♠

(1) Let V, W be irreducible representations of a Lie group G. Show that $(V\otimes W^*)^G=0$ if V is non-isomorphic to W and that $(V\otimes V^*)^G$ is canonically isomorphic to $\mathbb{C}$.
(2) Let V be an irreducible representation of a Lie algebra $\mathfrak{g}$. Show that V^* is also irreducible and deduce from this that a space of $\mathfrak{g}$-invariant bilinear forms on V is either 0 or 1-dimensional.

Exercise 4.5. ♠ For a representation V of a Lie algebra $\mathfrak{g}$ define the space of coinvariants by $V_{\mathfrak{g}}=V/\mathfrak{g}V$, where $\mathfrak{g}V$ is the subspace spanned by $xv, x\in\mathfrak{g}$, $v\in V$.

(1) Show that if V is completely reducible then the composition $V^{\mathfrak{g}}\to V\to V_{\mathfrak{g}}$ is an isomorphism.
(2) Show that in general case it is not so (Hint: take $\mathfrak{g}=\mathbb{R}$)

Exercise 4.6. ♠ Prove that if $A\colon\mathbb{C}^n\to\mathbb{C}^n$ is an operator of finite order, $A^n=\mathrm{Id}$, then A is diagonalizable.

Exercise 4.7. ♠ Let C be the standard cube in $\mathbb{R}^3 : C=\{|x_i|\le 1\}$, and let S be the set of faces of C ($|S|=6$). Consider the six-dimensional complex vector space V of functions on S and define $A\colon V\to V$ by

$$(Af)(\sigma)=\frac{1}{4}\sum_{\sigma'}f(\sigma')$$

where the sum is taken over all faces σ' which share an edge with σ. We want to diagonalize A

(1) Let $G=\{g\in O(3,\mathbb{R})\,|\,g(C)=C\}$ be the group of symmetries of C. Show that A commutes with the natural action of G on V.
(2) Let $z=-\mathrm{Id}\in G$. Show that as a representation of G, V can be decomposed into a direct sum

$$V=V_-\oplus V_+\quad V_\pm=\{f\in V\,|\,zf=\pm f\}.$$

(3) Show that as a representation of G, V_+ can be decomposed in the direct sum

$$V_+ = V_+^0 \oplus V_+^1, \quad V_+^0 = \{f \in V_+ \mid \sum_\sigma f(\sigma) = 0\}, \quad V_+^1 = \mathbb{C} \cdot 1,$$

where 1 denotes the constant function 1 on S.

(4) Find the eigenvalues of A on V_-, V_+^0 and V_+^1.

Exercise 4.8. ♠ Show that if V is a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ then $V \simeq \bigoplus V_k^{n_k}$ and $n_k = \dim V[k] - \dim V[k+2]$ (the notation V_k for the irreducible representation of highest weight k and $V[k]$ for the subspace of vectors of weight k as on the lecture). Show also that $\sum n_{2k} = \dim V[0]$ and $\sum n_{2k+1} = \dim V[1]$.

Exercise 4.9. ♠ Decompose symmetric powers of the standard representation $\mathbb{C}^2$ of $\mathfrak{sl}(2,\mathbb{C})$ into irreducible representations.

Exercise 4.10. ♠ Show that

(1) Every finite-dimensional complex representation of $\mathfrak{so}(3,\mathbb{R})$ admits a weight decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V[n]$$

where $V[n] = \{v \in V \mid J_z V = \frac{in}{2} v\}$, where $J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(2) A representation of $\mathfrak{so}(3,\mathbb{R})$ can be lifted to a representation of $\mathrm{SO}(3,\mathbb{R})$ is and only if all weight are even, i.e. $V[k] = 0$, for k odd.

Exercise 4.11. ♠ Let $\mathfrak{g}$ be a Lie algebra and $(,)$ a symmetric, ad-invariant bilinear form on $\mathfrak{g}$. Show that the element $\omega \in (\mathfrak{g}^*)^{\otimes 3}$ given by

$$\omega(x,y,z) = ([x,y],z)$$

is skew-symmetric and ad-invariant.

Exercise 4.12. Let $G = SU(2)$. Recall that we have a diffeomorphism $G \simeq S^3$

(1) Show that the left action of G on $G \simeq S^3 \subset \mathbb{R}^4$ can be extended to an action of G by linear orthogonal transformations on $\mathbb{R}^4$.

(2) Let $\omega \in \Omega^3(G)$ be a left-invariant 3-form whose value at $1 \in G$ is defined by

$$\omega(x,y,z) = ([x,y],z)$$

Show that $\omega = \pm 4dV$ where dV is the volume form on S^3 induced by the standard metric in $\mathbb{R}^4$ (hint: let x,y,z be some orthonormal basis in $\mathfrak{su}(2)$ with respect to $\frac{1}{2}\mathrm{tr}(a,\overline{b}^T)$).

(3) Show that $\frac{1}{8\pi^2}\omega$ is a bi-invariant form on G such that, for an appropriate choice of orientation, $\frac{1}{8\pi^2}\int_G \omega = 1$.

5. Structure theory of Lie algebras

Exercise 5.1. ♠ Let $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ and B_V a bilinear form on $\mathfrak{g}$ defined as $B_V(x,y) = \mathrm{tr}(\rho(x)\rho(y))$. Let $W \subset V$ be a subrepresentation. Show that $B_V = B_W + B_{V/W}$.

Exercise 5.2. ♠ Let $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$ be the subspace consisting of block triangular matrices:

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \right\},$$

where A is $k \times k$ matrix, B a $k \times (n-k)$ matrix and C a $(n-k)\times(n-k)$ matrix.

(1) Show that $\mathfrak{g}$ is a Lie subalgebra.
(2) Calculate the radical of $\mathfrak{g}$ and describe the quotient $\mathfrak{g}/\mathrm{rad}\mathfrak{g}$.

Exercise 5.3. ♠ Show that the bilinear form $\mathrm{tr}(xy)$ on $\mathfrak{sp}(n,K)$ is non-degenerate.

Exercise 5.4. ♠ Let V be a finite-dimensional complex vector space and $A\colon V \to V$ an strictly upper-triangular operator. Let $F^k \subset \mathrm{End}(V)$, $-n \le k \le n$ be the subspace spanned by matrix units E_{ij} with $i - j \le k$. Show that then $A.F^k \subset F^{k-1}$ an thus $\mathrm{Ad}\,A\colon \mathrm{End}(V) \to \mathrm{End}(V)$ is nilpotent. (For a matrix E, $A.E = AE - EA$).

Exercise 5.5 (Fitting's lemma)**.** ♠ Let $\varphi\colon V \to V$ be a linear transformation of a finite-dimensional vector space V. Show that $V = V_0 \oplus V_1$, where V_0, V_1 are φ-invariant, $\varphi|_{V_0}$ is nilpotent and $f|_{V_1}$ is an isomorphism.

Exercise 5.6. ♠ Let $\mathfrak{g}$ be a Lie algebra and $I_1, I_2 \subset \mathfrak{g}$ ideals. Assume that there exists a bilinear $B\colon I_1 \times I_2 \to K$ which is non-degenerate and, for any $x_1 \in I_1$, $x_2 \in I_2$, $y \in \mathfrak{g}$

$$B([x_1,y],x_2) + B(x_1,[x_2,y]) = 0.$$

Let $x_1 \ldots, x_m$ be a basis of I_1 and $x^1, \ldots, x^m$ the dual basis of I_2. For $y \in \mathfrak{g}$, let $[x_i, y] = \sum \alpha_{ij} x_j$ and $[x^i, y] = \sum \beta_{ij} x^j$.

(1) Show that $\alpha_{ik} = -\beta_{ki}$.
(2) Show that the Casimir element $T = \sum_{i=1}^m x_i x^i$ is central in the universal enveloping algebra $U_{\mathfrak{g}}$.

Exercise 5.7. ♠ Let $\mathfrak{g}$ be a semi-simple Lie algebra with an invariant, non-degenerate bilinear form. Show that the inclusion of any ideal $I \subset \mathfrak{g}$ can be completed to a decomposition $\mathfrak{g} = I \oplus J$, for an ideal J.

Exercise 5.8. Prove the first Whitehead lemma:

Let $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ be a finite-dimensional representation of a finite-dimensional semi-simple Lie algebra (of characteristic zero) and $f\colon \mathfrak{g} \to V$ a linear map such that

$$f([x,y]) = f(x)y - f(y)x.$$

Then there exists $v \in V$ such that $f(x) = vx$.

(1) Consider the kernel I of ρ and an ideal $J \subset \mathfrak{g}$ such that $\mathfrak{g} = I \oplus J$. Show that $(x,y) = \mathrm{tr}(\rho(x)\rho(y))$ is non-degenerate on J. Find the Casimir operator T for J and this bilinear form and show that $\rho(T)$ is a morphism of V considered as $\mathfrak{g}$ module.

(2) Show that if the endomorphism $\rho(T)$ of V is nilpotent then $v = 0$ is the sought element.

(3) Show that if the endomorphism $\rho(T)$ of V is an isomorphism then $\rho(T)^{-1}(\sum_{i=1}^{m} f(x_i)x^i)$ is the sought element, where $(x_1, \ldots, x_m)$, $(x^1, \ldots, x^m)$ are dual bases of J.

(4) Conclude the proof.

Exercise 5.9. Prove the second Whitehead lemma:

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra of characteristic zero, $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ a finite-dimensional representation and $g\colon \mathfrak{g} \times \mathfrak{g} \to V$ a bilinear map such that

(i) $g(x,x) = 0$, for any $x \in \mathfrak{g}$

(ii) $g([x,y],z) + g(x,y)z + g([y,z],x) + g(y,z)x + g([z,x],y) + g(z,x)y = 0$, for any $x,y,z \in \mathfrak{g}$.

Then there exists a linear map $p\colon \mathfrak{g} \to V$ such that

(iii) $g(x,y) = p(x)y - p(y)x - p([x,y])$, for $x,y \in \mathfrak{g}$.

(1) Consider, I, $J = \mathrm{span}\{x_1, \ldots, x_m\} = \mathrm{span}\{x^1, \ldots, x^m\}$ as in the previous exercise. Let T be the Casimir operator for J. Show that

$$-g(z,y)T = \sum g([z,y],x_i)x^i + \sum (g(y,x_i)x^i)z + \sum (g(x_i,z)x^i)y,$$

for any $z,y \in \mathfrak{g}$.

(2) Show that if $\rho(T)\colon V \to V$ is an isomorphism, then the map $p(y) = \rho(T)^{-1}(\sum g(y,x_i)x^i)$ satisfies (iii).

(3) Show that if $\rho(T)$ is nilpotent then (ii) reduces to

$$g([x,y],z) + g([y,z],x) + g([z,x],y) = 0.$$

Show that $\varphi x(y) = -\varphi([x,y])$ defines a structure of $\mathfrak{g}$ module on $\mathrm{Hom}_K(\mathfrak{g}, V)$.

(4) For $x \in \mathfrak{g}$, let $\varphi_x \in \mathrm{Hom}_K(\mathfrak{g}, V)$ be defined as $\varphi_x(y) = g(x, y)$. Under the assumption that $\rho(T)$ is nilpotent use the first Whitehead lemma to find linear map $p\colon \mathfrak{g} \to V$ as in (iii).

(5) Conclude the proof of the lemma.

Exercise 5.10. ♠ Let $I \subset \mathfrak{g}$ be an ideal. Show that the restriction of the Killing form of $\mathfrak{g}$ to I coincides with the Killing form of I.

Exercise 5.11. ♠ Show that for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ the Killing form is given by $K(x, y) = 2n\mathrm{tr}(xy)$.

Exercise 5.12. ♠ Let $\mathfrak{g}$ be a real Lie algebra with a positive definite Killing form. Show that then $\mathfrak{g} = 0$.

Exercise 5.13. ♠ Let $\mathfrak{g}$ be a simple Lie algebra

(1) Show that an invariant bilinear form is unique up to a factor.

(2) Show that $\mathfrak{g} \simeq \mathfrak{g}^*$ as representations of $\mathfrak{g}$.

6. Jordan decomposition

Let V be a finite dimensional complex vector space and $A\colon V \to V$ a linear map.

A is *nilpotent* if there exists N such that $A^N = 0$. A is *semi-simple* if for every $W \subset V$ such that $AW \subset W$ there exists $Z \subset V$ such that $AZ \subset Z$ and $V = W \oplus Z$.

Exercise 6.1. ♠ Show that $A\colon V \to V$ is semi-simple if and only if it is diagonalizable.

Exercise 6.2. ♠ Show that the sum of two commuting semi-simple operators is semi-simple and the sum of two commuting nilpotent operators is nilpotent.

Exercise 6.3. ♠ Show that $A\colon V \to V$ can be uniquely written as a sum of commuting semi-simple and nilpotent operators:

$$A = A_s + A_n.$$

Show that there exists a polynomial $p \in \mathbb{C}[t]$ such that $A_s = p(A)$.

Exercise 6.4. ♠ Define $\mathrm{ad}A\colon \mathrm{End}\, V \to \mathrm{End}(V)$ via $\mathrm{ad}A(B) = AB - BA$. Show that $(\mathrm{ad}A)_s = \mathrm{ad}A_s$ and $\mathrm{ad}A_s = P(\mathrm{ad}A)$, for some polynomial $P \in \mathbb{C}[t]$ such that $P(0) = 0$.

Exercise 6.5. ♠ Define $\overline{A}_s$ as the linear map which has the same eigenspaces as A_s but complex conjugate eigenvalues. Show that $\mathrm{ad}\overline{A}_s = Q(\mathrm{ad}A)$, for some polynomial $Q \in \mathbb{C}[t]$ such that $Q(0) = 0$.

7. Abelian categories and extensions

Let $\mathcal{A}$ be an abelian category. For objects $A, B \in \mathcal{A}$ define $\mathrm{Ext}^1(A,B)$ as the set of equivalence classes of short exact sequences $0 \to B \to C \to A \to 0$ with equivalence relation given by commuting diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & B & \longrightarrow & C' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

We say that a class in $\mathrm{Ext}^1(A,B)$ is *trivial* or *splits* if it is of the form

$$0 \to B \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} A \oplus B \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} A \to 0.$$

Exercise 7.1. ♠ Show that a class of a short exact sequence $0 \to B \xrightarrow{i} C \xrightarrow{p} A \to 0$ is trivial in $\mathrm{Ext}^1(A,B)$ if and only if there exists $s\colon A \to C$ such that $ps = \mathrm{Id}_A$ if and only if there exists $\pi\colon C \to B$ such that $\pi i = \mathrm{Id}_B$.

Exercise 7.2. ♠ Define Yoneda composition $\mathrm{Hom}(B,D) \otimes \mathrm{Ext}^1(A,B) \to \mathrm{Ext}^1(A,D)$ (hint: use push-outs) and $\mathrm{Ext}^1(A,B) \otimes \mathrm{Hom}(D,A) \to \mathrm{Ext}^1(D,B)$ (hint: use pull-backs).

Exercise 7.3. ♠ Define sum of two elements in $\mathrm{Ext}^1(A,B)$ and show that this operation endows $\mathrm{Ext}^1(A,B)$ with a commutative group structure.

8. Complex semi-simple Lie algebras

Exercise 8.1. ♠ Show that the space of diagonal matrices of trace zero is Cartan subalgebra of $\mathfrak{sl}(n,\mathbb{C})$.

Exercise 8.2. ♠ Show that for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ an element $x \in \mathfrak{g}$ is semi-simple/nilpotent (i.e. $\mathrm{ad}x$ is semi-simple/nilpotent operator) if and only if x is semi-simple/nilpotent as a matrix.

Exercise 8.3. ♠ Show that if $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra in a complex semi-simple LIe algebra then $\mathfrak{h}$ is a nilpotent subalgebra which coincides with its normalizer $n(\mathfrak{h}) = \{x \in \mathfrak{g} \,|\, [x,\mathfrak{h}] \subset \mathfrak{h}\}$.

Exercise 8.4. ♠ Let $\mathfrak{g}$ be a complex Lie algebra which has a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $R \subset \mathfrak{h}^* \setminus \{0\}$ is a finite subset and for $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$ $[h,x] = \alpha(h)x$. Show that then $\mathfrak{g}$ is semi-simple and $\mathfrak{h}$ is a Cartan subalgebra.

Exercise 8.5. ♠ Let $\mathfrak{h} \subset \mathfrak{so}(4, \mathbb{C})$ be the subalgebra consisting of matrices of the form

$$\begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix},$$

for $a, b \in \mathbb{C}$. Show that $\mathfrak{h}$ is a Cartan subalgebra and find the corresponding root decomposition.

Exercise 8.6. ♠ Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. Suppose that $\mathfrak{g}$ has a faithful representation (i.e. $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is injective) on which $x \in \mathfrak{g}$ acts diagonalisably. Show that x is a semi-simple element in $\mathfrak{g}$ and hence acts diagonalisably on any representation of $\mathfrak{g}$.

Exercise 8.7. ♠ Suppose that $\mathfrak{g}_1$, $\mathfrak{g}_2$ are complex semi-simple Lie algebras and that $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a surjective homomorphism. Let $x = x_s + x_n$ be Jordan decomposition of $x \in \mathfrak{g}_1$. Show that $\varphi(x) = \varphi(x_s) + \varphi(x_n)$ is Jordan decomposition of $\varphi(x) \in \mathfrak{g}_2$.

Exercise 8.8. ♠ Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra in a complex semi-simple Lie algebra and $\alpha \in R$ a root. Show that $x \in \mathfrak{g}_\alpha$ is nilpotent.

Exercise 8.9. ♠ Consider $\alpha = e_1 - e_2 \in \mathfrak{sl}(3, \mathbb{C})^*$. Show that α is a root for the Cartan subalgebra $\mathfrak{h}$ consisting of diagonal matrices with trace zero. Describe the subalgebra $\mathfrak{sl}(2, \mathbb{C})_\alpha$ and decompose $\mathfrak{sl}(3, \mathbb{C})$ into irreducible $\mathfrak{sl}(2, \mathbb{C})_\alpha$ modules.

Exercise 8.10. ♠ Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra in a complex semi-simple Lie algebra. Show that the root space $R \subset \mathfrak{h}^*$ spans $\mathfrak{h}$.

9. Root systems

Exercise 9.1. ♠ Consider $\mathbb{R}^n$ with the usual inner product and the orthonormal basis $\{e_i\}_{i=1}^n$. Let $E = \{(x_1, \ldots, x_n) \,|\, \sum x_i = 0\}$ and $R = \{e_i - e_j \,|\, 1 \leq i, j \leq n, i \neq j\} \subset E \backslash \{0\}$. Show that R is a reduced root system of the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$.

Exercise 9.2. ♠ Let $R \subset \mathbb{R}^n$ be given by

$$R = \{\pm e_i, \pm 2e_i \,|\, 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \,|\, 1 \leq i, j \leq n, i \neq j\},$$

where e_i is the standard basis in $\mathbb{R}^n$. Show that R is a non-reduced root system (of type BC_n).

Exercise 9.3. ♠ Let $R \subset E$ be a root system. Show that the set

$$R^\vee = \{\alpha^\vee \,|\, \alpha \in R\} \subset E^*$$

where $\alpha^\vee$ is the coroot, is also a root system.

Exercise 9.4. ♠ Let $R \subset E$ be a root system. Let $E' = \text{span}\{\alpha, \beta\}$ be the plane spanned by non-collinear roots $\alpha, \beta \in R$. Show that $R' = R \cap E$ is a root system.

A subset B of R is a *base* of a root system R if

(1) B is a basis for E and
(2) Any $\alpha \in R$ can be written as $\alpha = \sum_{\beta \in B} c_\beta \beta$ with $c_\beta \in \mathcal{Z}$ where all the non-zero coefficients c_β have the same sign.

Exercise 9.5. ♠ Show that if B is a base for a root system then the angle between any two distinct elements of B is obtuse (greater or equal $\pi/2$).

Exercise 9.6. ♠ Find a base for the root system from Exercise 9.1.

Exercise 9.7. ♠ Show that any root system has a base.

Exercise 9.8. ♠ Describe the root lattice Q, the coroot lattice $Q^\vee$ and the weight lattice P for all root systems of rank 2. Calculate P/Q.

Exercise 9.9. ♠ Let R be the root system in Exercise 9.1. Calculate the Weyl group and describe the Weyl chambers.

Exercise 9.10. ♠ Let $\Pi \subset R$ be the set of simple roots. Show that $\Pi^\vee \subset R^\vee$ is the set of simple roots in $R^\vee$ as in Exercise 9.3.

Exercise 9.11. ♠ Let $v_1, \ldots, v_n$ be non-zero vectors in an euclidean space E such that $(v_i, v_j) \le 0$, for $i \ne j$ and $(v_i, t) > 0$, for some $t \in E$. Show that $v_1, \ldots, v_n$ are linearly independent.

Exercise 9.12. ♠ Consider an irreducible reduced root system R, the root lattice Q and the weight lattice P. Show that $|P/Q|$ is the determinant of the Cartan matrix.

Exercise 9.13. ♠ Let $s_1, \ldots, s_r$ be simple reflections in a Weyl group W. Assume $w = s_{i_1} \ldots s_{i_l}$ and let $\beta_k = s_{i_1} \ldots s_{i_{k-1}}(\alpha_k)$ be the walls that a path from C_+ to $w(C_+)$ crosses. Show that if $\beta_k = \pm\beta_j$ for some $j \le k$ then $w = s_{i_1} \ldots \widehat{s_{i_j}} \ldots \widehat{s_{i_k}} \ldots s_{i_l}$.

Exercise 9.14. ♠

(1) Let R be the reduced root system of rank 2 with simple roots α and β. Show that the longest element in the Weyl group is a product of m factors $s_1 s_2 s_1 \dots$ where $\pi - \frac{\pi}{m}$ is the angle between α and β.

(2) Let R be a reduced root system and $s_1, \dots s_r$ simple reflections in the Weyl group W. Show that $(s_i s_j)^{m_{ij}} = 1$, where m_{ij} is as in (1).

Exercise 9.15. ♠ Let $\mathfrak{n}_\pm \subset \mathfrak{g}$ be the subalgebras $\bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$. Show that $\mathfrak{n}_\pm$ are nilpotent. Show that $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$ are solvable.

Exercise 9.16. ♠

(1) Show that if two vertices in a Dynkin diagram are connected by a single edge, then the corresponding simple roots are in the same W-orbit.

(2) Show that for a reduced irreducible root system, the Weyl group acts transitively on the set of all roots of the same length.

Exercise 9.17. ♠ Let $R \subset E$ be an irreducible root system. Show that then E is an irreducible representation of the Weyl group W.

Exercise 9.18. ♠ Show that the root systems B_n and C_n are dual to one another in the sense of Exercise 9.3.

Exercise 9.19. ♠ Show that Lie algebras $\mathfrak{sp}(4, \mathbb{C})$ and $\mathfrak{so}(5, \mathbb{C})$ are isomorphic.

10. Representations of semi-simple Lie algebras

Exercise 10.1. ♠ Let $Q_+ \subset Q$ be the cone $\bigoplus \mathbb{Z}_+[\alpha_i]$ and let $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$. Show that for any $t \in \mathbb{R}_+$ the set $\{\lambda \in Q_+ \mid (\lambda, \rho) < t\}$ is finite.

Exercise 10.2. ♠ Show that for any $\lambda \in P_+$ the set $\{\mu \in P_+ \mid \mu \preceq \lambda\}$ is finite.

Exercise 10.3. ♠ Let $\{\omega_i\}_{i=1}^r$ be the basis of P. Show that $\mathbb{C}[P]$ is isomorphic to $\mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$, for $x_i = e^{\omega_i}$.

Exercise 10.4. ♠ Let $\mathfrak{g}$ be a simple complex Lie algebra and $\theta \in R$ the maximal root of $\mathfrak{g}$.

(1) Show that any $\alpha \in R \cup \{0\}$ can be written as $\alpha = \theta - \sum n_i \alpha_i$, $n_i \in \mathcal{Z}_+$ and that this condition uniquely determines θ.

(2) Show that height of θ is maximal possible. The number $\mathrm{ht}(\theta) + 1$ is called the *Coxeter number* of $\mathfrak{g}$.

Exercise 10.5. Let $\mathfrak{g}$ be a simple complex Lie algebra and $(-, -)$ a non-degenerate invariant bilinear symmetric form on $\mathfrak{g}$.

(1) Show that the Casimir element C can be written

$$C = \sum_{\alpha\in R_+}(e_\alpha f_\alpha + f_\alpha e_\alpha) + \sum_i h_i^2$$

where h_i is an orthonormal basis in $\mathfrak{h}$.

(2) Show that on any highest weight module with highest weight λ C acts by $(\lambda, \lambda+2\rho)$.

(3) Show that if $(-,-)$ is the Killing form then the Casimir element acts by 1 on the adjoint representation.

(4) Let θ be the maximal root. Show that $K(\theta, \theta+2\rho) = 1$ and deduce that

$$K(\theta,\theta) = \frac{1}{2h^\vee}, \qquad h^\vee = 1 + \langle\rho, \theta^\vee\rangle.$$

The number $h^\vee$ is known as the *dual Coxeter number*.

Exercise 10.6. Let $k \geq 0$. Consider the representation $V = S^k\mathbb{C}^n$ of $\mathfrak{sl}(n,\mathbb{C})$.

(1) Compute all weights of V and describe the corresponding weight subspaces.
(2) Show that V contains a unique (up to a factor) vector v such that $\mathfrak{n}_+v = 0$, namely $v = x_1^k$, and deduce from this that V is irreducible.
(3) Find the highest weight of V and draw the corresponding Young diagram.

Exercise 10.7. Let $1 \leq k \leq n$. Consider the representation $V = \Lambda^k\mathbb{C}^n$ of $\mathfrak{sl}(n,\mathbb{C})$.

(1) Compute all weights of V and describe the corresponding weight subspaces.
(2) Show that V contains a unique (up to a factor) vector v such that $\mathfrak{n}_+v = 0$, namely $v = x_1 \wedge \ldots \wedge x_k$, and deduce from this that V is irreducible.
(3) Find the highest weight of V and draw the corresponding Young diagram.

Exercise 10.8. Let V be an irreducible representation of $\mathfrak{sl}(n,\mathbb{C})$ with given Young diagram. Describe the Young diagram of V^*.

Exercise 10.9. Let $V = \mathbb{C}^3$ be the fundamental representation of $\mathfrak{sl}(3,\mathbb{C})$. Decompose $S^2V \otimes V^*$ into a direct sum of irreducible representations.

Exercise 10.10. Consider the Lie algebra $\mathfrak{g} = \mathrm{sp}(4,\mathbb{C})$ and the fundamental representation $V = \mathbb{C}^4$. Check that $V \simeq V^*$ as $\mathfrak{g}$-representations and $\Lambda^2V = W \oplus \mathbb{C}$.

Exercise 10.11. Find the character of the adjoint representation of $\mathrm{sp}(4,\mathbb{C})$.

Exercise 10.12. (1) Let $f(x)$, $x = (x_1, \ldots, x_n)$ be a polynomial in n variables. Show that if $f(x) = 0$ for all $x \in \mathbb{Z}_+^n$ then $f = 0$.
(2) Show that if $f_1, f_2 \in s\mathfrak{h}$ are such that $f_1(\lambda) = f_2(\lambda)$ for all $\lambda \in P_+$ then $f_1 = f_2$.

Exercise 10.13. Let V_n be the irreducible $n+1$-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$. Show that

$$V_n \otimes V_m \simeq \bigoplus V_k,$$

where the direct sum is over all $k \in \mathcal{Z}_+$ such that

$$|n-m| \leq k \leq n+m,$$
$$n+m-k \in 2\mathbb{Z}.$$

Exercise 10.14. Let V be the adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$. Find the highest weight of V and the corresponding Young diagram.

Exercise 10.15. Using the formula for $\mathrm{ch}(L_\lambda)$ calculate the character of the fundamental representation $\mathbb{C}^3$ of $\mathfrak{sl}(3,\mathbb{C})$.