Algebraic Topology I problems

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Problems with ♠ are already done, with **L** were done on the lecture, with !!! should be done next time

1) ♠A functor $F : C \to D$ is an equivalence iff

1. $F : Mor_C(X_1, X_2) \to Mor_D(F(X_1), F(X_2))$ is an bijection for any pair of objects $X_1, X_2 \in Ob(C)$
2. Any object $Y \in Ob(D)$ is isomorphic to the object of the form $F(X)$ for some $X \in Ob(C)$

Written homework: we have defined the inverse of F, called G by a choice of an isomorphism $f_Y : Y \to F(X)$. Define a transformation of functors $G \circ F \to Id_C$ and check the compatibility of the relevant diagram.

2) ♠[Yoneda1] Given a category C. For $X \in Ob(C)$ let $h^X(-) = Mor_C(-, X) \in Func(C^{op}, S)$.
Show that $h_X : C \to Func(C, S)$ is a full embedding.

3) ♠[Yoneda2] Given a category C. For $X \in Ob(C)$ and $F \in Func(C^{op}, S)$.
Show that the map $NatTr(h^X, F) = Mor_{Func(C^{op},S)}(h^X, F) \to F(X)$ defined by $\Phi \mapsto \Phi(X)(id_X)$ is a bijection.

4) ♠A small category C in which for any pair of objects there exists at most one morphism is equivalent to the category defined by a partially ordered set.

5) If in a category every morphism is an isomorphism and every two objects are isomorphic, then this category is equivalent to the category defined by a group. (Example: the fundamental groupoid of a topological space.)

6) Show that the functor $h^X \times h^Y : C \to S$ is representable by the cartesian product (if it exists in C).

7) **L** Construct the left adjoint functor to $Hom(V, -) : Vect_k \to Vect_k$ and the right adjoint if $\dim V < \infty$.

8) **L** Let $F : I \to C$ be a functor from a small category I (i.e. a diagram in C). Show that $L \in Ob(C)$ is an inverse limit $\lim_I F$ iff the functor $X \mapsto \lim_I Mor_C(X, F(-))$ is representable by L.

9) What is an analogue of the previous statement for the direct limit $\mathrm{colim}_I F$?

10) ♠Show that if in C there exist equalizers of any pair of morphism and there exist products of any family of objects, then in C exist limits indexed by any category.

11) ♠A functor $G : C \to D$ which is left adjoint preserves colimits. This means that for any diagram $F : I \to C$ the natural map $\mathrm{colim}_I G \circ F \to G(\mathrm{colim}_I F)$ is an isomorphism. (see 9))

12) Consider the following obvious functors and check whether there exist right and left adjoint functors:
a) Topological Hausdorff spaces → Compact spaces
b) Topological spaces → Sets
c) Groups → Sets
d) Topological groups → Topologicals spaces
e) Abelian groups → Groups
f) R-modules → Abelian groups
g) ...

13) ??? Let $L : C \rightleftharpoons D : R$ be a pair of adjoint functors. By adjunction we have natural maps $\tau_X : X \to RL(X)$ for $X \in Ob(C)$ and $\sigma_Y : LR(Y) \to Y$ for $Y \in Ob(D)$ (in fact transformations of functors). Show that the composition

$$R(Y) \xrightarrow{\tau_{R(Y)}} RLR(Y) \xrightarrow{R(\sigma_Y)} R(Y)$$

is the identity. Similarly for the compositions with interchanged roles of functors (see the next exercise).

14) ??? Let $L : C \to D$ and $R : D \to C$ be two functors. Suppose that there are given transformation of functors $\sigma : LR \to Id_D$ and $\tau : Id_C \to RL$. Composing with L we define transformation of functors:

$$F * \tau : L \to LRL \quad \text{and} \quad \sigma * L : LRL \to L.$$

Composing with R we define transformation of functors:

$$\tau * R : R \to RLR \quad \text{and} \quad R * \sigma : RLR \to R.$$

Show that if

$$(\sigma * L) \circ (L * \tau) = Id_L \in NatTr(L, L) \quad \text{and} \quad (R * \sigma)(\tau * R) = Id_R \in NatTr(R, R)$$

then L is left adjoint to R.

15) Let R be a commutative ring. Show that there is natural isomorphism of R-modules: $(A \oplus B) \otimes C \simeq (A \otimes C) \oplus (B \otimes C)$.

16) ♠(written) Let $f : R \to S$ be a ring homomorphism. (Then the ring S is a R-module via f.) For a S module N define a R-module f^*N, which is equal to N as a group with multiplication $r \cdot n = f(r)n$ for $r \in R$. For a R-module M define $f_*M = S \otimes_R M$ with multiplication given by $s_1 \cdot (s_2 \otimes m) = (s_1 s_2) \otimes m$. Show that f_* and f^* are adjoint. Does one have to assume that the rings are commutative?

17) (written) Let (X, A) be a pair of topological spaces. Suppose that there exist a neighbourhood $U \supset A$ and a deformation of U to A in X constant on A (i.e. a homotopy $h_t : U \to X$ such that $h_0(u) = u$, $h_t(a) = a$, $h_1(u) \in A$ for $t \in [0, 1]$, $u \in U$, $a \in A$) and a continious function $X \to [0, 1]$, $f^{-1}(1) = A$, $f_{|X-U} = 0$. Prove that $A \hookrightarrow X$ is a cofibration.

18) ♠Let C be a category. Let $(A, A \to A \sqcup A)$ be a cogroup object, $(B, B \times B \to B)$ a group object. Suppose that two group structures on $Mor_C(A, B)$ have the same neutral objects. (If C has an object which is both initial and final then this assumption is satisfied.) Prove that these structures coincide. Moreover the common group structure is abelian.

Hint: show $(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$.

19) ♠Show that the functor defined by the smash product in Top_*

$$F_Y(X) := X \wedge Y = X \times Y/(X \times * \cup * \times Y)$$

is adjoint to $\mathrm{Map}_{Top_*}(Y, -)$. (Be careful with topology, assume that Y is locally compact, Z Hausdorff.)

20) ♠Show $S^1 \wedge S^n \simeq S^{n+1}$.

21) ♠Let $X \to Y$ be a continuous map of topological spaces which is surjective. Show that if it is open or closed, then the topology on Y is a quotient topology. Show an example of a quotient topology which is not open nor closed.

22) ♠Show that 1-point compactification of the Möbius strip is homeomorphic to $\mathbf{RP}^2$.

23) Let X be a topological space which is arc-connected. Let p and q be two points. Assume that $\{p\} \to X$ and $\{q\} \to X$ are cofibtations. Show that $(X, \{p\})$ is homotopy equivalent to $(X, \{q\})$, i.e. an isomorphism in $hTop_*$.

24) ♠Suppose $A \hookrightarrow X$ is a cofibration and A is contractible. Then X and X/A are homotopy equivalent.

25) ♠Show that the empty torus with one disc glued to the meridian (i.e $S^1 \times S^1 \cup D^2 \times \{pt\}$) and $S^2 \vee S^1$ are homotopy equivalent.

26) There exist following homotopy equivalences:
a)♠$S^n/S^k \sim S^n \vee S^{k+1}$
b)♠$(S^n \times S^m)/(S^n \times \{s_0\}) \sim S^{n+m} \vee S^m$
c)!!! $\Sigma(S^n \times S^m) \sim S^{n+1} \vee S^{m+1} \vee S^{n+m+1}$
d)!!! $\Sigma P_g \sim S^3 \vee \bigvee_{2g} S^2$, where P_g is a compact surface of genus g.

27) If $X = A \cup B$ where A, B closed subsets and $A \cap B \subset A$ is a Borsuk pair, then $B \subset X$ is a Borsuk pair.

28) ♠Let $f : A \to X$ be a map and suppose that X is contractible. Then the cone of the inclusion $CA \cup_f X$ is homotopy equivalent to the suspension.

29) (written)♠Let $E \to B$ be a map and suppose that E is contractible. Then the homotopy fiber $F_b := \{(e,\gamma) \in E \times Map(I,B) \ : \ \gamma(0) = f(e)\,,\ \gamma(1) = b\}$ is homotopy equivalent to the loop space ΩB.

30) ♠What is the homotopy cofiber of $X \to pt$?

31) ♠What is the homotopy fiber of $pt \to X$?

32) ♠Let $f : E \to B$ be a fibration over a path-connected space.
a) if there exists $b \in B$ such that $f^{-1}(b)$ is path-connected, then E is path connected.
b) if E is path-connected and B simplyconnected, then the fibres are path-connected.

33) ♠Find fibrations
a) $S^0 \hookrightarrow S^1 \twoheadrightarrow S^1$
b) $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$
c) $S^3 \hookrightarrow S^7 \twoheadrightarrow S^4$
d) $S^7 \hookrightarrow S^{15} \twoheadrightarrow S^8$

34) ♠
a) Find a fibration $E \longrightarrow B$ such that $B \sim \mathbf{RP}^\infty$, $E \sim S^1$, and the fiber is homotopy equivallent to S^1.
b) Find a fibration $E \longrightarrow B$ such that $B \sim \mathbf{CP}^\infty$, $E \sim S^2$, and the fiber is homotopy equivallent to S^3.
c) Find a fibration $E \longrightarrow B$ such that $B \sim \mathbf{HP}^\infty$, $E \sim S^4$, and the fiber is homotopy equivallent to S^7.

35) ♠Show that a locally trivial fibration is a fibration (in the sense of Hurewicz). (Assume e.g. that the base is paracompact).

36) ♠Find the homtopy fiber of the map $\mathbf{CP}^\infty \vee \mathbf{CP}^\infty \to \mathbf{CP}^\infty$ identifying two copies of $\mathbf{CP}^\infty$.

37) Show that the homotopy fiber of $X \vee X \to X$ is $\Sigma\Omega X$.
(*Technical step*: Let $f : P \to X$ be a map from a contractible space. Then the push-out of mapping cylinders $\begin{pmatrix} P & \to & Z(f) \\ \downarrow & & \\ Z(f) & & \end{pmatrix}$ is homotopy equivalent to $X \vee X$.)

38) ? What is the relation of the fiber inclusion map $\Sigma\Omega X \hookrightarrow X \vee X \twoheadrightarrow X$ with the map $\Sigma\Omega X \to X$ adjoint to two natura maps $id, inverse : \Omega X \to \Omega X$.

39) ♠Suppose that in the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \end{array}$$

$i\,,j$ are cofibrations, f and g are homotopy equivalences. Show that (f,g) is a homotopy equivalence of pairs.

40) If $p : E \to B$ is a fibration, then for any locally compact space X the natural map $Map(X,E) \to Map(X,B)$ is a fibration. What can you say about the map $Map(B,X) \to Map(E,X)$?

41) ♠Is the following statement true? Suppose that $E_1 \to B$ and $E_2 \to B$ are fibrations and E_1 is homotopy equivalent to E_2. Then there exist a fiberwise homotopy equivalence

$$\begin{array}{ccc} E_1 & \overset{\sim}{\longrightarrow} & E_2 \\ & \searrow \ \swarrow & \\ & B & \end{array}$$

42) The homotpy coequalizer of two maps $f,g : X \to Y$ is defined as the push-out

$$\begin{array}{ccc} X \vee X & \overset{f\vee g}{\longrightarrow} & Y \\ \downarrow & & \downarrow \\ X \wedge I^+ & \longrightarrow & hcoeq(f,g). \end{array}$$

Dually the homotopy equalizer is defined by the pull-back

$$\begin{array}{ccc} Y \times Y & \overset{f\times g}{\longleftarrow} & X \\ \uparrow & & \uparrow \\ P(X) & \longleftarrow & heq(f,g). \end{array}$$

Consider the circle S^1 as the unit complex numbers. Let $f,g : S^1 \to S^1$, $f = id$, $g(z) = \overline{z}$. Describe the homotopy coequalizer of f and g. What is the homotopy type of the homotopy equalizer? (??)

43) Let $f : X \to Y$ be a map of pointed spaces. Compare $C(Sf)$ with $S(Cf)$ and the dual objects involving homotopy fiber and loop spaces.

44) ♠Compute homotopy groups of surfaces of positive genus.

45) Compute $\pi_n(S^n)$ (♠),
♠(written) $\pi_3(S^2)$, show $\pi_n(S^3) \simeq \pi_n(S^2)$ for $n > 2$. What are consequences for homotopy groups from Problem 33?

46) ♠Compute homotopy groups of $\mathbf{RP}^\infty$ and $\mathbf{CP}^\infty$. What do you know about $\pi_i(\mathbf{HP}^\infty)$?

47) ♠If the homotopy groups of the basis and the fiber are finite, then the homotopy groups of the total space of the fibration are also finite.

48) ♠If the fibration has a section, then $\pi_n(E) \simeq \pi_n(B) \oplus \pi_n(F)$. (Assumption: F connected, $n > 1$)

49) ♠If the fiber contracts to a point in E, then $\pi_n(B) \simeq \pi_n(E) \oplus \pi_{n-1}(F)$. (Assumption: $n > 1$.)

50) ♠Compute homotopy groups π_i of the Stiefel manifold $V_k(\mathbf{R}^n)$ for $i \leq n-k$. (Assumption: $n > 1$.)

51) Whitehead product:
♠naturality with respect to maps and the action of the fundamental groupoid
♠$[a,b] = (-1)^{pq}[b,a]$
♠$[a,\gamma] = a - [h_\gamma(a)]$ for $\gamma \in \pi_1(X)$
*** show the Jacobi rule:

$$(-1)^{pq}[[a,b],c] + (-1)^{qr}[[b,c],a] + (-1)^{rp}[[c,a],b] = 0\,.$$

52) ♠If A is a retract of X, then $\pi_n(X) \simeq \pi_n(A) \oplus \pi_n(X, A)$

53) ♠If A is contractible in X then $\pi_n(X, A) \simeq \pi_n(X) \oplus \pi_{n-1}(A)$

54) ♠$\pi_n(X \vee Y) \simeq \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y)$

55) ♠Decompose into cells the projective spaces $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$

56) !!! Decompose the lens space $L(n; k) = \mathbf{CP}^2/\mathbf{Z}_n$ (action of the generator of $\mathbf{Z}_n$: $[z_1, z_2] \mapsto [\xi z_1, \xi^k z_2]$, where $(n, k) = 1$, ξ is an n-th primitive root of 1.

57) !!! Prove that any finite CW-complex can be embedded into an Euclidian space.

58) !!! HELP
A) Let $Y \subset Z$ be a cofibration. Show that two conditions are equivalent:
   a) $\pi_n(Z, Y) = 0$
   b) given a map $f : D^n \to Z$ and a homotopy $h_t : S^{n-1} \to Z$ such that $h_0 = f_{|S^{n-1}}$ and $h_1(S^{n-1}) \subset Y$; it is possible to extend the homotopy to $H_t : D^n \to Z$ in a way that $H_0 = f$ and $H_1(D^n) \subset Y$.
B) Generalize the statement for any map $Y \to Z$ instead of an inclusion.
C) Replace the pair (D^n, S^{n-1}) by any cofibration (X, A), the condition a) by ,,$Y \to Z$ is an n-equivalence" and show a) $\Rightarrow$ b).

59) Whitehead theorem
A)♠Assume that $Y \to Z$ is an n-equivalence. Show that for a CW-complex X of dimension $\leq n$ the map $[X, Y] \to [X, Z]$ is surjective. (Use HELP with $A = \emptyset$.)
B) For $\dim U < n$ the map $[U, Y] \to [U, Z]$ is injective. (In HELP take $X = U \times I$, $A = U \times \{0, 1\}$, $f =$ homotopy between maps $U \to Z$, $h_t = h_0$.)

60) *** Show that any Lie group G (e.g. $G = U(n)$ or $SO(n)$ if you are not familiar with a general theory) we have $\pi_2(G) = 0$. Hint: assume that G is compact and simply connected and consider the maximal torus T and the exact sequence

$$0 \to \pi_2(G) \to \pi_2(G/T) \to \pi_1(T) \to 0.$$

61) *** Let (X, A) be a relative CW-complex obtained from A by adding one n-cell. Show, that $\pi_n(X, A, a_0)$ is a free $\pi_1(A, a_0)$-module on one generator. (Here a_0 is a distinguished point in A.)

62) ♠Let $X = \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n \ : \ z_i \neq z_j \text{ for } i \neq j\}$. Compute $\pi_k(X)$ for $k > 1$. Do you know $\pi_1(X)$?
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63) !!! (compare Problem 49) For $n > 1$ we have $\pi_n(S^4) \simeq \pi_n(S^7) \oplus \pi_{n-1}(S^3)$ and $\pi_n(8) \simeq \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$.

64) !!! Let P_n be the space of complex polynomials od degree n without multiple roots. Compute $\pi_i(P_n)$ for $i \geq 0$.

65) !!! Suppose that X is a CW complex of dimension $\leq 2n + 2$ for some $n \geq 0$. Prove that the diagonal map $d : X \to X \times X$ is homotopic to a map d' such that $d'(X) \subset (X \times X^n) \cup (X^n \times X)$, where X^n is the n-th skeleton of X.

66) !!! If X is a CW-complex of type $K(\pi, n)$ for $n > 1$ and Y is an arbitrary CW-complex, then $\pi_n(X \vee Y) \simeq \pi_n(Y) \oplus \bigoplus_{\lambda \in \pi_1(Y)} \pi_\lambda$ where $\pi_\lambda = \pi$ for each λ.

67) **(written homework)** Given a sequence of groups $\{\pi_q\}_{q>0}$ with π_q abelian for $q > 1$, and given an action of π_1 as a group of operators on each π_q for $q > 1$, prove that there is a space Y which realizes this sequence (that is $\pi_q(Y) \simeq \pi_q$ and $\pi_l(Y)$-actinon on $\pi_q(Y)$ corresponds to the action of π_1 on π_q).