Some problems

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1.1 Exercises from Fulton-Harris §7

1.2 Let G be a topological group. Show that if a subgroup of G is open, then it is closed. Show that component of 1 is a subgroup.

1.3 Show that $\pi_1(G)$ is abelian.

1.4 Let G be a connected topological group. Let $p : \widetilde{G} \to G$ be a connected covering. Choose an inverse image of 1. Show that $\widetilde{G}$ has a natural group structure, such that p is a homomorphism. Interpret the kernel as the group of deck transformation.

1.5 Check basic properties of quaternions (octonions) e.g.: $a\bar{a} = |a|^2 \in \mathbb{R}$, $|ab| = |a||b|$ for $a, b \in \mathbb{H}$ (or $\in \mathbb{O}$).

1.6 Check that the set of the maps of algebras $map(C, H)$ is the 2-dimensional sphere.

1.7 Prove that $Aut(H) = \mathbb{H}^*/\mathbb{R}^*$.

1.8 Show that the only R-division (associative) algebra are $\mathbb{C}$ and $\mathbb{H}$.

1.9 Show that $Sp(n) \subset SU(2n)$.

1.10 Show, that $Sp(n)$ is the maximal compact subgroup of $Sp(n, \mathbb{C})$.

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2.1 Exercises from Fulton-Harris §8

2.2 Show that $\mathbb{R}^3$ with the vector product $\times$ is a Lie algebra isomorphic to $so(3)$.

2.3 Compare the Lie algebra of upper-triangular 3×3 matrices with 0's on the diagonal with the Lie algebra generated by x and $\frac{d}{dx}$ acting on the polynomial ring $\mathbb{C}[x]$.

2.4 Compute explicitly exp for the algebras above.

2.5 Check that the commutator of two derivations of an algebra (not necessarily associative) is a derivation.

2.6 For any $\mathbb{R}$-algebra compare $Lie(Aut(A))$ and $Der(A)$.

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3.1 Compute $Hom(S^3, S^3)$.

3.2 Check that the complexifications of $U(n)$, $SU(n)$, $O(n)$, $SO(n)$, $Sp(n)$ are the groups $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $O(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$.

3.3 Compute Lie algebras of the classical compact groups, and find their complexifications.

3.4 Compute the differential of the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^2$ in the direction X. Show that it does not vanish if A and X are symmetric, A positive definite.

3.5 Compute few terms of Baker-Campbell-Hausdorff formula. (At least the third term.)

3.6 Check the formula

$$\frac{d}{dt}e^{A+tB} = e^A \left(B - \frac{[A,B]}{2!} + \frac{[A,[A,B]]}{3!} - \frac{[A,[A,[A,B]]]}{4!} + \dots \right).$$

3.7 Show that exp for $SU(2)$ is surjective. At which points it is a submersion?

3.8 Let $G \subset GL_n(\mathbb{C})$ be a reductive group. Define a hermitian product in $\mathbf{g}$ by the formula $\langle\langle X, Y\rangle\rangle = tr(XY^*)$. The hermitian product in $\mathbf{g}$ allows to define the Cartan involution in $GL(\mathbf{g})$. Show that $Ad(G) \subset GL(\mathbf{g})$ is a reductive subgroup. (Show that $(ad_X)^* = ad_{X^*}$.)

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4.1 For which groups: $GL_n^+(\mathbb{R})$, $SL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, B_n (upper triangular), N_n (upper triangular with 1's on the diagonal) exp is surjective?

4.2 Compute the Killing form for the classical algebras (in particular for $sl_n(\mathbb{C})$, $gl_n(\mathbb{C})$) and for b_n. Show, that for $sl_n(C)$ the Killing form is equal up to a constant to $B_0(X,Y) = Tr(XY)$.

4.3 Let $\mathbf{g} \subset End(\mathbb{C}[z])$ be the Lie subgroup generated by the multiplication by z and $\frac{\partial}{\partial z}$. Show that $\mathbf{g}$ has a finite dimension. Find a group G which has Lie algebra $\mathbf{g}$. Does G act on $\mathbb{C}[z]$?

4.4 Exercises from Fulton-Harris §9.

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5.1 Compute what are the maximal tori in $U(n)$, $SU(n)$, $SO(n)$ and $Sp(n)$. What are the normalizers $N(T)$ and the Weyl groups.

5.2 Show that in $U(n)$ every commutative subgroup is included in a maximal torus.

5.3 Show that the same statement is not true for $SO(3)$.

5.4 Let T be a torus (compact connected commutative Lie group). Show that there exists $g \in T$ such that $\langle g\rangle$ is dense in T.

5.5 Show that for any element g of a topological group G $closure\langle g\rangle$ is abelian. For $G = U(n)$ characterize those elements for which $closure\langle g\rangle$ is a maximal torus.

5.6 Jordan decomposition. Let a semisimple Lie algebra acts on a vector space $\rho : \mathbf{g} \to End(V)$, and let $a \in \mathbf{g}$. Decompose $\rho(a) = d + n$ where d is diagonal in a certain basis, and n is upper-triangular with 1's at the diagonal. Show that there exist elements a_d and a_n in $\mathbf{g}$ such that $\rho(a_d) = d$, $\rho(a_n) = n$ and $a_d + a_n = a$.

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6.1 Decompose $Hom(V, V)$ into irreducible representations of $G = GL(V)$, where G acts on $Hom(V, V)$
– left multiplication
– conjugation.

6.2 Show that the natural representation of $SL_2(\mathbb{C})$ is isomorphic to its dual. (This is not true for $GL_2(\mathbb{C})$.)

6.3 Decompose bilinear forms on V into irreducible representations of $GL(V)$.

6.4 Show that irreducible representations of $G \times H$ are of the form $V \otimes W$, where V is a irreducible representation of G and W is a irreducible representation of H.

6.5 Let V be irreducible real representation of odd dimension. Show that $V_{\mathbb{C}}$ is irreducible. If the dimension is even it can happen that $V_{\mathbb{C}} \simeq W \oplus \overline{W}$.

6.6 Show that two real representation are isomorphic if and only if their complexification are isomorphic.

6.7 Give the precise formula for the action of the Lie algebra $\mathbf{g}$ on $Hom_G(V, W)$.

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7.1 Decompose into a direct sum of irreducible subrepresentations $Sym^3(\mathbb{C}^2) \otimes Sym^2(\mathbb{C}^2)$.

7.2 Show that $Sym^n(Sym^2(\mathbb{C}^2)) \simeq \bigoplus_{s=0}^{[n/2]} Sym^{2n-4s}(\mathbb{C}^2)$.

7.3 Decompose $Sym^2 Sym^3(\mathbb{C}^2)$.

7.4 What are the irreducible representations of $GL_2(\mathbb{C})$?

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Fulton-Harris: exercises §13-§14.

8.1 Decompose $Sym^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$ into irreducible representations of $SL_2(\mathbb{C})$.

8.2 Show $\Lambda^{n-1}\mathbb{C}^n = (\mathbb{C}^n)^*$ as representations of $SL_n(\mathbb{C})$.

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Except from Fulton-Harris:

9.1 Show that for any simple Lie group G the quotient Λ/Λ_R is isomorphic to the center of G. Here Λ is the lattice of weights, and Λ_R is the sublattice generated by roots. First check the claim for $SL_n(\mathbb{C})$ and $SO(n)$.

9.2 Show that for a compact simple Lie group there exists only one up to a constant invariant scalar product, which is the –Killing form.

9.3 Check by examples ($\mathbb{C}^3$, $(\mathbb{C}^3)^*$, $Sym^2(\mathbb{C}^3)$, etc.) what is the kernel of the map $M(\omega) \to V(\omega)$ from the Verma module the to irreducible representation associated to a weight ω. Then find a formula in a book.

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10.1 Find the maximal weight of the dual representation of the irreducible representation of $SL_n(\mathbb{C})$ corresponding to the diagram λ.

10.2 Find Kostka numbers of the irreducible representation of $SL_n(\mathbb{C})$ corresponding to the diagram $\lambda = (n-1, n-2, n-3, \dots, 1, 0)$.

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11.1 Suppose that $H \subset G$ and $rank\, H = rank\, G$. Show that every root of H is a root of G. Give interesting examples ($GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \subset GL_{m+n}(\mathbb{C})$ is a trivial example). Compute Weyl groups.

11.2 For $SL_n(\mathbb{C})$: can one split the homomorphism $NT \to NT/T = W$?

11.3 Construct a 2-fold coverings $SU(2) \times SU(2) \to SO(4)$, $Sp(2) \to SO(5)$, $SU(4) \to SO(6)$.

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Exercises and examples from Fu-Ha SS16-17

12.1 Check how Weyl character formula works for $sp(n)$.

12.2 Assume that in the category $\mathcal{C}$ the isomorphism classes of objects forms a set X. Assume that the direct sum exist. Define an relation X^2:

$$([V],[W]) \sim ([V'],[W']) \qquad \text{if} \qquad \exists Z \in Ob(\mathcal{C}) \quad V \oplus W' \oplus Z \simeq V' \oplus W \oplus Z\,.$$

Check that it is an equivalence relation and that the in the set of equivalence classes one there is a natural stucture of an abelian group.

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13.1 „Bott periodicity" for complex Clifford algebras: Check that $C_{n+2} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the algebra of 2×2 matrices with coefficients in $C_n \otimes_{\mathbb{R}} \mathbb{C}$.

13.2 Compute the group of invertible elements C_2^* of the real Clifford algebra and the Clifford group Γ_2. Which two circles in Γ_2 form $Pin(2)$?

13.3 Find explicite isomorphisms or show that it does not exist between representations of $Spin(n)$:
– spinors S and S^* for n odd
– spinors $S^\pm$ and $(S^\pm)^*$ for n even

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14.1 Which spinors are complexifications of real representations of $Spin(n)$? The answer depends on the deisibility of n by 8.

14.2 Check the isomprphism of $Spin(2n)$ representations

$$Sym^2(S^+) = (\lambda^n)^+ + \lambda^{n-4} + \lambda^{n-8} + \dots$$
$$\Lambda^2(S^+) = \lambda^{n-2} + \lambda^{n-6} + \lambda^{n-10} + \dots$$
$$Sym^2(S^-) = (\lambda^n)^- + \lambda^{n-4} + \lambda^{n-8} + \dots$$
$$\Lambda^2(S^-) = \lambda^{n-2} + \lambda^{n-6} + \lambda^{n-10} + \dots$$

Here λ^k is the k-th exterior power of the natural representation of the ortogonal group.

14.3 Check the isomprphism of $Spin(2n+1)$ representations

$$Sym^2(S) = \lambda^n + (\lambda^{n-3} + \lambda^{n-4}) + \ldots + (\lambda^{n-4i-3} + \lambda^{n-4i-4}) + \ldots$$

$$\Lambda^2(S) = (\lambda^{(}n-1) + \lambda^{n-2}) + (\lambda^{n-5} + \lambda^{n-6}) + \ldots + (\lambda^{n-4i-1} + \lambda^{n-4i-2}) + \ldots$$

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15.1 Show that the stabilizer in $GL_7(\mathbb{R})$ of the 3-form $\Phi \in \bigwedge^3 \mathbb{R}^n$ defining multiplication of octonions is equal G_2.