Some problems

1 Show, that $Sp(n)$ is the maximal compact subgroup of $Sp(n, \mathbb{C})$.

2 Show that $\mathbb{R}^3$ with the vector product $\times$ is a Lie algebra isomorphic to $so(3)$.

3 Compare the Lie algebra of upper-triangular 3×3 matrices with 0's on the diagonal with the Lie algebra generated by x and $\frac{d}{dx}$ acting on the polynomial ring $\mathbb{C}[x]$.

4 Compute explicitly exp for the algebras above.

5 Check that the commutator of two derivations of an algebra (not necessarily associative) is a derivation.

6 For any $\mathbb{R}$-algebra compare $Lie(Aut(A))$ and $Der(A)$.

7 Compute $Hom(S^3, S^3)$.

8 Compute the differential of the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^2$ in the direction X. Show that it does not vanish if A and X are symmetric, A positive definite.

9 Compute few terms of Baker-Campbell-Hausdorff formula. (At least the third term.)

10 Check the formula

$$\frac{d}{dt} e^{A+tB} = e^A \left(B - \frac{[A,B]}{2!} + \frac{[A,[A,B]]}{3!} - \frac{[A,[A,[A,B]]]}{4!} + \dots \right).$$

11 Show that exp for $SU(2)$ is surjective. At which points it is a submersion?

12 Let $G \subset GL_n(\mathbb{C})$ be a reductive group. Define a hermitian product in $\mathbf{g}$ by the formula $\langle\langle X, Y \rangle\rangle = tr(XY^*)$. The hermitian product in $\mathbf{g}$ allows to define the Cartan involution in $GL(\mathbf{g})$. Show that $Ad(G) \subset GL(\mathbf{g})$ is a reductive subgroup. (Show that $(ad_X)^* = ad_{X^*}$.)

13 For which groups: $GL_n^+(\mathbb{R})$, $SL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, B_n (upper triangular), N_n (upper triangular with 1's on the diagonal) exp is surjective?

14 Compute the Killing form for the classical algebras (in particular for $sl_n(\mathbb{C})$, $gl_n(\mathbb{C})$) and for b_n. Show, that for $sl_n(C)$ the Killing form is equal up to a constant to $B_0(X, Y) = Tr(XY)$.

15 Let $\mathbf{g} \subset End(\mathbb{C}[z])$ be the Lie subgroup generated by the multiplication by z and $\frac{\partial}{\partial z}$. Show that $\mathbf{g}$ has a finite dimension. Find a group G which has Lie algebra $\mathbf{g}$. Does G act on $\mathbb{C}[z]$?

16 Show that in $U(n)$ every commutative subgroup is included in a maximal torus.

17 Show that the same statement is not true for $SO(3)$.

18 Let T be a torus (compact connected commutative Lie group). Show that there exists $g \in T$ such that $\langle g \rangle$ is dense in T.

19 Show that for any element g of a topological group G $closure\langle g \rangle$ is abelian. For $G = U(n)$ characterize those elements for which $closure\langle g \rangle$ is a maximal torus.

20 Decompose $Hom(V,V)$ into irreducible representations of $G = GL(V)$, where G acts on $Hom(V,V)$
– left multiplication
– conjugation.

21 Show that the natural representation of $SL_2(\mathbb{C})$ is isomorphic to its dual. (This is not true for $GL_2(\mathbb{C})$.)

22 Decompose bilinear forms on V into irreducible representations of $GL(V)$.

23 Let V be an irreducible real representation of odd dimension. Show that $V_{\mathbb{C}}$ is irreducible. If the dimension is even it can happen that $V_{\mathbb{C}} \simeq W \oplus \overline{W}$.

24 Show that two real representation are isomorphic if and only if their complexification are isomorphic.

25 Give the precise formula for the action of the Lie algebra $\mathbf{g}$ on $Hom_G(V,W)$.

26 What are the irreducible representations of $GL_2(\mathbb{C})$?

27 Show that for a compact simple Lie group there exists only one up to a constant invariant scalar product, which is the –Killing form.

28 Check by examples ($\mathbb{C}^3$, $(\mathbb{C}^3)^*$, $Sym^2(\mathbb{C}^3)$, etc.) what is the kernel of the map $M(\omega) \to V(\omega)$ from the Verma module the to irreducible representation associated to a weight ω. Then find a formula in a book.

29 Find Kostka numbers of the irreducible representation of $SL_n(\mathbb{C})$ corresponding to the diagram $\lambda = (n-1, n-2, n-3, \dots, 1, 0)$.

30 Suppose that $H \subset G$ and $rank\, H = rank\, G$. Show that every root of H is a root of G. Give interesting examples ($GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \subset GL_{m+n}(\mathbb{C})$ is a trivial example). Compute Weyl groups.

31 For $SL_n(\mathbb{C})$: can one split the homomorphism $NT \to NT/T = W$?

32 Construct a 2-fold coverings $SU(2) \times SU(2) \to SO(4)$, $Sp(2) \to SO(5)$, $SU(4) \to SO(6)$.

33 Assume that in the category $\mathcal{C}$ the isomorphism classes of objects forms a set X. Assume that the direct sum exist. Define an relation X^2:

$$([V],[W]) \sim ([V'],[W']) \qquad \text{if} \qquad \exists Z \in Ob(\mathcal{C}) \quad V \oplus W' \oplus Z \simeq V' \oplus W \oplus Z\,.$$

Check that it is an equivalence relation and that the in the set of equivalence classes one there is a natural stucture of an abelian group.

34 ,,Bott periodicity" for complex Clifford algebras: Check that $C_{n+2} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the algebra of 2×2 matrices with coefficients in $C_n \otimes_{\mathbb{R}} \mathbb{C}$.

35 Compute the group of invertible elements C_2^* of the real Clifford algebra and the Clifford group Γ_2. Which two circles in Γ_2 form $Pin(2)$?

36 Find explicite isomorphisms or show that it does not exist between representations of $Spin(n)$:
– spinors S and S^* for n odd
– spinors $S^{\pm}$ and $(S^{\pm})^*$ for n even

37 Which spinors are complexifications of real representations of $Spin(n)$? The answer depends on the deisibility of n by 8.

38 Check the isomprphism of $Spin(2n)$ representations

$$Sym^2(S^+) = (\lambda^n)^+ + \lambda^{n-4} + \lambda^{n-8} + \dots$$

$$\Lambda^2(S^+) = \lambda^{n-2} + \lambda^{n-6} + \lambda^{n-10} + \dots$$

$$Sym^2(S^-) = (\lambda^n)^- + \lambda^{n-4} + \lambda^{n-8} + \dots$$

$$\Lambda^2(S^-) = \lambda^{n-2} + \lambda^{n-6} + \lambda^{n-10} + \dots$$

Here λ^k is the k-th exterior power of the natural representation of the ortogonal group.

39 Check the isomprphism of $Spin(2n+1)$ representations

$$Sym^2(S) = \lambda^n + (\lambda^{n-3} + \lambda^{n-4}) + \ldots + (\lambda^{n-4i-3} + \lambda^{n-4i-4}) + \dots$$

$$\Lambda^2(S) = (\lambda^(n-1) + \lambda^{n-2}) + (\lambda^{n-5} + \lambda^{n-6}) + \ldots + (\lambda^{n-4i-1} + \lambda^{n-4i-2}) + \dots$$

40 Show that the stabilizer in $GL_7(\mathbb{R})$ of the 3-form $\Phi \in \bigwedge^3 \mathbb{R}^n$ defining multiplication of octonions is equal G_2.