1 Lecture 15 i ostatni

Triality

Data: $V_1 \times V_2 \times V_3 \xrightarrow{\Phi} K$ trilinear form

Q_i – a nondegenerate form of V_i

$\rightsquigarrow f_i : V_i \times V_j \longrightarrow V_k^* \underset{Q_k}{\approx} V_k$ $\quad i \neq j \neq k \neq i$

So far we have one example in dim 8, $K = \mathbb{C}$

$V_1 = \mathbb{C}^8 \quad V_2 = S^+ \quad V_3 = S^-$

and the map

$\mathbb{C}^8 \times S^+ \longrightarrow S^-$

is the action of the Clifford algebra C_8 (which is generated by $\mathbb{C}^8$) on spinors $S = S^+ \oplus S^-$.

Another example of triality in dim 8
$K = \mathbb{R} \quad V_1 = V_2 = V_3 = \mathbb{O}$ – octonions

$V_1 \times V_2 \to V_3$ octonion multiplication

(or $\phi(x, y, z) = -\mathrm{Re}\,(xyz)$)

the formula for ortogonality is satisfied

$$Q_3(f_3(x,y)) = Q_1(x)\, Q_2(y)$$

$$\|xy\|^2 = \|x\|^2 \|y\|^2 .$$

Exercise Find an isomorphisms

$$\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^8, \quad \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} \simeq S^{\pm}$$

preserving the triality.

②

The multiplication of octonions is determined by the multiplication of the imaginary part

$$\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}\, \mathbb{O}$$

$\mathrm{Im} \circ \mathrm{mult} = f : \mathrm{Im}\, \mathbb{O} \times \mathrm{Im}\, \mathbb{O} \to \mathrm{Im}\, \mathbb{O}$

or by adjunction

$$\phi_{/\mathrm{Im}\,\mathbb{O}} : (\mathrm{Im}\, \mathbb{O})^{\otimes 3} \longrightarrow \mathbb{R}$$

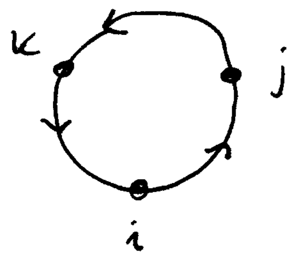
$$\phi(x,y,z) = -\mathrm{Re}(xyz) = \langle x, yz \rangle$$

This form is antisymmetric:

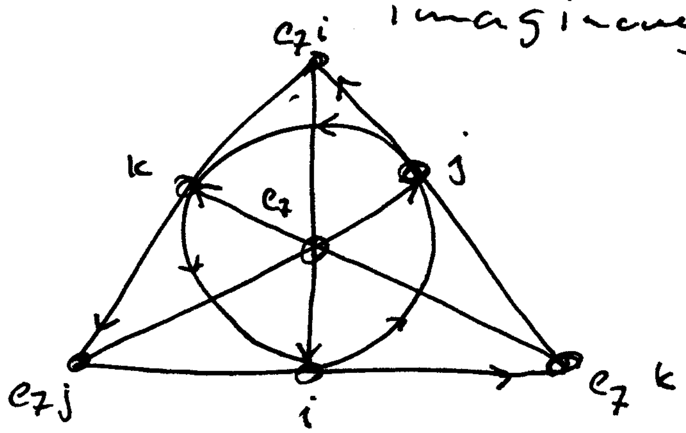
the shortest definition of $\mathbb{O}$:

$\mathbb{O} = \mathbb{H} \oplus e_7 \mathbb{H}$: $e_7{}^2 = -1$

e_7 anticommutes with imaginary quaternions



quaternions

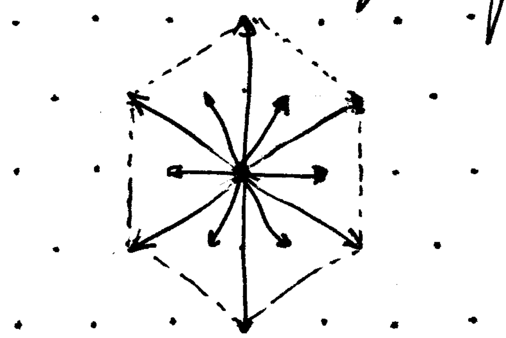


$$\phi_{/\mathrm{Im}\,\mathbb{O}} = \sum_{\substack{\text{ordered} \\ \text{edges } (a,b,c)}} e_a^* \wedge e_b^* \wedge e_c^*$$

③ Conclusion

The group of automorphisms of $\mathbb{O}$ is the subgroup of $SO(7)$ preserving the 3-linear form ϕ.

We will show that the root system of this group looks like this:



rk $G_2 = 2$

$\mathfrak{g}_2 = sl_3 \oplus V \oplus V^*$ (after complexification)

sl_3 subalgebra spanned by longer roots

$\Lambda^*_W = \Lambda^* = \Lambda^*_R$ $\qquad V = \mathbb{C}^3$ the natural rep. of sl_3

(The multiplication table for $\mathfrak{g}_2$ see Fulton § 22.2 on better p.360)

Construction of G_2 from spinors

[Adams: Lectures on Exceptional Lie groups)

We work with compact groups

1. $\boxed{Spin(6) = SU(4)}$ both have diagram

2. $\boxed{Spin(5) = Sp(2)}$ — " —

(To distinguish spinors from spheres we write $\mathcal{S}$ for spinors.)

Corollary $Spin(5)$ acts transitively on the sphere $S^7 \subset$ spinor representation $\mathcal{S}$ ($\dim_{\mathbb{C}} \mathcal{S} = 4$)

Proof: the tautological representation $\mathbb{H}^2$ of $Sp(2)$ correspond to the $\frac{1}{2}$(longer root) of $Sp(2)$. For spinor group this representation is $\mathcal{S}$. Since $Sp(2)$ acts transitively on the unit sphere in the tautological representation $Spin(5)$ acts transitively on $S^7 \subset \mathcal{S}$ $\square$

<u>lemma</u>

The representation $\mathbb{S}^+$ of $Spin(6)$ is the same as the <u>natural representation</u> after identification $Spin(6) \simeq SU(4)$.

<u>Proof</u>: the weights of $\mathbb{S}^+$ are the shortest in Λ_w^*

$\frac{1}{2}(\pm L_1 \pm L_2 \pm L_3)$ (even number of $-$).

They correspond to weights L_i $i=1..4$ for $SU(4)$, also the shortest in the lattice ■

The stabilizer of a point $z \in S^7 \subset \mathbb{S}^+$ is equal to $SU(3) \subset SU(4) \simeq Spin(6)$.

We need real spinors for $Spin(7)$

<u>lemma</u> The spinors $\mathbb{S}$ for $Spin(2n+1)$ admit an <u>invariant</u> nondegenerate symmetric form $\bar{\beta}$ if $n = 0$ or 3 mod 4.

<u>Proof</u> $\bar{\beta}(v\,w) = \int \bar{v} \wedge w$ where:

- $\wedge$ also it is the Clifford multiplication in $C(Q|W) = \Lambda^{\cdot}(W)$ (W is maximal isotropic in $\mathbb{C}^7$)
- $\bar{v} = t\alpha(v) = \alpha t(v)$ (inversion $\circ$ minus on vectors)
- $\int$ depends on the choice of $vol \in \Lambda^n W$.

<u>Invariance</u> of $\bar{\beta}$ — direct check.

<u>Symmetry</u> we check for $2n+1 = 7$, $\dim W = 3$

for $\bar{v} \wedge w \in \Lambda^3 W$ we have to check that $\bar{v} \wedge w = \bar{w} \wedge v$. That is for $v \in \Lambda^3 W$? $\bar{v} = v$.

$\overline{(\bar{v} \wedge w)}$

$\overline{e_1 \wedge e_2 \wedge e_3} = (-e_3) \wedge (-e_2) \wedge (-e_1) = -e_3 \wedge e_2 \wedge e_1 =$

$= e_1 \wedge e_2 \wedge e_3$

(5) This way we have an action of the compact group Spin(7) on S preserving the symmetric form $\bar{\beta}$ (S is a complex space, and $\bar{\beta}$ is a complex form)

<u>General statement</u> If a compact group G acts on a complex space V preserving a symmetric nondegenerate form, then there exists a real vector space $V_{\mathbb{R}}$ and an action of G on $V_{\mathbb{R}}$, such that

$V \simeq V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ as representations of G.

<u>Proof</u> The action given by $G \longrightarrow SO_n(\mathbb{C})$ after an identification $V \simeq \mathbb{C}^n$. The image of G is compact, therefore up to a conjugation it is contained in the maximal compact subgroup of $SO_n(\mathbb{C})$ which is $SO(n)$. This means that after a change of coordinates (i.e. after a change of the identification $V \simeq \mathbb{C}^n$) G preserves $\mathbb{R}^n$.

This way we obtain $S_{\mathbb{R}}$ for Spin$(2n+1)$ $n = 0, 3 \mod 4$.
(Also $S^+_{\mathbb{R}}$ for Spin$(2n)$ $n = 0 \mod 4$.)

6 In our case $S_{\mathbb{R}}$ can be described explicitly: it is spanned by the vectors:

$$1 - i\, f_1 \wedge f_2 \wedge f_3 \quad , \quad i - f_1 \wedge f_2 \wedge f_3$$
$$f_1 - i\, f_2 \wedge f_3 \qquad\qquad i f_1 - f_2 \wedge f_3$$

(+ cyclic renumberation) where $f_k := \frac{1}{\sqrt{2}} e_k$

We check directly that this space is preserved by $Spin(7)$. Computation involve action of the generators of $Pin(7)$

$$f_k + f_{k+3} = \frac{e_k + e_{k+3}}{\sqrt{2}} \text{ acting via } L_{f_k} + D_{f_k}$$

$$i(f_k - f_{k+3}) = i\left(\frac{e_k - e_{k+3}}{\sqrt{2}}\right) \text{ -- " -- } i(L_{f_k} - D_{f_k})$$

and e_7 acting by change of the sign on $\Lambda^{odd} W$.

The generators of $Pin(7)$ exchange $S_{\mathbb{R}} \longleftrightarrow i S_{\mathbb{R}}$

<u>Note that</u> $S_{\mathbb{R}}$ when projected to $\Lambda^{even} W$ or $\Lambda^{odd} W$ is <u>surjective</u>.

The spaces $\Lambda^{even/odd} W$ are the spinors $S^{\pm}$ for $Spin(6)$.

Therefore we have an identification $S_{\mathbb{R}}$ for $Spin(7)$ with $S^{\pm}$ for $Spin(6)$.

Everything works equally well for $Spin(n)$ <u>$n = 7 \mod 8$</u>.

7 <u>Corollary</u> Spin(6) acts transitively on pairs (x,z) : $x \in S^5 \subset \mathbb{R}^6$, $z \in S^7 \subset \mathbb{S}^+$

<u>Proof</u> Spin(6) covers $SO(6)$ which acts transitively on S^5. Let us see how the stabilizer of a point $x \in S^5$ acts on $\mathbb{S}^+$ (it should act transitively). We can assume that $x = (0,0,0,0,0,1)$. Then Spin(5) fixes x:

$$\boxed{\text{Stabiliser}(x) = \text{Spin}(5) \subset \text{Spin}(6)}$$

We claim: the representation of Spin(6) $\mathbb{S}^+$ restricted to Spin(5) is $\mathbb{S}$. This is so since $\mathbb{S}^+$ for Spin(6) has the maximal weight $\frac{1}{2}(L_1+L_2+L_3)$. After restriction (to the subtorus given by $L_3=0$) we obtain the weight $\frac{1}{2}(L_1+L_2)$. This is the weight of $\mathbb{S}$ for Spin(5). ∎

<u>Lemma</u> Spin(7) acts transitively on the triples $(x,y,z) \in \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{S}_{\mathbb{R}}$
x, y – unit vectors $x \perp y$, z – unit spinor.

<u>Proof</u> Fix y e.g. $y = (0,0..0,1)$. Then the stabiliser of y contains Spin(6).
Now $\mathbb{S}$ restricted to Spin(6) decomposes as $\mathbb{S}^+ \oplus \mathbb{S}^-$ over $\mathbb{C}$, but $\mathbb{S}_{\mathbb{R}} \subset \mathbb{S} \twoheadrightarrow \mathbb{S}^+$ is an isomorphism. By the <u>Corollary</u> Spin(6) acts transitively on the pairs $x \in \mathbb{R}^6$ $|x|=1$ and $z \in \mathbb{S}^+$.

8) Let $G \subset \mathrm{Spin}(7)$ be the group which fixes a point $z \in S^7 \subset \mathbb{S}_{\mathbb{R}}$. Then G is compact, connected, simply-connected of rank 2, dimension 14 with the Dynkin diagram .
<u>In short</u> $G = G_2$.

<u>Proof</u> Since $\mathrm{Spin}(7)$ acts transitively on $S^7 \subset \mathbb{S}_{\mathbb{R}}$ we have

$$\dim G = \dim \mathrm{Spin}\,7 - 7 = \frac{7 \cdot 6}{2} - 7 = 14$$

Let $H \subset G$ be the subgroup fixing $y = (0,0,0 \ldots -1) \in S^6$; then $H \subset \mathrm{Spin}(6)$ fixes $z \in \mathbb{S}^+$. Hence $H \cong SU(3)$.

$\mathrm{Spin}(6)$ fixes y $\subset$ $\mathrm{Spin}(7)$ acts transitively on $(x, y, z) \in \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{S}_{\mathbb{R}}$, $x \perp y$

$SU(3) \cong H$ fixes y, z $\subset \mathrm{Spin}(6)$

$H \subset G$ fixes z $\subset \mathrm{Spin}(7)$

$\Rightarrow$ G acts transitively on $S^6 \subset \mathbb{R}^7$ with stabilizer $SU(3)$.
We have a fibration

$$SU(3) \subset G \twoheadrightarrow S^6 = G/SU(3).$$

It follows that G is connected and simply-connected.

$$T_e G = \mathfrak{g} = \underset{\mathfrak{su}(3)}{T_e SU(3)} \oplus T_y S^6$$

$$T_y S^6 = T_y\, \mathrm{Spin}(7)/\mathrm{Spin}(6) = \underline{\mathfrak{so}(7)/\mathfrak{so}(6)}$$

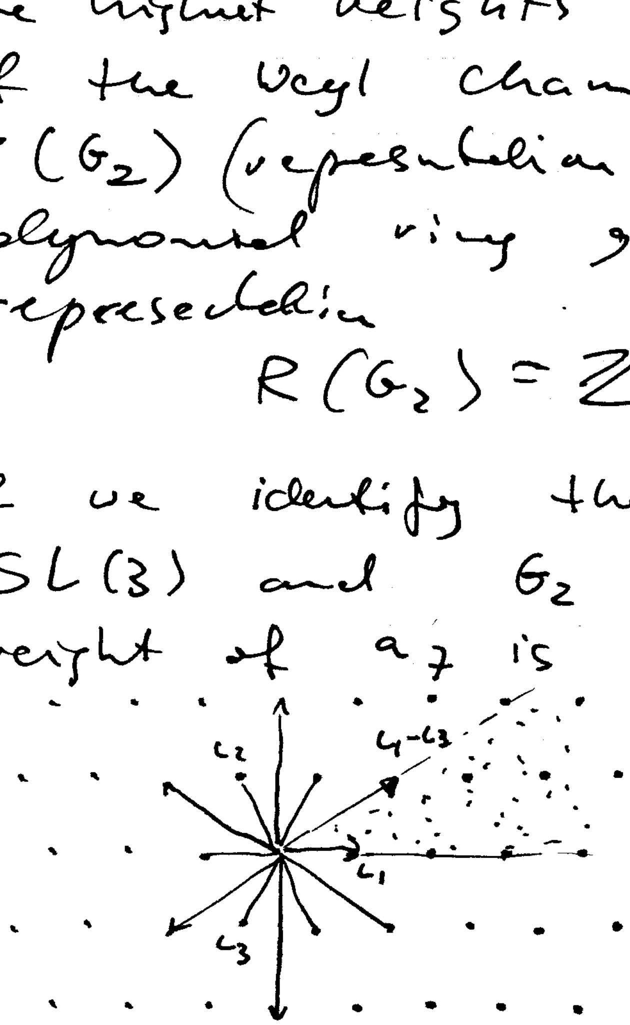
The weights of the action of $\mathfrak{h} \subset \mathfrak{so}(6)$ are $\pm L_i$
$\mathfrak{h} = \mathfrak{su}(3)$

We conclude that the weights have the shape G_2 as desired. ∎

Note G_2 comes with two representations: $\mathbb{R}^7$ via the action of $Spin(7)$, but the action factors through $SO(7)$, and the adjoint action on $\mathfrak{g}_2 \cong \mathbb{R}^{14}$. The highest weights generate the rays of the Weyl chamber. Therefore $R(G_2)$ (representation ring) is the polynomial ring generated by these 2 representations

$$R(G_2) = \mathbb{Z}[a_7, a_{14}] .$$

If we identify the maximal tori of $SL(3)$ and G_2, then the highest weight of a_7 is L_1 and of a_{14} is $L_1 - L_3$



Note that $G_2 \hookrightarrow Spin(7) \xrightarrow[2:1]{} SO(7)$ is mono.

(since $Z(G_2) = \{1\} \Rightarrow G_2$ does not cover anything.)

Defining G_2 via octonions

Therefore G_2 can be considered as a subgroup of $SO(7)$. We claim that G_2 fixes certain 3-form on $\mathbb{R}^7$. We leave it as an exercise.

(Note: general stabilizer of $\phi \in \Lambda^3 \mathbb{R}^7$ in $GL_7(\mathbb{R})$ has $\dim = 7^2 - \binom{7}{3} = 14$.)

Hint $Sym^2(S) = \Lambda^3 \mathbb{C}^7 \oplus \mathbb{1}$