Wykład 14

1) The maximal tori:

in $SO(2n)$, rank $= n$:
$$\begin{pmatrix} \boxed{R_1} & & & 0 \\ & \boxed{R_2} & & \\ & & \ddots & \\ 0 & & & \boxed{R_n} \end{pmatrix}$$
R_i a rotation in the plane e_{2i}, e_{2i+1}

Similarly for $SO(2n+1)$, rank $= n$:
$$\begin{pmatrix} \boxed{R_1} & & & & 0 \\ & \boxed{R_2} & & & \\ & & \ddots & & \\ & & & \boxed{R_n} & \\ 0 & & & & 1 \end{pmatrix}$$

If we pass to the complexification it is more convenient to chang coordinates, so the quadratic form has the matrix
$$\left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)$$ in the even case or $$\left(\begin{array}{c|c|c} 0 & I & 0 \\ \hline I & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$ odd case,
(hyperbolic form)
the maximal tori: even case
$\mathrm{diag}(x_1, \dots x_n, x_1^{-1}, \dots x_n^{-1})$ as in $Sp_n(\mathbb{C})$
$\mathrm{diag}(x_1 \dots x_n, x_1^{-1}, \dots x_n^{-1}, 1)$ - odd case.
Similarly the Lie algebras of the tori
even case: $\mathrm{diag}(x_1 \dots x_n, -x_1, \dots -x_n)$.
odd case: $\mathrm{diag}(x_1 \dots x_n, -x_1 \dots -x_n, 0)$.

The inclusion $SO(2n) \hookrightarrow SO(2n+1)$ preserves the maximal tori, hence for Weyl groups we have an inclusion
$$W_{SO(2n)} \hookrightarrow W_{SO(2n+1)}.$$
Moreover $W_{SO(2n+1)} = W_{Sp(n)}$ is generated by Σ_n (permutation of x_i's) and $x_i \mapsto -x_i$.

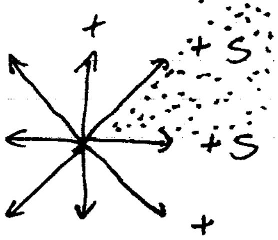
2) The transposition of x_i with x_j is realized by the block transposition (which has $\det = 1$)
The transposition of x_i with $-x_i$ has $\det -1$, but we write -1 in $(2n+1)$-th place and get an element of $SO(2n+1)$.
This is impossible in $SO(2n)$ hence:

$$\begin{array}{ccccc}
(\mathbb{Z}/2)^{n-1} \simeq K & \subset & W_{SO(2n)} & \twoheadrightarrow & \Sigma_n \\
\cap & & \cap & & \parallel \\
(\mathbb{Z}/2)^n & \hookrightarrow & W_{SO(2n+1)} & \twoheadrightarrow & \Sigma_n
\end{array}$$

$K = \{(a_1 \dots a_n) \in (\mathbb{Z}/2)^n : \Sigma a_i = 0\}$

Let us study the odd case $SO(2n+1)$ pictures for $n=2$

Roots $\pm L_i \pm L_j$, L_i (here is a difference with $Sp(n)$, where there is $2L_i$)



Root spaces:

$E_{ij} - E_{n+j\, n+i}$	for $L_i - L_j$
$E_{i,n+j} - E_{j,n+i}$	for $L_i + L_j$
$E_{n+i,j} - E_{n+j\, i}$	for $-(L_i + L_j)$
$E_{i,2n+1} - E_{2n+1,n+i}$	for L_i
$E_{n+i,2n+1} - E_{2n+1,i}$	for $-L_i$

The difference between $SO(2n+1)$ and $Sp(n)$ is well seen when we look at the block spanned by e_i, e_{n+i}, e_{2n+1} in the compact, real form. In another words we can assume that $n=1$ and we study the difference between $SO(3)$ and $Sp(2) = SU(2)$

3) The Lie algebra: $so(3)$ = antisymmetric matrices

Maximal torus $\begin{bmatrix} R & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & 1 \end{bmatrix} = R$ $\qquad SO(2)$

root space $\begin{bmatrix} 0 & 0 & x \\ 0 & 0 & y \\ -x & -y & 0 \end{bmatrix} = X$

action via $R X R^{-1}$ = usual action of $SO(2)$ on $\mathbb{R}^2$

<u>While</u> for $SU(2)$

Maximal torus $\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$ $\qquad z \in S^1 \subset \mathbb{C}$

root space $\begin{bmatrix} 0 & z \\ -\bar{z} & 0 \end{bmatrix}$

action via z^2.

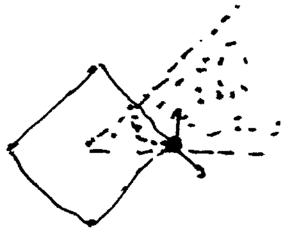
<u>Representations</u>

The natural representation $\mathbb{C}^{2n}$

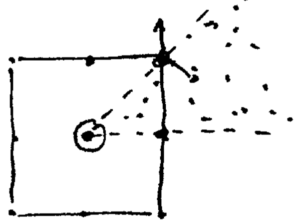
e_i of the weight L_i $\quad i \leq n$

e_{i+n} $\quad -L_i$ $\quad i > n$

L_1 the maximal weight



Adjoint representation $L_1 + L_2$ – maximal weight



Exterior power $\Lambda^2 \mathbb{C}^{2n+1}$: weights

	$\pm(L_i + L_j)$ $\; i \neq j$	$e_i \wedge e_j$
	$L_i - L_j$	$e_i \wedge e_{j+n}$
n times	0	$e_i \wedge e_{i+n}$
	L_i	$e_i \wedge e_{2n+1}$
	$-L_i$	$e_{i+n} \wedge e_{2n+1}$

We see that $\Lambda^2 \mathbb{C}^{2n+1} \simeq$ adjoint representation

4) This is another similarity with $Sp(n)$:

$SO(2n+1)$ = group preserving a symmetric nondegenerate 2-tensor
$so(2n+1)$ = antisymmetric tensors (as representation)
$Sp(n)$ = group preserving an antisymmetric nondegenerate 2-tensor
$sp(n)$ = symmetric tensors (as representation)

The explicite isomorphism $\Lambda^2 \mathbb{C}^m \to so(m)$ (working also for m even)

$v \wedge w \mapsto \varphi_{v\wedge w}$ $\qquad \varphi_{v\wedge w}(z) = 2(Q(w,z)\cdot v - Q(v,z)\, w)$

"2" is introduced because $\Lambda^2 \mathbb{C}^m$ can be considered as a subspace of the Clifford algebra

$v \wedge w \mapsto \frac{1}{2}(v\cdot w - w\cdot v) = ab - Q(a,b)$

Exercise: this identification

$$so(m) \simeq \Lambda^2 \mathbb{C}^n \subset C_m = C(Q)$$

preserves Lie algebra structure, in particular $\Lambda^2 \mathbb{C}^m$ is closed under $[\,,\,]$ is the Clifford algebra.

Exercise $\Lambda^2 \mathbb{C}^m$ is the Lie algebra $spin(m) \subset C_m$.
(Note $Spin(m) \subset C_m^*$, $Lie(C_m^*) = C_m$).

Importance of the latice

So far we have 2 latices in h^*: Λ^*, Λ^*_R - spanned by roots. Let us define Λ^*_W: called weight latice:

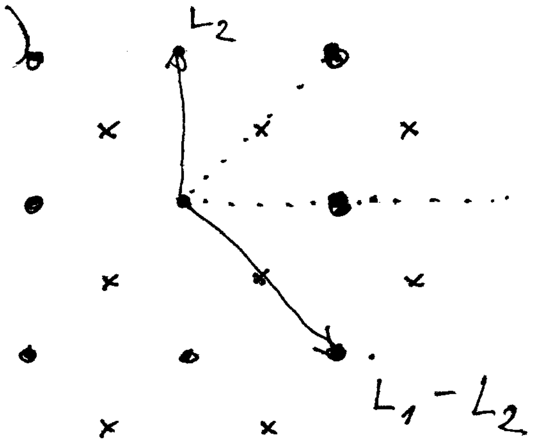
$$\Lambda^*_W = \{ \omega \in h^* : \forall \alpha\text{-root } \omega(H_\alpha) \in \mathbb{Z} \}$$

here $H_\alpha \in h$ (this time it is not a hypersurface)
H_α is the $\overbrace{\text{basis}}^{\text{distinguished}}$ element of $sl_2(\mathbb{C}) \cap h$ characterized by $H_\alpha \in [g_\alpha, g_{-\alpha}]$ $\alpha(H_\alpha) = 2$.

$$H_\alpha = 2\frac{\langle \alpha, - \rangle}{\langle \alpha, \alpha \rangle} \in (h^*)^* = h$$

5) <u>Example</u> $so(5)$ (we identify $h \simeq h^*$ via the standard scalar product)

simple root α	H_α
$L_1 - L_2$	$L_1 - L_2$
L_2	$2L_2$



Therefore $\Lambda_R^* = \Lambda^*$

$$\Lambda_w^* = \{ aL_1 + bL_2 : \quad \langle aL_1 - bL_2 , L_1 - L_2 \rangle \in \mathbb{Z}, \ \langle aL_1 - bL_2 , 2L_2 \rangle \in \mathbb{Z} \}$$

i.e $2a \in \mathbb{Z}$

$a - b \in \mathbb{Z}$ $\Leftrightarrow$ both integers or both half-integers
(denoted by $\times$)

<u>Since</u> $H_\alpha \in \Lambda$ and H_α's span $\Lambda \otimes \mathbb{R}$ we <u>have in general</u> latices $\Lambda_R^* \subset \Lambda^* \subset \Lambda_w^*$ or for dual latices $\Lambda_w \subset \Lambda \subset \Lambda_R$
(notation opposite to Fulton-Harris)

<u>Theorem</u> 1) $\Lambda / \Lambda_w \simeq \pi_1(G)$

(The surjection $\Lambda \longrightarrow \Lambda/\Lambda_w$ corresponds to the surjection

$$\Lambda = \pi_1(T) \twoheadrightarrow \pi_1(G))$$

2) $\Lambda_R / \Lambda \simeq Z(G)$ - the center.

In our example $\Lambda / \Lambda_w = \mathbb{Z}/2 = \pi_1(SO(5))$
(ok for any n)

On the other hand for $Sp(4)$ $\Lambda = \Lambda_w$

but $\Lambda_R / \Lambda = \mathbb{Z}/2 = Z(Sp(n)) = \{ I, -I \}$
(Another incarnation of duality $Sp(n)$ & $SO(2n+1)$.)

6)

Other representations of $SO(5)$ we obtain taking tensor products:
if $\omega = b_1 L_1 + b_2 (L_1+L_2)$ then
$$V(\omega) \subset \mathrm{Sym}^{b_1} \underset{\substack{\shortparallel \\ \mathbb{C}^5}}{V_{L_1}} \otimes \mathrm{Sym}^{b_2} \underset{\substack{\shortparallel \\ \Lambda^2\mathbb{C}^5}}{V_{L_1+L_2}}$$
And similarly for any $SO(2n+1)$, $n \geq 1$

<u>the representation ring</u>
$$R(SO(2n+1)) = \mathbb{Z}[\lambda_1, \lambda_2 \dots \lambda_n]$$
where $\lambda_i = \Lambda^i \mathbb{C}^{2n+1}$. Moreover $\Lambda^i \mathbb{C}^{2n+1}$ is irreducible.

<u>Representations of $Spin(2n+1)$</u>:
there is an additional representation
$S = \Lambda^{\cdot} W$ $\qquad \mathbb{C}^{2n+1} = W \oplus W^* \oplus U$
maximal isotropic

What are the weights? We have to recognize the maximal torus of $Spin(2n+1)$:
$$so(2n+1) \simeq \Lambda^2 \mathbb{C}^{2n+1} \hookrightarrow C_{2n+1} = C(Q)$$
$$E_{ii} - E_{n+i\,n+i} \leftrightarrow \tfrac{1}{2}(e_i \wedge e_{n+i}) \leftrightarrow \tfrac{1}{2}(e_i \cdot e_{n+i} - 1)$$
The action on $\Lambda^{\cdot} W$ via $\frac{1}{2}(L_{e_i} \circ D_{2e_i^*} - \mathrm{Id}) =$
$= L_{e_i} \circ D_{e_i^*} - \frac{1}{2}\mathrm{Id}$.

We check:
$$(L_{e_i} \circ D_{e_i^*} - \tfrac{1}{2}\mathrm{Id})(e_I) = \begin{cases} \frac{1}{2} e_I & i \in I \\ -\frac{1}{2} e_I & i \notin I. \end{cases}$$
Weights: $\frac{1}{2}(\varepsilon_1 L_1 + \varepsilon_2 L_2 + \dots + \varepsilon_n L_n)$ $\qquad \varepsilon_i = \pm 1$

The maximal weight $\frac{1}{2}(L_1 + L_2 + \dots + L_n)$ "half weights"
$$R(Spin(2n+1)) = \mathbb{Z}[\lambda_1, \lambda_2 \dots \lambda_{n-1}, S]$$

7 $SO(2n)$ $n \geq 4$

Positive roots: $L_i - L_j$, $i<j$ and $L_i + L_j$

Simple roots $L_i - L_{i+1}$, $L_{n-1} + L_n$.

Weight in the Weyl chamber

$$a_1 L_1 + a_2 L_2 + \dots + a_n L_n$$

$$\underset{\substack{\uparrow \\ \text{scalar product with } L_i - L_{i+1}}}{a_1 \geq a_2 \geq \geq \quad \geq a_n} \qquad \underset{\substack{\uparrow \\ \text{scalar product} \\ \text{with } L_{n-1}+L_n}}{a_{n-1} \geq -a_n}$$

Theorem: For a connected Lie group G $R(G)$ is a polynomial algebra if the lattice points of the rays generate $\Lambda^* \cap W$.

Here rays are:

$a_1 = a_2 = \dots = a_i$ $a_{i+1} = a_{i+2} = \dots = a_n = -a_n$

$(1\ 1 \dots 1\ 0\ 0\ 0\ 0)$

$a_1 = a_2 = \dots = a_{n-1}$, $a_{n-1} = -a_n$

$(1\ 1\ 1 \dots 1\ {-1})$ and the last $(1\ 1\ 1\ 1 \dots 1)$

Note that $(1\ 1 \dots 1\ 0) \in \Lambda^* \cap W$ but it is not a sum of the ray elements.

Corollary $R(SO(2n))$ is not a polynomial algebra. It is a free module generated by 1 and $V(\omega)$ for $\omega = (1 1 \dots 1)$ over the ring $\mathbb{Z}[\lambda_1, \dots, \lambda_{n-1}, \lambda_n^-]$, $\lambda_i = \Lambda^i \mathbb{C}^{2n}$ $i<n$ and $\lambda_n^- = V_{(1,1 \dots 1,-1)}$.

$$\underbrace{V_{(1 1 \dots 1,-1)}}_{\lambda_n^-} \oplus \underbrace{V_{(1 1 \dots 1)}}_{\lambda_n^+} = \Lambda^n \mathbb{C}^{2n}.$$

$\boxed{\text{splitting defined by the involution } \Lambda^n \mathbb{C}^{2n} \underset{\text{pairing } \Lambda^k \times \Lambda^{2n-k}}{\cong} (\Lambda^n \mathbb{C}^{2n})^* \underset{Q}{\cong} \Lambda^n \mathbb{C}^{2n}}$

relation

$$(\lambda_n^+ + \lambda_{n-2} + \lambda_{n-4} + \dots)(\lambda_n^- + \lambda_{n-2} + \lambda_{n-4}) = (\lambda_{n-1} + \lambda_{n-3} + \dots)^2$$

8) <u>The representation ring of Spin(2n)</u> is more regular: $R(\mathrm{Spin}(2n))$ is a <u>polynomial ring</u>.

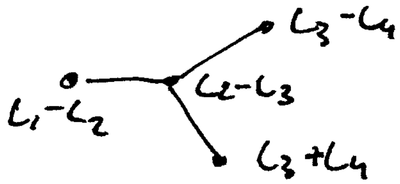
The weight lattice is bigger than Λ^*:
again $H_\alpha = L_i - L_j$ for $\alpha = L_i - L_j$ and Λ^*_W consists of $\sum a_i L_i \ : \ 2a_i \in \mathbb{Z}, \ a_i + a_j \in \mathbb{Z}$.

New elements of Λ^*_W correspond to $\mathrm{Spin}(2n)$ representations

$$\left.\begin{array}{lcl} \frac{1}{2}(L_1 + L_2 + \dots + L_n) & \longleftrightarrow & \Lambda^{even} W = S^+ \\ \frac{1}{2}(L_1 + L_2 + \dots - L_n) & \longleftrightarrow & \Lambda^{odd} W = S^- \end{array}\right\} \text{highest weight vectors } e_1 \wedge \dots \wedge e_n \text{ or } e_1 \wedge \dots \wedge e_{n-1}$$

$(0 \dots \underset{i}{1} \dots 0 \ 0 \dots \underset{n+i}{-1} \dots 0) \in h$ corresponds to $\frac{1}{2}(e_i \cdot e_{n+i} - 1) \in C_n$

<u>The case $n = 4$ $SO(8)$</u>



Dynkin diagram has Σ_3-symmetry.

Three representations
$\mathbb{C}^8$ (natural representation)
S^+ $\dim = 2^{n-1} = 8$
S^- — " —

Highest weights:

$\mathbb{C}^8$	L_1
S^+	$\frac{1}{2}(L_1 + L_2 + L_3 + L_4)$
S^-	$\frac{1}{2}(L_1 + L_2 + L_3 - L_4)$

These representations are not isomorphic but there exists an automorphism of $so(8)$ exchanging these representations.

<u>Lemma</u> The form $\beta(x, y) = \int t(x) \wedge y$ is symmetric and nondegenerate on $\Lambda^{\cdot} W \subset C_n$.
transposition of factors as in C_n

Here $\int : \Lambda^{\cdot} W \to \mathbb{C}$ depends on the choice of $\mathrm{vol} \in \Lambda^4 W \to \mathbb{C}$.

(The construction works for $\mathrm{Spin}(2n)$, but this form is symmetric if ($n = 0, 1 \bmod 4$))

Moreover β is $so(8)$-invariant.

Therefore we get an identification after a choice of an orthonormal basis $S^+ = \mathbb{C}^8$ $\quad S^- = \mathbb{C}^8$.

Let's make it explicit: a basis of S^+

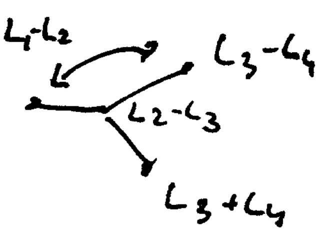
~~[illegible]~~	1 weight	$-\frac{1}{2}(L_1+L_2+L_3+L_4)$
2	$e_1 \wedge e_2$	$\frac{1}{2}(L_1+L_2-L_3-L_4)$
3	$e_1 \wedge e_3$	$\frac{1}{2}(L_1 \quad L_2+L_3-L_4)$
4	$e_1 \wedge e_4$	$\frac{1}{2}(L_1-L_2-L_3+L_4)$
	$e_2 \wedge e_3$	$\frac{1}{2}(-L_1+L_2+L_3-L_4)$
	$e_2 \wedge e_4$	$\frac{1}{2}(-L_1+L_2-L_3+L_4)$
	$e_3 \wedge e_4$	$\frac{1}{2}(-L_1-L_2+L_3+L_4)$
1	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$\frac{1}{2}(L_1+L_2+L_3+L_4)$

basis of S^-

We decompose S^+ into the sum of isotropic spaces
$\langle e_1\wedge e_2 \wedge e_3 \wedge e_4,\ e_1\wedge e_2,\ e_1 \wedge e_3,\ e_1\wedge e_4 \rangle = W'$
$\langle$ without e_1 $\rangle \cong (W')^*$

With the same maximal torus in $so(8)$ but with the natural representation replaced by S^+ and with the basis ① ② ③ ④ we obtain the simple roots:

instead of $L_1 - L_2$ we get $\frac{1}{2}(L_1+L_2+L_3+L_4) - \frac{1}{2}(L_1+L_2-L_3-L_4)$
$= L_3 + L_4$



$L_2 - L_3$ — $L_2 - L_3$

$L_3 - L_4$ — $L_3 - L_4$

$L_3 + L_4$ — $L_1 - L_2$

Triality (trójność?)

What is duality:

V_1, V_2 vector spaces with nondegenerate symmetric bilinear forms q_1 q_2

$+$ bilinear form $\phi : V_1 \times V_2 \to \mathbb{C}$

s.t. the associated maps

$$f : V_1 \xrightarrow[\phi]{} V_2^* \xleftarrow[q_2]{\approx} V_2$$

and its transpose

$$f^t : V_2 \xrightarrow[\phi]{} V_1^* \xleftarrow[q_1]{\approx} V_1$$

are isomorphisms of quadratic forms: $q_2(f(V_1)) = q_1(V_1)$

Similarly $\dim V_1 = \dim V_2 = \dim V_3$ q_i - bilinear nondegenerate symmetric forms

$+$ trilinear form $\phi : V_1 \times V_2 \times V_3 \to \mathbb{C}$

such that the associated maps

$$f_k : V_i \times V_j \xrightarrow[\phi]{} V_k^* \xleftarrow[q_k]{} V_k$$

satisfy the condition ("ortogonal maps"):

$$q_k(f(v_i, v_j)) = q_i(v_i)\, q_j(v_j) .$$

Examples 1) $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \xrightarrow{\phi} \mathbb{C}$ or $f_k : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $x, y \mapsto xy$

2) $V_1 = V_2 = V_3 = \mathbb{C}^2$

$q_i(x, y) = xy$ $\quad f_k((x_1, x_2)(y_1, y_2)) = (x_1 y_1, x_2 y_2)$

3) $V_i = M_{2 \times 2}(\mathbb{C})$ $\quad q_i = \det$

$f_k =$ multiplication of matrices

Theorem The dimensions of V which admit triality are $1, 2, 4, 8$

In the dimension 8

$$f_3 : \mathbb{C}^8 \times S^+ \longrightarrow S^-$$

given by the operation $L_w + 2D_\xi$ for $w + \xi \in W \oplus W^* = \mathbb{C}^8$

$$f_3((w + \xi), x) := w \wedge x + 2 D_\xi x \in \Lambda^{odd} W$$

$$\Big(\mathbb{C}_8 = \underset{w}{W} \oplus \underset{\xi}{W^*} \qquad x \in \Lambda^{even} W = S^+ \Big) .$$

Exercise Check that we obtain a triality

See [Varsfoli, Notes on Clifford algebras ... § 10]