Lecture 12

1) How to pass from a root system to Dynkin diagram

<u>Definition</u>: <u>Simple roots</u> Let V be a root system. Fix positive roots. Let $\Delta \subset R_+$ be the set of roots which cannot be presented as a sum of positive roots i.e $\nexists$ a presentation $\alpha = \sum_{i=1}^{k} n_i \beta_i$ $k \geq 2$, $n_i > 0$, $\beta_i \in R_+$

<u>Theorem</u>: Δ is a basis of V. Every positive root is a sum $\sum_{\alpha \in \Delta} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N} \cup \{0\}$.

1. Proof of the second part: Suppose there exists $\beta \in R_+$ such that β is not a sum.
The distinction between positive and negative roots is accomplished by $\langle g, - \rangle > 0$.
We can assume that β has minimal $\langle g, \beta \rangle$. Since $\beta \notin \Delta$ it can be written as $\beta = \alpha_1 + \alpha_2 + \dots + \alpha_k$ $\alpha_i \in R_+$ $k \geq 2$.
Now for some i the root α_i cannot be written as a sum of simple roots. But for all i have $\langle g, \alpha_i \rangle$ smaller then $\langle g, \beta \rangle$. $\lightning$
$\Rightarrow \Delta$ spans V.
To prove that Δ is linearly independent we need a lemma about possible lengths of roots and about possible angles between roots.

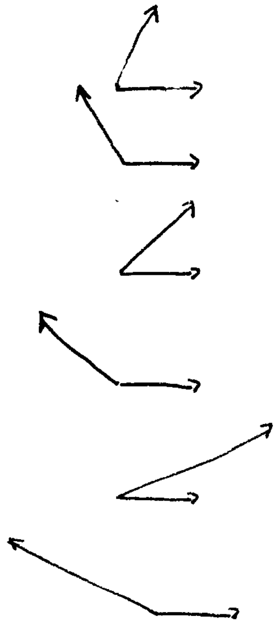
<u>Def</u> $n_{\alpha\beta} = 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ $\alpha, \beta \in R$ - Cartan numbers.

Note: $n_{\alpha\beta} = 2 \frac{|\beta|}{|\alpha|} \cdot \frac{\langle \alpha, \beta \rangle}{|\alpha| \cdot |\beta|} = 2 \frac{|\beta|}{|\alpha|} \cos(\alpha, \beta)$

2) So $n_{\alpha\beta} \cdot n_{\beta\alpha} = 4\cos^2(\alpha\beta) \in \mathbb{Z}$

the possible values of $|\cos(\alpha,\beta)|$: $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$

$n_{\alpha\beta}$	$n_{\beta\alpha}$	$\measuredangle(\alpha,\beta)$	$\lvert\alpha\rvert / \lvert\beta\rvert$ (assume $\lvert\alpha\rvert \geq \lvert\beta\rvert$)
0	0	$\pi/2 = 90°$	?
1	1	$\frac{\pi}{3} = 60°$	1
-1	-1	$\frac{2}{3}\pi = 120°$	1
1	2	$\pi/4 = 45°$	$\sqrt{2}$
-1	-2	$\frac{3}{4}\pi = 135°$	$\sqrt{2}$
1	3	$\frac{\pi}{6} = 30°$	$\sqrt{3}$
-1	-3	$\frac{5}{6}\pi = 150°$	$\sqrt{3}$



We need: <u>Proposition</u> If $\alpha \neq \beta$ $\langle\alpha,\beta\rangle > 0$ then $\alpha-\beta$ is also a root.

<u>Proof</u> $\langle\alpha,\beta\rangle > 0 \iff n_{\alpha\beta} > 0 \iff n_{\beta\alpha} > 0$ (acute angle)

Since $n_{\alpha\beta} \cdot n_{\beta\alpha} = 4\cos^2(\alpha\beta) < 4$ we have

$n_{\alpha\beta} = 1$ or $n_{\beta\alpha} = 1$.

If $n_{\beta\alpha} = 1$ then $\alpha - \beta = \alpha - n_{\beta\alpha}\beta = S_\beta \alpha$ - a root

(similarly if $n_{\alpha\beta} = 1 \Rightarrow \beta - \alpha = S_\alpha \beta$.)

<u>Lemma</u> If $\alpha,\beta \in \Delta$ then $\langle\alpha,\beta\rangle \leq 0$

(obtuse angle)

<u>Proof</u> If not then $\alpha-\beta$ and $\beta-\alpha$ are roots,

Say $\alpha - \beta \in R_+$. Then $\alpha = (\alpha-\beta)+\beta$ ↯

<u>Lemma</u> A set $\Delta \subset V$ lies in the same half-space. If for any two $\alpha,\beta \in \Delta$ $\langle\alpha,\beta\rangle \leq 0$, then S is linearly independent.

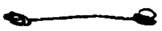
<u>Proof</u>: $\gamma = \sum m_\alpha \alpha = \sum n_\beta \beta$ $\quad m_\alpha, n_\beta \geq 0$, α's & β's distinct.

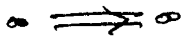
$|\gamma|^2 = \sum m_\alpha n_\beta \langle\alpha,\beta\rangle \leq 0 \Rightarrow |\gamma| = 0 \Rightarrow m_\alpha, n_\beta = 0$ ∎

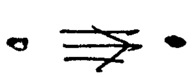
3) Dynkin diagram

• - for an element of Δ

 no edge if $\alpha \perp \beta$

 if $\measuredangle(\alpha,\beta) = 120^\circ$

 if $\measuredangle(\alpha,\beta) = 135^\circ$ $|\alpha| > |\beta|$

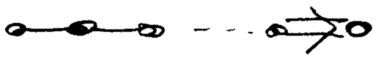
 if $\measuredangle(\alpha,\beta) = 150^\circ$ $|\alpha| > |\beta|$

<u>Theorem</u> Dynkin diagram determines the root system. (since it defines the Weyl chamber.)

<u>Theorem</u> Root system determines the Lie algebra

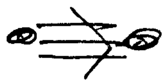
<u>Theorem</u> The only possible root systems have the following Dynkin diagrams

A_n  (n dots) $SL_n(\mathbb{C})$ U_n

B_n  $n \geq 2$ $SO_{2n+1}(\mathbb{C})$

C_n  $n \geq 3$ $Sp_n(\mathbb{C})$

D_n  $n \geq 4$ $SO_{2n}(\mathbb{C})$

G_2 

F_4 

E_6 

E_7 

E_8 

Exceptional Lie groups/algebras

4) The assignment:

irreducible representation $\mapsto$ highest weight $\in \Lambda^* \cap W_0$

is exactly as in the case of $SL_n(\mathbb{C})$.

For a weight $\omega \in \Lambda^* \cap W_0$ it is possible to construct the irreducible representation $V(\omega)$ (Borel-Bott-Weil-Schmit) as

$$V(\omega) = H^0_{\bullet}(G_{\mathbb{C}}/B_{\mathbb{C}}; L_{-\omega}) \quad \text{(global sections)}$$

where $L_{-\omega}$ is the line bundle associated with the representation $B_{\mathbb{C}} \to T_{\mathbb{C}} \xrightarrow{-\omega} \mathbb{C}^*$.

We will give explicit constructions of $V(\omega)$ in some cases.

<u>Weyl character formula</u> takes form

$$\chi_{V(\omega)} = \frac{A_{\omega+\rho}}{A_\rho} \quad \text{where } \rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$$

which is always a weight.

$$A_\mu = \sum_{w \in W} (-1)^{\ell(w)} x^{w(\mu)} \in \mathbb{Z}[\Lambda] \quad \text{(group ring)}$$

Here $\ell(w)$ = number of reflections in the shortest presentation of w.

The monomial:

$$x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n} : \mathfrak{h} \longrightarrow \mathbb{C}$$

is the product of the linear forms x_i, obtained by a choice of a basis of Λ:

$e_1 \dots e_n$ basis of Λ

and $\mu_i = \mu(e_i)$.

5)

Group of symplectic transformation of $(\mathbb{C}^{2n}, \omega)$
$Sp_n(\mathbb{C})$ (sometimes denoted by $Sp(2n.\mathbb{C})$)
Natural representation in $\mathbb{C}^{2n} = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$
Basis $e_1\, e_2, \dots e_n, f_1, f_2 \dots f_n$ $\quad f_i = e_{n+i} = e_i^*$
$\omega((V,V'),(W,W')) = V'(W) - W'(V)$, in coordinates $\omega = \sum f_i^* \wedge e_i^*$
Embedding: $Gl_n(\mathbb{C}) \to Sp_n(\mathbb{C})$

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix}$$

The maximal torus in $Gl_n(\mathbb{C})$ is mapped to the maximal in $Sp_n(\mathbb{C})$

$$T = diag(x_1 \dots x_n, x_1^{-1} \dots x_n^{-1})$$

Weyl groups:

$$\Sigma_n = W_{Gl_n\mathbb{C}} \hookrightarrow W_{Sp_n\mathbb{C}}$$

Also $(\mathbb{Z}/2)^n$ is a subgroup of $W_{Sp_n\mathbb{C}}$
$\tau_i = (0, \dots \underset{i}{1}, \dots 0) \in (\mathbb{Z}/2)^n$, corresponds to $\pm$ transposition e_i with $-f_i$; it is an element of NT. Hence τ_i acts on T by inverting x_i.

$$(\mathbb{Z}/2)^n \hookrightarrow W_{Sp_n\mathbb{C}} \twoheadrightarrow \Sigma_n$$

Lie algebra: $A^T J + JA = 0$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

In the block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $\quad$ B, C symmetric, $D = -A^T$

Roots: $L_i - L_j, \pm(L_i + L_j), \pm 2L_i$, where L_i

$$\mathbb{C}^n \simeq \mathfrak{t}_{\mathbb{C}} \xrightarrow{L_i} \mathbb{C}$$

$(x_1, \dots x_n) \leftrightarrow (x_1 \dots x_n, -x_1, \dots -x_n) \longmapsto x_i$
Weyl group permutes x_i's and changes signs

6) Root spaces:

$L_i - L_j$ $\qquad E_{ij} \underbrace{- E_{n+j,n+i}}_{-(E_{ij})^T}$ $\qquad i \neq j$

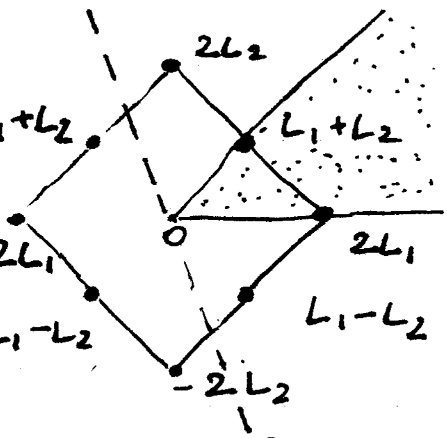
$L_i + L_j$ $\qquad E_{i,n+j} + E_{j,n+i}$ $\qquad i \neq j$

$-L_i - L_j$ $\qquad E_{n+i,j} + E_{n+j,i}$ $\qquad i = j$

$2L_i$ $\qquad E_{i,n+i}$

$-2L_i$ $\qquad E_{n+i,i}$

Picture for $n = 2$



simple roots
$2L_2 \qquad L_1 - L_2$

In general
positive roots: $\begin{cases} 2L_i, & L_i + L_j, \\ L_i - L_j & i < j, \end{cases}$
simple roots: $L_i - L_{i+1}$ $\quad i = 1 \ldots n-1$, $\quad 2L_n$

(every positive root is a sum of simple roots, eg $2L_{n-1} = 2(L_{n-1} - L_n) + 2L_n$)

Dynkin diagram:

last root is longer

7)

Irreducible representations ~ latice points in W

$\omega = \sum a_i L_i$ st. $(\omega, L_i - L_{i+1}) = a_i - a_{i+1} \geq 0$

$(\omega, 2L_n) = 2a_n \geq 0$

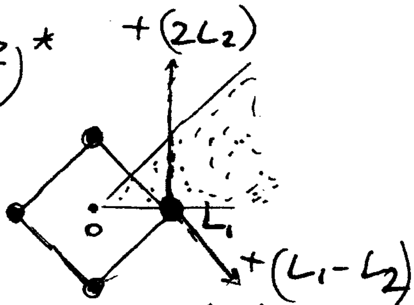
So: $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$.

For $n=2$:

Natural representation: $V = \mathbb{C}^2 \oplus (\mathbb{C}^2)^*$

veights $L_1, L_2, -L_1, -L_2$

highest veight L_1.

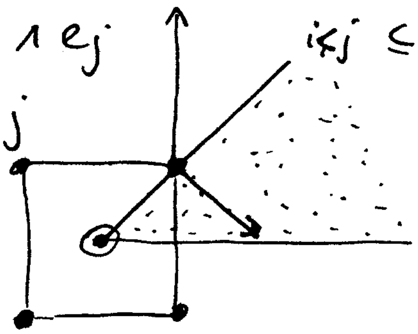


Dual representation $V^* \cong$ natural representation (iso given by ω)

$\Lambda^2 V$ basis $e_i \wedge e_j$ $i \leq j \leq 4$ $i \neq j$

veights $\pm L_i \pm L_j$

Picture:



0 is double $0 = L_1 - L_1 = L_2 - L_2$

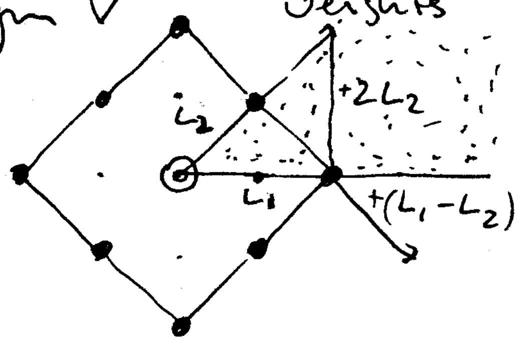
$e_1 \wedge f_1$ $e_2 \wedge f_2$

There is a map of representations

$$\Lambda^2 V \xrightarrow{\omega} \mathbb{C} = \mathbb{1}$$

$\ker(\omega)$ is a irreducible representation with the highest weight $L_1 + L_2$

$\mathrm{Sym}^2 V$ weights $\pm L_i \pm L_j$ (i can be equal j)



Highest weight $2L_1$
(killed by simple roots)

<u>Claim</u> $\mathrm{Sym}^2 V$ is isomorphic to the adjoint representation

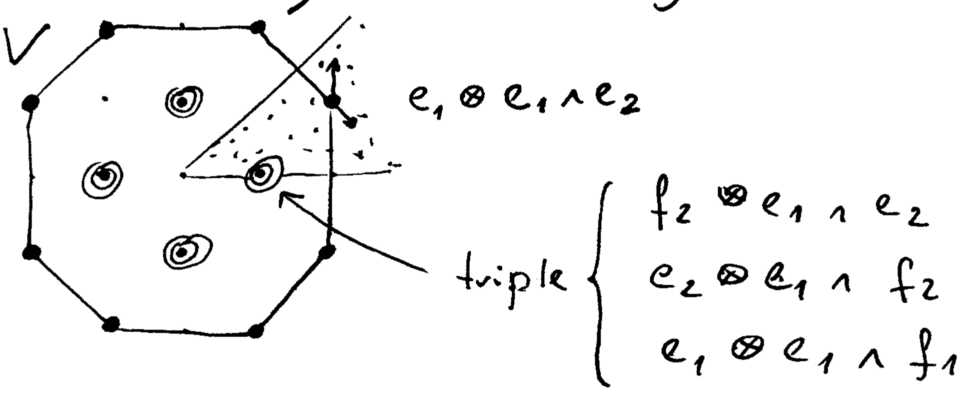
$$\mathrm{Sym}^2 V \longrightarrow sp_n(\mathbb{C}) \subset \mathrm{Hom}(V, V)$$
$$v_1 \cdot v_2 \longmapsto \left(A_{v_1 v_2} : u \mapsto \omega(u, v_1)\cdot v_2 + \omega(u, v_2) v_1\right)$$

<u>Exercise</u>: check that it is an isomorphism. (first one has to check
$$\omega(A_{v_1 v_2} u_1, u_2) + \omega(u_1, A_{v_1 v_2} u_2) = 0)$$

To get the irreducible representation with the highest weight.
$$a_1 L_1 + a_2 L_2 = b_1 L_1 + b_2 (L_1 + L_2)$$
we consider the representation

$$W = \mathrm{Sym}^{b_1} V \otimes \mathrm{Sym}^{b_2} \Lambda^2 V$$
and the vector $v = e_1^{b_1} \otimes (e_1 \wedge e_2)^{b_2}$
which has the highest weight there.
W is reducible in general, so we take the subrepresentation generated by v.

<u>Example</u> $V \otimes \Lambda^2 V$



But we have the map

$$V \otimes \Lambda^2 V \xrightarrow{\wedge} \Lambda^3 V \simeq V^* \overset{\omega}{\simeq} V$$

<u>Exercise</u> $\ker(\wedge)$ is the irreducible representation of the weight $2L_1 + L_2$.

In general: the <u>highest weight</u> in $\Lambda^k V$ (where V is the natural representation $\mathbb{C}^n \oplus (\mathbb{C}^n)^*$) is $L_1 + L_2 + \dots + L_k$.

This representation is not irreducible.

The contraction with ω

$$\phi_k : \Lambda^k V \longrightarrow \Lambda^{k-2} V$$

(from $e_I \wedge f_J$ we remove the pairs $e_i \wedge f_i$)

<u>Exercise</u>:

$\ker(\phi_k)$ is in fact irreducible.

$e_1 \wedge e_2 \wedge \dots \wedge e_k \in \ker(\phi_k)$

To find the irreducible representation of the weight

$$b_1 L_1 + b_2 (L_1 + L_2) + \dots + b_n (L_1 + L_2 + \dots + L_n)$$

we look at $\bigotimes_{i=1}^{n} \mathrm{Sym}^{b_i} \Lambda^i V$

and inside we find the representation spanned by $\bigotimes_{i=1}^{n} (e_1 \wedge \dots \wedge e_i)^{b_i}$

The explicite construction of the representation corresponding to the given weight see Fulton-Harris - formula (17.10)

To compute the character apply Weyl Character formula (for $n=3$ the functions A_ω have $3! \cdot 2^3 = 48$ summands)

10) Another approach to the representation theory
REPRESENTATION RING

Virtual representations: construction as from $(\mathbb{N},+) \rightsquigarrow \mathbb{Z}$

$[V]-[W]$ isomorphism classes (we will drop [ ])

$[V]-[W]=[V']-[W']$ if $V+W'+W'' \cong V'+W+W''$ for some W''

If G is compact (or G reductive complex group and we consider only holomorphic representations) then any formal difference can be uniquely written as

$\sum a_i V_i$ where V_i are distinct irreducible representations, $a_i \in \mathbb{Z}$

We get $R(G)$ (also denoted $K(G)$)

- an abelian group (free for G compact) with respect to $\oplus$
- a commutative ring with respect to $\otimes$

<u>We claim</u> $R(Sp_n\mathbb{C})$ is the polynomial <u>ring</u>

$$\phi : \mathbb{Z}[\lambda_1, \lambda_2, \dots \lambda_n] \xrightarrow{\cong} R(Sp_n\mathbb{C})$$
$$\lambda_i \longmapsto \Lambda^i V$$

The map ϕ is <u>injective</u>: for $f(\lambda)$ consider a monomial $a_b \lambda^b$ with the highest weight (there can be few). Then $\phi(f(\lambda)) = a_b V_{b_1 L_1 + b_2(L_1+L_2)+\dots b_n(L_1+\dots L_n)} +$ representations of lower or incomparable highest weights

The map ϕ is surjective since each irreducible representation is in the image, modulo lower weight.

Here the ring $R(G)$ is easy, but we loose the information which element corresponds to a true representation.

11) The same construction can be applied to $G = SL_n(\mathbb{C})$

<u>Then</u> $R(SL_n(\mathbb{C})) = R(SU(n))$ is the polynomial ~~algebra~~ ring

$$\mathbb{Z}[\lambda_1 \dots \lambda_{n-1}] \qquad \lambda_i \leftrightarrow \Lambda^i \mathbb{C}^n.$$

In both cases the map

$$R(G) \longrightarrow R(T)^W \qquad (*)$$

is an isomorphism.

On the level of characters:

$$R(SL_n(\mathbb{C})) = \left(\mathbb{Z}[x_1 \dots x_n] / _{x_1 \cdots x_n = 1}\right)^{\Sigma_n}$$

i.e. symmetric polynomials.

<u>Theorem</u> 1) If G is <u>connected</u> then the map $(*)$ is iso.

2) If G is <u>simply-connected</u> $(\pi_1 G = 1)$ Then $R(G)$ is a polynomial ring. The generators are the irreducible (or any other) representations with highest weights ω_i, where ω_i span$_{\mathbb{Z}}$ the edges of the Weyl chamber.