0 Wykład 11

Representation theory of $SL_n(\mathbb{C})$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Irreducible representations} \arrow[rr, bend left, "\text{character}"] \arrow[d, bend right, "\text{highest weight}"'] & & \\
& & \text{Schur polynomials} \arrow[dll, bend left, "\text{exponents of the leading monomial}"] \\
\text{Young diagrams} = \text{partitions} \arrow[u, bend right, "\text{Schur functors } S^{\lambda} \text{ applied to } \mathbb{C}^n"'] \arrow[urr, "\text{Weyl character formula}"] & &
\end{tikzcd}
$$

1) Fundamental problem in representation theory:

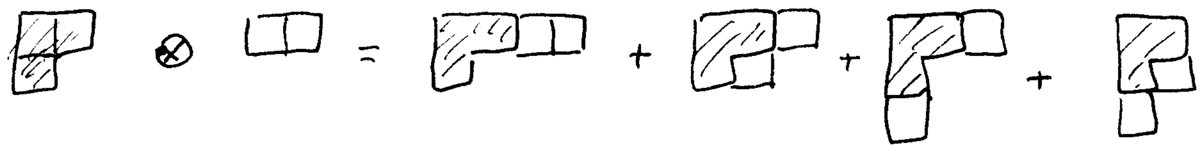
For two irreducible representations $V(\lambda)$ and $V(\mu)$ the tensor product decomposes

$$V(\lambda) \otimes V(\mu) = \sum_{\substack{V(\nu) \\ \text{irreducible}}} n_{\lambda\mu}^{\nu} \, V(\nu)$$

Find $n_{\lambda\mu}^{\nu}$. Answer: Littlewood-Richardson formula

If $\mu = (k, 0, 0 \ldots 0)$ then formula is a bit simpler: Pieri Formula

e.g



Rule: all $n_{\lambda\mu}^{\nu} = 0$ or 1

To the diagram λ we have to add k boxes an obtain the diagram ν. It is forbidden to put 2 boxes in the same column.

To justifying this rule one has to make the calculation on characters, that is to show

$$S_\lambda \cdot S_\nu = \text{suitable sum of } S_\nu$$

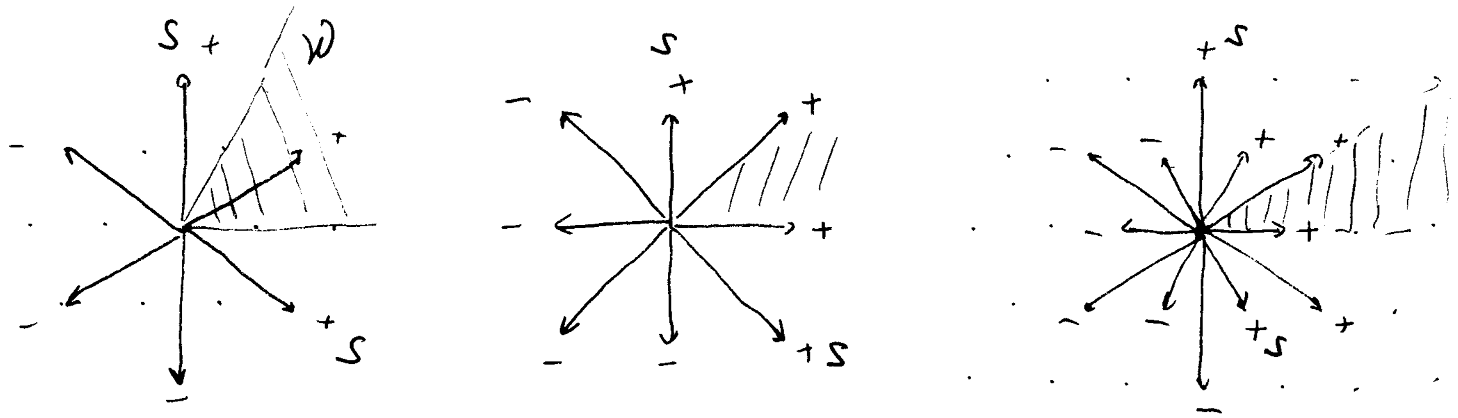
We have studied SL_n. There we have

- $\mathfrak{h}$ = diagonal matrices, roots $\in \mathfrak{h}^* = i\mathfrak{t}^*$
- Weyl group $W = NT/T = \Sigma_n$ acts on $\mathfrak{h}^*$,
  <u>W permutes the roots</u>
  We divided the roots into R_+ and R_-
  Weyl Chamber $\mathcal{W} = \{ w \in \mathfrak{h}^* : \langle w, \alpha \rangle \geq 0 \text{ for } \alpha \in R_+ \}$
  is the closure of one component $\mathfrak{h}^* - \bigcup_{\alpha \in R} H_\alpha$
- W permutes Weyl Chambers, <u>W is generated by the reflection in H_α</u>.
- W acts <u>transitively</u> on the set of Weyl Chambers. This action is <u>free</u>.
- Irreducible representations correspond to the integral weights $\Lambda^* \cap \mathcal{W}_0$

The <u>same statements</u> are true for arbitrary simple groups

<u>Warning</u> In general W does not act transitively on roots.

Roots for Lie groups with $\dim T = 2$.
Simple root marked by "s" (see p. 8).



Complex $SL_3(\mathbb{C})$ $\quad$ $Sp_2(\mathbb{C}) \xrightarrow{2:1} SO_5(\mathbb{C})$ $\quad$ $G_2(\mathbb{C})$

Compact $SU(3)$ $\quad$ $Sp(2) \longrightarrow SO(5)$ $\quad$ G_2

(Exercise: construct the covering)

Groups of rank 1

G - compact Lie group, $T \subset G$ a maximal torus
$\operatorname{rank} G = \dim T$
(since all maximal tori are conjugate $T' = gTg^{-1}$ the definition does not depend on the choice of T)

__Theorem__ If $\operatorname{rk} G = 1$ then $G = S^1$ or $G = SU(2)$ or $G = SO(3)$. Equivalently: $\dim \mathfrak{g} = 1$ or $\mathfrak{g} = su(2)$.

__Proof__ Lie algebra $\mathfrak{t}$ acts on $\mathfrak{g}$.
$\mathfrak{t} = \operatorname{lin}\{X_0\}$, X_0 has eigenvalues on $\mathfrak{g}_{\mathbb{C}}$ purely imaginary.

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus \mathfrak{g}_\alpha \qquad [X_0, X_\alpha] = i n_\alpha X_\alpha \quad \text{for } X_\alpha \in \mathfrak{g}_\alpha.$$

(we can rescale X_0 to have $n_\alpha \in \mathbb{Z}$)

Since the action is real, the eigenspaces come in pairs: if $X_\alpha \in \mathfrak{g}_\alpha$ then $\overline{X_\alpha} \in \mathfrak{g}_{-\alpha}$ with $n_{-\alpha} = -n_\alpha$

__Step 1__ $\mathfrak{h}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ $\qquad \mathfrak{h} = \mathfrak{t} \oplus \operatorname{lin}\{X_\alpha + \overline{X}_\alpha, i(X_\alpha - \overline{X}_\alpha)\}$

$\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ since $[X_\alpha, \overline{X}_\alpha] \in \mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$.

K is the corresponding Lie group $\quad \operatorname{rk} K \leq \operatorname{rk} G = 1$.

__Claim__ $[X_\alpha, \overline{X_\alpha}] \neq 0$.

If not, then K would have at least 2-dimensional maximal torus.

(after rescaling X_α: $\quad [X_\alpha, \overline{X}_\alpha] = i X_0$ and get an isomorphism $\mathfrak{h} \simeq su(2)$ $\quad X_0 \leftrightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = iH$

$$X_\alpha \leftrightarrow \tfrac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X$$

$$\overline{X}_\alpha \leftrightarrow \tfrac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -Y \ )$$

Consider $V \subset \mathfrak{g}_{\mathbb{C}}$ for a fixed $X_{-\beta} \in \mathfrak{g}_{-\beta}$, β such that $0 < n_\beta$ smallest

$$V = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\substack{\text{positive} \\ n_\alpha}} \mathfrak{g}_\alpha \oplus \operatorname{lin} X_{-\beta}$$

$$\operatorname{Tr}(\operatorname{ad} H_{|V}) = 0 + \sum_{\substack{\alpha \\ \text{positive} \\ \alpha \neq \beta}} i\, n_\alpha \dim L_\alpha + i n_\beta \dim L_\beta - i n_\beta$$

but since $X_0 = [X_\beta, \bar{X}_\beta]$ and <u>$\operatorname{ad} X_\beta$, $\operatorname{ad} \bar{X}_\beta$ preserve V</u>

we have $\operatorname{Tr}(\operatorname{ad} X_{0|V}) = 0$.

It follows that $\dim L_\alpha = 0$ for $\alpha \neq \beta$ and $\dim L_\beta = 1$. ▨

Let $\mathfrak{g}$ be a Lie algebra of arbitrary rank $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$

<u>Theorem</u>: $\dim \mathfrak{g}_\alpha = 1$. If α is proportional to β occurring in the decomposition, then $\alpha = \beta$ or $\alpha = -\beta$.

<u>Proof</u> Let $k_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\beta \sim \alpha} \mathfrak{g}_\beta$ (for a fixed α occurring in the decomposition of $\mathfrak{g}$). Then all the roots of K have common kernel $\mathfrak{u}_\alpha \subset \mathfrak{t}$. Then the subtorus $U_\alpha \subset T$ lies in the center of K. The quotient K/U_α is of rank 1. The Lie algebra of K/U_α is $\mathfrak{t}/\mathfrak{u}_\alpha \oplus \bigoplus_{\beta \sim \alpha} \mathfrak{g}_\beta$. Hence $\beta = \pm\alpha$ $\dim \mathfrak{g}_\beta = 1$. as in the case of rank 1 groups. ⊠

<u>Remark</u>: Centralizers of element $t \in T$:

Let $I = \{\alpha : t \in U_\alpha\}$.

Centralizer of t has the Lie algebra $\mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in I} \mathfrak{g}_\alpha$

If $I = \emptyset$ then we say that t is <u>regular</u>, otherwise <u>singular</u>

Weyl group and Weyl chambers

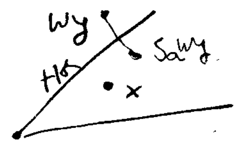
We fix a G-invariant scalar product in $\mathfrak{g}$
⇝ identification $\kappa : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ $(\mathfrak{h} = i\mathfrak{t})$
Given a root $\alpha \in \mathfrak{h}^*$ ⇝ symmetry of $\mathfrak{h}$ (reflection in H_α)

1) Claim: There is an element $w_\alpha \in W$, such that w_α induces the symmetry in H_α.
(W acts on T, so W acts on its Lie algebra)

Proof $K = Z(U_\alpha) \subset G$
K/U_α is of rank 1. isomorphic to $SU(2)$ or $SO(3)$

$W(K/U_\alpha) = \mathbb{Z}/2$ and it is isomorphic to
$W(K) = N_K T/T \hookrightarrow N_G T/T = W$
The nontrivial element of $W(K)$ induces the reflection in U_α. ⊠

2) Claim The subgroup W' generated by s_α acts transitively on Weyl chambers.
Proof



for $y \in \mathcal{W}_1$ and $x \in \mathcal{W}_0$
let $w \in W$ be such that $\mathrm{dist}(x, wy)$ is minimal. We claim that $w(\mathcal{W}_1) = \mathcal{W}_0$.
If wy and x are not on the same side of H_α then $s_\alpha wy$ is closer to x than wy. (contr.) ⊠

3) Claim W acts freely on the set of Weyl chambers.
Proof Fix $\mathcal{W}_0$. Let ρ be the sum of α s.t. H_α is a wall of $\mathcal{W}_0$. Suppose w fixes $\mathcal{W}_0$.

Then $w\xi = \xi$. Let $g \in NT$ represents $w \in NT/T$, g commutes with $\exp(t\xi)$.

<u>Lemma</u> There exists a torus T' s.t. g and $\exp(t\xi)$ are contained in T'.

If so then $T' = T$ since $\exp(t\xi)$ is a regular element and it belongs to only one maximal torus,

<u>Proof of the lemma</u>: The closure of the group generated by g and $\exp(t\xi)$ is isomorphic to $\mathbb{Z}/n \times S$, where S is a torus, $n \in \mathbb{N}$. Let $h = (h_1, h_2)$, h_1 - generator of $\mathbb{Z}/n$ and h_2 topological generator of S, i.e. closure $\langle h_2^m : m \in \mathbb{Z} \rangle = S$. We use (again) the fact that there exists a maximal torus T' containing any element - for example h. Then $T' \supset \langle g, \exp(t\xi) \rangle$
$\Rightarrow [g] = 1 \in W$ ⊠

<u>Conclusion</u> Since $W' \subset W$ generated by the reflections acts transitively and W acts freely we deduce:

<u>Theorem</u> W acts <u>freely</u> and <u>transitively</u> on the set of Weyl chambers.

Now we clarify the identification $k : h \xrightarrow{\simeq} h^*$.

<u>Proposition</u> For every $\alpha \in R \subset h^*$ let $\alpha^* = 2k^{-1}(\alpha)/\langle \alpha, \alpha \rangle \in h$. (It is called inverse root.) Then $k^{-1} S_\alpha k : h \to h$ is given by $x - \alpha(x)\alpha^*$. The inverse root does not depend on the metric

<u>Proof</u>: Clear: symmetry is given by $kx \mapsto kx - 2\frac{\langle kx, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$.
and $\langle kx, \alpha \rangle = \alpha(x)$.

If we rescale $\langle , \rangle$ by λ then α^* does not change. ⊠
since k^{-1} changes by λ.

7) <u>Proposition</u> Inverse roots are integral i.e. $R^* \subset \Lambda$

<u>Proof</u>

$$\begin{tikzcd} \Lambda \arrow[d] \arrow[r, phantom, "\subset"] & \mathfrak{h} \arrow[d, "\alpha"'] \arrow[r, "\exp"] & T \arrow[d, "\bar{\alpha}"] & \supset U_\alpha = \operatorname{Ker} \bar{\alpha} \\ \mathbb{Z} \arrow[r, phantom, "\subset"] & \mathbb{R} \arrow[r] & S^1 & \end{tikzcd}$$

$$\alpha(\alpha^*) = \alpha\left(2k^{-1}(\alpha)/\langle\alpha,\alpha\rangle\right) = 2\,\frac{\alpha(k^{-1}\alpha)}{\langle\alpha,\alpha\rangle} = 2$$

therefore

$$\alpha\left(\frac{\alpha^*}{2}\right) = 1 \iff \exp\left(\frac{\alpha^*}{2}\right) \in \operatorname{Ker}\bar{\alpha} = U_\alpha$$

the symmetry s_α acting on T fixes U_α

therefore $\exp\left(s_\alpha \frac{\alpha^*}{2}\right) = s_\alpha \exp\left(\frac{\alpha^*}{2}\right) = \exp\left(\frac{\alpha^*}{2}\right)$.

On the other hand $s_\alpha \frac{\alpha^*}{2} = -\frac{\alpha^*}{2}$

hence $\exp\left(-\frac{\alpha^*}{2}\right) = \exp\left(\frac{\alpha^*}{2}\right)$

$\Rightarrow \exp(\alpha^*) = 1 \Rightarrow \alpha^* \in \ker \exp = \Lambda$ ∎

<u>Corollary</u> The reflection in $\mathfrak{h}^*$ is of the form:

$$s_\alpha : \omega \longmapsto \omega - 2\frac{\langle\omega,\alpha\rangle}{\langle\alpha,\alpha\rangle}\cdot\alpha = \omega - \omega(\alpha^*)\cdot\alpha$$

For integral weigh (e.g. a root) $\omega(\alpha^*) \in \mathbb{Z}$.

We arrived to the <u>axioms of a root system</u>. The root system of a Lie group satisfies these axioms: R is a finite set in V with $\langle\,,\rangle$

1. R spans V
2. If $\alpha \sim \beta$ then $\alpha = \beta$ or $\alpha = -\beta$
3. The symmetry in $H_\alpha = \alpha^\perp$ maps R to R
4. $s_\alpha(\beta) - \beta$ is an integer multiple of α.

Simple Lie algebras $\longleftrightarrow$ <u>indecomposible</u> root systems $\longleftrightarrow$ Dynkin diagrams

Indecomposible means that (V, R) is not of the form $(V_1 \overset{\perp}{\oplus} V_2,\ R_1 \sqcup R_2)$.