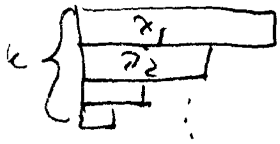
Wykład 10

1) Schur Functors

$\lambda = (5, 4, 2, 1)$

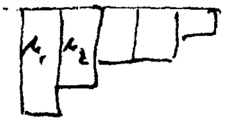
Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$



It is called a partition:

$$\underbrace{|\ \lambda_1\ |\ \lambda_2\ |\ \lambda_3\ |\ \dots\ |\ \cdot\ |}_{d = \lambda_1 + \lambda_2 + \dots \lambda_k}$$

Also we have a dual partition



$\mu_1 \geq \mu_2 \geq \mu_3 \dots \mu_l$

$\mu = (4, 3, 2, 2, 1)$

Consider the space $V^{\otimes d}$ for any v.sp. V

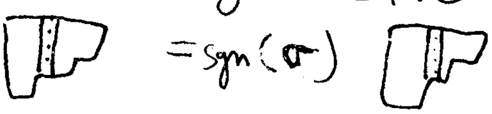
$$\begin{array}{ccccccccc} V & \otimes & V & \otimes & V & \otimes & V & \otimes & V \\ \otimes & & \otimes & & & & & & \\ V & \otimes & V & \otimes & V & \otimes & V & & \\ \otimes & & \otimes & & & & & & \\ V & \otimes & V & & & & & & \\ \otimes & & & & & & & & \\ V & & & & & & & & \end{array}$$

$V^{\otimes \lambda} \simeq V^{\otimes d}$

$$S_\lambda(V) = V^{\otimes \lambda} \big/ \sim$$

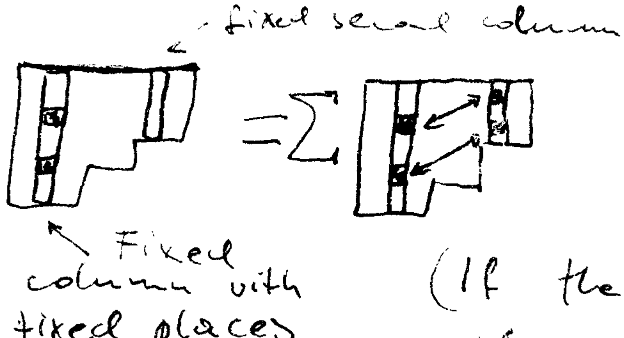
We divide $V^{\otimes \lambda}$ by the relations of two types

1 - antisymmetric in columns i.e.



columns differ by a permutation σ

2



sum taken over all exchanges

(If the fixed column has k fixed places and the second column has μ_j boxes then there are $\binom{\mu_j}{k}$ exchanges

2) Example

$$
\begin{array}{|c|c|}\hline 1 & 5 \\ \hline \boxed{2} & 6 \\ \hline 3 & 7 \\ \hline \boxed{4} & \\ \hline \end{array}
=
\begin{array}{|c|c|}\hline 1 & \boxed{2} \\ \hline \boxed{5} & \boxed{4} \\ \hline 3 & 7 \\ \hline \boxed{6} & \\ \hline \end{array}
+
\begin{array}{|c|c|}\hline 1 & \boxed{2} \\ \hline \boxed{5} & 6 \\ \hline 3 & \boxed{4} \\ \hline \boxed{7} & \\ \hline \end{array}
+
\begin{array}{|c|c|}\hline 1 & 5 \\ \hline \boxed{6} & \boxed{2} \\ \hline 3 & \boxed{4} \\ \hline \boxed{7} & \\ \hline \end{array}
$$

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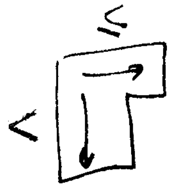
number $\boxed{i}$ stands for the vector V_i.

Def A numbering of a diagram is admissible if the numbers
- grow downwards
- non decrease rightwards



We will find a basis of $S_\lambda(V)$. Let $e_1 \dots e_n$ be a basis of V. Then for a numbering T of the diagram λ we construct an element $e_T \in S_\lambda(V)$.

Theorem $\{e_T\}_{T\text{-admissible numbering}}$ is a basis of $S_\lambda(V)$.

Example If $\lambda = \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}}_{k}$ then $S_\lambda V = \mathrm{Sym}^k V$

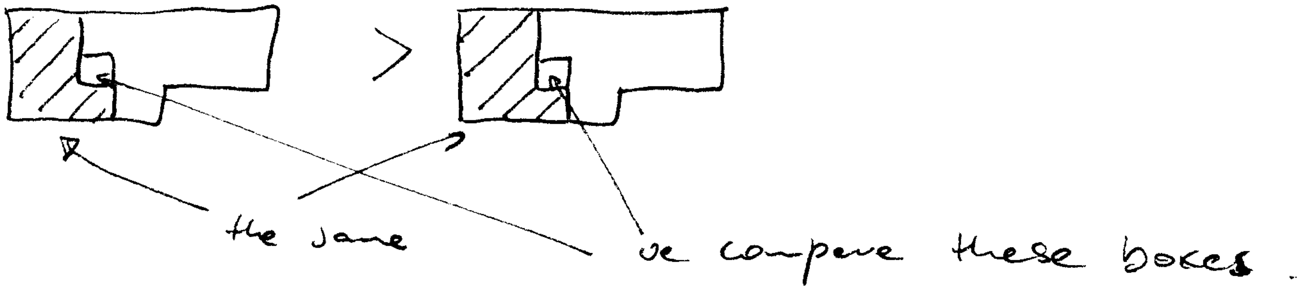
$T = \boxed{i_1}\boxed{i_2}\boxed{i_3}\boxed{i_4}$ $\quad i_1 \leq i_2 \leq i_3 \dots \quad e_T = e_1^{i_1} \cdot e_2^{i_2} \dots e_k^{i_k}$

If $\lambda = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$ $\quad S_\lambda V = \Lambda^k V$

$T = \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline \end{array}$ $\quad i_1 < i_2 < \dots < i_k \quad e_T = e_{i_1} \wedge e_{i_2} \dots \wedge e_{i_k}$

3)

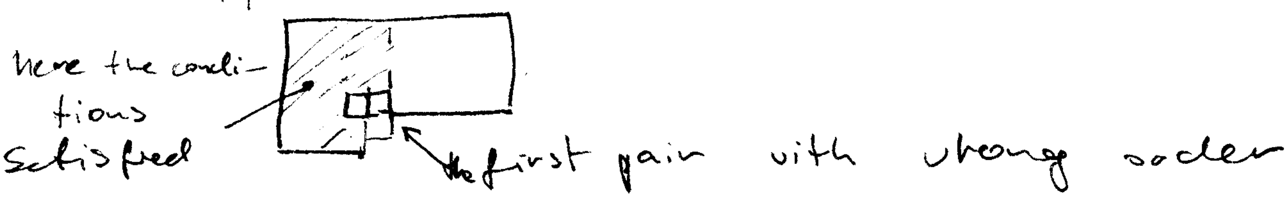
Proof We will introduce an order of numberings (a lexicographic order)



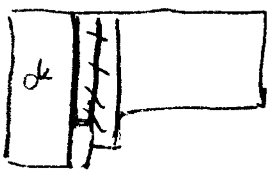
Claim If e_T is obtained from T - not admissible, then e_T is a sum of $e_{T'}$ with $T' > T$.

1. If we replace a column by the ordered column, then we get bigger T'.
   (if the entries are the same, then $e_T = 0$)

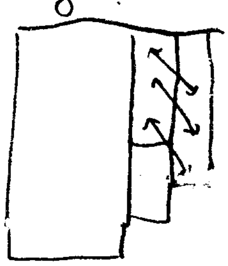
2. Suppose



We exchange the top part of the first column



We do not spoil the part with good ordering:



get bigger boxes in the first column

We skip the proof that e_T are linearly independent.

4)

S_λ is a functor

Vector spaces $\longrightarrow$ Vector spaces

We look just at $S_\lambda(\mathbb{C}^n)$.
This is a representation of $Gl_n(\mathbb{C})$.

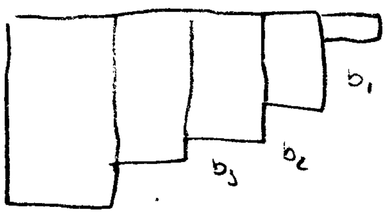
<u>Theorem</u> $S_\lambda(\mathbb{C}^n)$ is an irreducible representation of $Gl_n(\mathbb{C})$.

(We will not give a proof. Another construction of $S_\lambda(\mathbb{C}^n)$ with a proof of irreducibility: see FuHa - Th. 6.3)

<u>Corollary</u> $S_\lambda(\mathbb{C}^n)$ is an irreducible representation of $Sl_n(\mathbb{C})$ since $Gl_n(\mathbb{C})$ is generated by center $\{a\cdot I : a\in\mathbb{C}^*\}$ and $Sl_n(\mathbb{C})$.

Note: we have a surjective map

$$\bigotimes_{i=1}^{n-1} Sym^{b_i}(\Lambda^i\mathbb{C}^n) \twoheadrightarrow S_\lambda(\mathbb{C}^n)$$



$b_i = \lambda_i - \lambda_{i+1}$

The highest weight vector corresponds to the numbering
$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & & \\ 3 & 3 & & & \end{array}$$
i.e. i in i'th row.

$$\boxed{S_\lambda(\mathbb{C}^n) = V(\lambda)}$$

5)

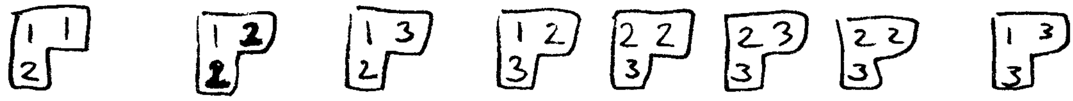
It is easy to compute other weights:

$$\mathrm{weight}(e_T) = \sum a_i L_i$$

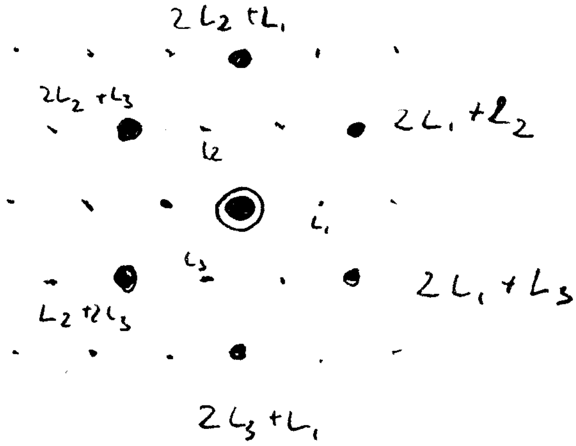
a_i = number of i's in the filling of the young diagram.

filling := numbering

<u>Example</u> $\lambda =$  $n = 3$



$2L_1+L_2$ L_1+2L_3 $\underbrace{\qquad\qquad}_{L_1+L_2+L_3}$ $2L_2+L_3$ L_2+2L_3 $2L_2+L_3$ L_1+2L_3



view of $\tilde{h}^* / L_1+L_2+L_3 = h^*$

This way we get a formula

$\dim V(\lambda)_\omega = K_{\lambda\omega}$ = number of fillings of λ with numbers given by $\omega = \{1^{a_1}, 2^{a_2}, \ldots n^{a_n}\}$

In fact $K_{\lambda\omega}$ are considered only for $\omega \in W$ since the multiplicities are symmetric with respect to action of the Weyl group.

$K_{\lambda\omega}$ are called Kostka numbers.

(6) If we restrict $Gl_n(\mathbb{C})$-representation to $Sl_n(\mathbb{C})$ then $S_\lambda(\mathbb{C}^n) \simeq S_{\lambda'}(\mathbb{C}^n)$ where $\lambda_i' = \lambda_i + c$ since $c(L_1 + L_2 + \dots L_n) = 0$ in $\mathfrak{h}^*$. Therefore we consider only λ with $\lambda_n = 0$.

Characters of representations

V representation $\rightsquigarrow$ function: $G \xrightarrow{\chi_V} \mathbb{C}$
$g \mapsto \mathrm{Tr}(g : V \to V)$

- If $G = T^n = \underbrace{S^1 \times S^1 \cdots S^1}_{n}$

then each character is of the form
$$\sum a_\omega z_1^{\omega_1} z_2^{\omega_2} \cdots z_n^{\omega_n} \qquad (a_\omega \in \mathbb{N} \cup \{0\})$$
From χ_V we esily read the decompsition of V into irreducible components
$$V = \bigoplus_\omega (\mathbb{C}_\omega)^{\oplus a_i}$$

- If $G = Gl_n(\mathbb{C})$

Consider the restriction $Gl_n(\mathbb{C})$ to $U(n)$.

Holomorphic representations $Sl_n(\mathbb{C})$ are determined by the restriction to $U(n)$. Further we restrict the representation to the maximal torus $T \subset U(n)$

Claim 1. For a representation V of $Gl_n(\mathbb{C})$ the character of the restriction to T is

7)

<u>invariant with respect to the action of the Weyl group $\Sigma_n = N(T)/T$.</u>

<u>Proof</u>: Let $g \in N(T)$, $z \in T$ then $\mathrm{Tr}(gzg^{-1}) = \mathrm{Tr}(z)$.

<u>Claim 2</u> If V_1, V_2 representations of $Gl_n(\mathbb{C})$

$\chi_{V_1}|_T = \chi_{V_2}|_T$ then $\chi_{V_1} = \chi_{V_2}$

<u>Proof</u> If $\chi_{V_1}|_T = \chi_{V_2}|_T$

then $\chi_{V_1}|_{gTg^{-1}} = \chi_{V_2}|_{gTg^{-1}}$ for any $g \in U(n)$

Now we use the following (which is valid for any compact connected Lie group):

for any element $g \in U(n)$ there exists h s.t. $g \in hTh^{-1}$.

Therefore $\chi_{V_1}|_{U(n)} = \chi_{V_2}|_{U(n)}$.

Since we assume that the representations are holomorphic then $\chi_{V_1} = \chi_{V_2}$. $\square$

So we have:

$$\text{Representations of } Gl_n(\mathbb{C}) \longrightarrow \left\{ \begin{array}{l} \Sigma_n\text{- invariant} \\ \text{Polynomial function} \\ \text{in } x_1\, x_2 \cdots x_n,\ x_1^{-1}, x_2^{-1} \cdots x_n^{-1} \end{array} \right\}$$

For Sl_n where we have $x_1 \cdot x_2 \cdots x_n = 1$ we get

$$\text{Representations of } Sl_n(\mathbb{C}) \longrightarrow \left\{ \begin{array}{l} \Sigma_n \text{ invariant polynomials} \\ \text{in } x_1\, x_2\, x_3 \cdots x_n \end{array} \right\}_{\big/ x_1 x_2 \cdots x_n = 1}$$

8) The rest follows from the general theory of characters for compact groups:

1. The characters of irreducible representations are ortonormal with respect to the product in $L^2(G)$

$$\circledast \quad \int_G \chi_{V_1}(g)\overline{\chi_{V_2}(g)}\,dg = \begin{cases} 1 & V_1 \simeq V_2 \\ 0 & V_1 \not\simeq V_2 \end{cases}$$

(in fact χ_V for V irreducible form a basis of the Hilbert space of the functions invariant with respect to conjugation - so called "class functions"

$$\left(L^2(G)\right)^G = \widehat{\bigoplus_{V \text{ irred.}}} \mathbb{C}\cdot\chi_V \quad \text{- Peter-Weyl th.})$$

2. Corollary V is determined by its character:
if $(\chi_W, \chi_V) = b_W$ for W irreducible
then $V \simeq \bigoplus_{W\text{ - irreducible}} W^{\oplus b_W}$
(in particular $b_W \in \mathbb{N} \cup \{0\}$)

The formula $\circledast$ follows from

$$\circledast\circledast \quad \int_G \chi_W\,dg = \dim W^G$$

since $\chi_{V_1}\cdot\overline{\chi_{V_2}} = \chi_{V_1}\cdot\chi_{V_2^*} = \chi_{V_1\otimes V_2^*} = \chi_{\mathrm{Hom}(V_2,V_1)}$

and $\dim \mathrm{Hom}(V_2,V_1) = \begin{cases} 1 & V_1 \simeq V_2 \\ 0 & V_1 \not\simeq V_2 \end{cases}$ (Schur Lemma)

9)

Proof of ⊛ Define the projection onto the invariant space:

$$\varphi: W \to W^G$$
$$v \mapsto \int_G g(v)\,dg \quad (\text{averaging } v)$$

$\varphi^2 = \varphi$ so $tr(\varphi) = \dim(\mathrm{im}\,\varphi) = \dim W^G$

$$tr\,\varphi = \int_G tr(g)\,dg = \int_G \chi_W\,dg \qquad \blacksquare.$$

We conclude:

Representations of $Sl_n(\mathbb{C})$ $\rightsquigarrow$ Symmetric polynomials $/ \prod x_i = 1$ is injection.

Not every symmetric polynomial is in the image: eg $n = 2$ (positivity of coefficients is not enough)

$$V = \bigoplus_{i=0}^{N} a_i \,\mathrm{Sym}^i(\mathbb{C}^2) \qquad \chi_V \in \mathbb{Z}[x, y = x^{-1}]$$

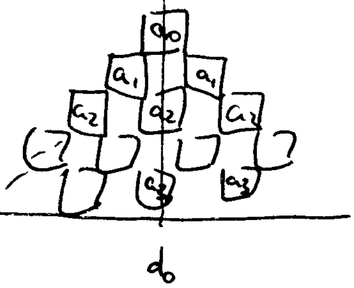
then χ_V is of the form: $\sum_{i=-N}^{+N} d_i x^i$

$a_0 + a_2 + a_4 + \dots = d_0$ $\longrightarrow$

$a_1 + a_3 + a_5 + \dots = d_1$

$a_2 + a_4 + a_6 \dots = d_2$

$\vdots$



$d_i = d_{-i}$

for $i \geq 0$ $\quad d_{i+2} < d_i$

But if we consider formally

$$\sum_{V\text{-irred}} a_V V \in Rep(SL_n(\mathbb{C})) \qquad a_V \in \mathbb{Z}$$

then $Rep(Sl_n(\mathbb{C})) \cong$ Symmetric Polynomials $/ \prod x_i = 1$

Since $\{s_\lambda = x^\lambda + \text{symmetrization} + \text{lower terms}\}$ is a basis of symmetric polynomials $/ \prod x_i = 1$

10) We see

$$\chi_{S_\lambda(\mathbb{C}^n)} = \sum K_{\lambda\omega}\, x^\omega =: s_\lambda \quad \text{Schur function}$$

where $x^\omega = x_1^{\omega_1} x_2^{\omega_2} \dots x_n^{\omega_n}$

Note that this formula does not depend on n provided that λ and ω have at most n rows.
If λ has more rows then $S_\lambda = 0$ and there is no way to fill the first column with entries $1 \dots n$ without repetition, so $K_{\lambda\omega} = 0$.

Weyl Character Formula

$$s_\lambda(x_1, \dots x_n) = \frac{\det\left(x_j^{\lambda_i + n - i}\right)_{i,j \in \{1 \dots n\}}}{\det\left(x_j^{n-i}\right)_{i,j \in \{1 \dots n\}}}$$

This formula (which is purely combinatorial statement) has generalizations for other simple Lie groups.

First consider an example $n = 3$ $\lambda = \begin{smallmatrix}\boxed{7} \\ \boxed{5}\end{smallmatrix} = (7,5)$

λ extended by 0
$(7,5,0)$

$$\begin{vmatrix} x_1^{7+3-1}, & x_2^{7+3-1}, & x_3^{7+3-1} \\ x_1^{5+3-2}, & x_2^{5+3-2}, & x_3^{5+3-2} \\ x_1^{0+3-3}, & x_2^{0+3-3}, & x_3^{0+3-3} \end{vmatrix}$$

11)

Denominator $\lambda = (0,0,0)$

$$\begin{vmatrix} x_1^{3-1} & x_2^{3-1} & x_3^{3-1} \\ x_1^{3-2} & x_2^{3-2} & x_3^{3-2} \\ x_1^{3-3} & x_2^{3-3} & x_3^{3-3} \end{vmatrix} = \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1^1 & x_2^1 & x_3^1 \\ 1 & 1 & 1 \end{vmatrix}$$

which is up to a sign Vandermonde determinant.

Other important expression for S_λ

$$\begin{pmatrix} c_{\lambda_1} & c_{\lambda_1+1} & c_{\lambda_1+2} & \cdots \\ c_{\lambda_2-1} & c_{\lambda_2} & c_{\lambda_2+1} & \cdots \\ c_{\lambda_3-2} & c_{\lambda_3-1} & c_{\lambda_3} & \cdots \\ & & \cdots & c_{\lambda_n} \end{pmatrix}$$

where $c_k = \sigma_k = \Lambda_k = E_k = \sum_{\substack{|I|=k \\ i_1 < i_2 \cdots < i_k}} x^{i_1} x^{i_2} \cdots x^{i_k}$

is the elementary symmetric function

$c_k = \chi_{\Lambda^k \mathbb{C}^n)}$

Generalization for arbitrary simple group:

λ dominant weight, $\chi_{V(\lambda)} = \frac{A_{\lambda+\rho}}{A_\rho}$ $\rho = \frac{1}{2}(\sum \text{positive roots})$

(for $sl_n(\mathbb{C})$ $\rho = \frac{1}{2}\sum_{i<j} L_i - L_j = \frac{1}{2}\sum_{i=1}^{n-1}(n+1-2i)L_i = \sum_{i=1}^{n}(n-i)L_i$)

$A_\mu = \sum_{w \in W} (-1)^{|w|} x^{w(\mu)}$

This generalizes the formula $\det(x_j^{\mu_i}) = \sum_{\sigma \in \Sigma} \operatorname{sgn} \sigma \prod_{i=1}^{n} x_{\sigma(i)}^{\mu_i}$

TBC.

12) In general μ is a weight i.e $\mu \in \Lambda^* \subset h^*$

When we identify $\Lambda^* \cong \mathbb{Z}^n$ $\mu = (\mu_1 \dots \mu_n)$ then x^μ means $x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3} \dots x_n^{\mu_n}$.

Let's come back to the simples case i.e $G = Sl_2(\mathbb{C})$ $\lambda = (k,0)$ $\underbrace{\boxed{}\boxed{}\boxed{}\boxed{}}_{k}$ $V_{(\lambda)} = Sym^k \mathbb{C}^2$

$$S_\lambda = \begin{vmatrix} x_1^{k+2-1}, & x_2^{k+2-1} \\ x_1^{2-2}, & x_1^{2-2} \end{vmatrix} \Big/ \begin{vmatrix} x_1, & x_2 \\ 1, & 1 \end{vmatrix}$$

We have the relation $x_1 x_2 = 1$.

Let $x_1 = x$ $x_2 = x^{-1}$

$$S_\lambda = \begin{vmatrix} x^{k+1} & x^{-(k+1)} \\ 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} x & x^{-1} \\ 1 & 1 \end{vmatrix} =$$

$$= \left(x^{k+1} - x^{-(k+1)}\right) \Big/ \left(x - x^{-1}\right) = \frac{x^{-(k+1)}}{x^{-1}} \frac{x^{2k+2} - 1}{x^2 - 1}$$

$$= x^{-k} \left(x^{2k} + x^{2k-2} + \dots x^2 + 1\right) =$$

$$= x^{-k} + x^{-k+2} + \dots \quad x^{k-2} + x^k$$

that is: weight appearing in $Sym^k \mathbb{C}^2$ are $-k, -k+2, \dots k-2, k$, each weight space is of dimension one.