Wykład 9

(1) $\mathfrak{g} = sl_n(\mathbb{C}) = \bigoplus \mathfrak{g}_\alpha \oplus \mathfrak{h}$ $\left(\alpha = L_i - L_j \Rightarrow \mathfrak{g}_\alpha = \mathrm{lin}\, E_{ij}\right)$

$\mathfrak{h}$ = diagonal matrices $\overset{tr=0}{=}$ Cartan algebra

$\Lambda \subset \mathfrak{h}^*$ lattice of weights

Λ generated by L_i $\quad L_i(\mathrm{diag}(a_1 \dots a_n)) = a_i$

Λ_R lattice generated by roots $L_i - L_j$

Exercise $\quad \Lambda/\Lambda_R \cong \mathbb{Z}/n \cong \mathrm{Center}(SL_n \mathbb{C})$

Proposition 1 Let V be an irreducible representation. Then any two weights occuring in V differ by a vector from Λ_R

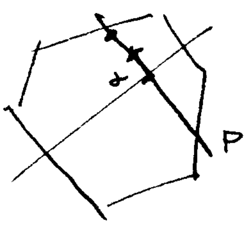
Proof α - a root $\quad X \in \mathfrak{g}_\alpha$

ω - a weight $\quad v \in V_\omega$

then $\quad Xv \in V_{\omega+\alpha}$

Therefore $X \in \mathfrak{g}$ does not mix the types of weights: $V = \bigoplus_{[\omega] \in \Lambda/\Lambda_R} V_{[\omega]}$ as representations. For irreducible representation only one type occurres. $\square$

Proposition 2 The set of weights is symmetric with respect to $H_\alpha = \alpha^\perp$ for any α - root.



For every root α the algebra generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ is isomorphic to $sl_2(\mathbb{C})$ (this is a general statement (for simple alg), but for $\mathfrak{g} = sl_n(\mathbb{C})$, $\alpha = L_i - L_j$ $\langle \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \rangle$ = Endomorphism of $\mathrm{lin}\{\varepsilon_i, \varepsilon_j\}$)

Let us fix a line P $\quad P \perp H_\alpha$

② The subspace $\bigoplus_{\alpha \in P} V_\alpha$ is a representation of $sl_2\mathbb{C} = \langle g_\alpha, g_{-\alpha} \rangle$. Therefore the weight of $sl_2\mathbb{C}$ are symmetric with respect to 0 = intersection point $P \cap H_\alpha$. ☐

Prop 3 There exists an eigenvector $v \in V_\omega$ s.t. $Xv = 0$ for $X \in g_\alpha$, α positive.
(The standard choice of positive roots $\alpha = L_i - L_j$ $i<j$)

Proof Take ω = the vertex which is the most distant from the hyperplane H dividing positive roots and negative roots. E.g. $H = \delta^\perp$ where $\delta = \frac{1}{2}$ sum of positive roots. here $\delta = \frac{1}{2} \sum_{i=1}^{n} (n+1-2i) L_i$. Take ω such that $\varphi(\omega) = \langle \omega, \delta \rangle$ maximal.
If $v \in V_\omega$ $X \in g_\alpha$ $\alpha = L_i - L_j$ $i<j$
then $Xv \in V_{\omega+\alpha}$ $\varphi(\omega+\alpha) = \varphi(\omega) + j - i > \varphi(\omega)$.
therefore $Xv = 0$. ③

Prop 4 V is generated by v and the elements of g_α for α negative.

Proof We have to show that for
$w = Y_1 Y_2 \dots Y_n v$ with $Y_i \in g_{\alpha_i}$ $\alpha_i < 0$
and $X \in g_\beta$ $\beta > 0$
the vector $X Y_1 \dots Y_n v$ can be expressed by Y_i' - $Y_i' \in g_{\alpha_i'}$ $\alpha_i' < 0$.
The proof is by induction: $w \in (g_-)^n v \Rightarrow X \in (g_-)^{n-1} v$
if $n = 0$ then $Xw = Xv = 0$
step $XYw = ?$ $w \in (g_-)^n v$
We have to consider two cases:
to show that $XYw \in (g_-)^n v$

3)

1) $X = E_{ij}$ $Y = E_{kl}$ if $j \neq k$ and $i \neq l$

$XYw = 0.$

2) $XYw = E_{ij} E_{ji} w = (E_{ii} - E_{jj}) w = \text{scalar} \cdot w \in (\mathfrak{g}_-)^n V$

3) $XYw = \overset{+}{E_{ij}} \overset{-}{E_{jk}} w =$ $i < j$ $k < j$

$= (\underbrace{[E_{ij}, E_{jk}]}_{+ \text{ or } -} + \overset{-}{E_{jk}} \overset{+}{E_{ij}}) w$

$= E_{ik} w + E_{jk} \underbrace{E_{ij} w}_{\in (\mathfrak{g}_-)^{n-1} V \text{ by induction}} \in (\mathfrak{g}_-)^n V$

4) $XYw = \overset{+}{E_{ij}} \overset{-}{E_{ki}} w =$ $i < j$ $i < k$

$= (\underbrace{[E_{ij}, E_{ki}]}_{+ \text{ or } -} + E_{ki} E_{ij}) w$

$= -E_{kj} w + \overset{-}{E_{ki}} \underbrace{E_{ij} w}_{\in (\mathfrak{g}_-)^{n-1} V \text{ by induction}} \in (\mathfrak{g}_-)^n V$

$\bigcup_n \mathfrak{g}_-^n V$ is a subrepresentation of V, so it is equal to V. ∎

<u>Corollary</u> There is only one weight satisfying the statement of Prop 3, $w \in W$. It is called <u>highest weight</u> (or dominant weight). Corresponding weight space is of dimension 1 (since if v_1 v_2 two vectors of the highest weight $v_1 \nsim v_2$ then $v_2 \notin (\mathfrak{g}_-)^\infty v_1$.

<u>Prop 5</u> Suppose V_1 V_2 two irreducible representations with the same highest weight. Then $V_1 \approx V_2$

4) <u>Proof</u> Let $v_i \in V_i$ highest weight vector

$v = (v_1, v_2) \in V_1 \oplus V_2$ a vector of the highest possible weight.

let $U = (\mathfrak{g}_-)^n v \subset V_1 \oplus V_2$.

U is irreducible.

There are maps

$$\begin{tikzcd} & U \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V_1 & & V_2 \end{tikzcd}$$

which are nontrivial.

Therefore $V_1 \simeq U \simeq V_2$ $\square$

<u>Construction of irreducible representations</u>

Given a weight $w \in \Lambda \cap W$ (W Weyl chamber associated to the ordering of roots)

$w = \sum a_i L_i \qquad a_1 \geq a_2 \geq a_3 \dots \geq a_{n-1} \geq 0$

$w = b_1 L_1 + b_2 (L_1 + L_2) + b_3 (L_1 + L_2 + L_3) + \dots + b_{n-1} L_{n-1}$

1. Find a representation with the highest weight $(L_1 + \dots + L_i)$ $\qquad \Lambda^i \mathbb{C}^n$

2. Take $\mathrm{Sym}^{b_i}(\Lambda^i \mathbb{C}^n)$ (or $(\Lambda^i \mathbb{C}^n)^{\otimes b_i}$)

There is a subrepresentation of the highest weight $b_i (L_1 + \dots + L_i)$

3° $\bigotimes_{i=1}^{n-1} \mathrm{Sym}^{b_i}(\Lambda^i \mathbb{C}^n)$ – here again is a subrepresentation of the highest weight w.

5) Exterior power $\left(\Lambda^1 V = V,\ \Lambda^0 V = \mathbb{C}\right)$

$\Lambda^i V$

generated by the symbols

$v_1 \wedge v_2 \wedge \cdots \wedge v_i \qquad v_i \in V$ (formally $V^{\otimes i}/\sim$

relations: $v_1 \wedge v_2 \wedge \cdots \wedge v_i = \operatorname{sgn}\sigma \, v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(i)}$

for σ – a permutation

linearity $(a v_1 + b v_1') \wedge v_2 \wedge \cdots \wedge v_i = a(v_1 \wedge \cdots v_i) + b(v_1' \wedge \cdots v_i)$

__Basis__ $e_I = e_{j_1} \wedge \cdots \wedge e_{j_i}$ for $I = (j_1 < j_2 < \cdots < j_i)$, $\{e_j\}$ basis of V.

__Universal property__ For any antisymmetric i – linear map ϕ

$$
\begin{tikzcd}
\underbrace{V \times \cdots \times V}_{i} \arrow[r, "\phi"] \arrow[d, "\iota"'] & W \\
\Lambda^i V \arrow[ur, dotted, "\tilde{\phi}"'] &
\end{tikzcd}
$$

$(v_1, \dots v_i) \mapsto v_1 \wedge \cdots \wedge v_i$

there exists a linear map $\tilde{\phi} : \Lambda^i V \to W$ s.t. $\phi = \tilde{\phi}\iota$. The map $\tilde{\phi}$ is unique.

This construction is natural (functor)

Any map $V \to V'$ induces a map $\Lambda^i V \to \Lambda^i V'$.

We get a representation $\Lambda^i \mathbb{C}^n$ of $SL_n(\mathbb{C})$.

$$g(v_1 \wedge \cdots \wedge v_i) = g(v_1) \wedge \cdots \wedge g(v_i)$$

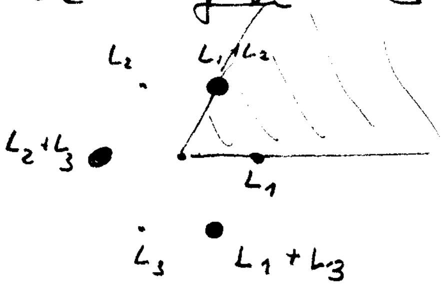
The action of the Lie algebra – have to apply the Leibniz rule:

$$X(v_1 \wedge \cdots \wedge v_i) = \sum_{k=1}^{i} v_1 \wedge \cdots \wedge \underset{\substack{\uparrow \\ k}}{X v_k} \wedge \cdots \wedge v_i .$$

6)

The vectors $e_I = e_{j_1} \wedge \dots e_{j_i}$ are eigen-vectors of the weight $L_{j_1} + \dots + L_{j_i}$. Only if $I = \{1, 2 \dots, i\}$ such weight belongs to W.

Picture for sl_3 $i = 2$



<u>Note</u> $\Lambda^n \mathbb{C}^n$ has dimension 1

$e_1 \wedge \dots \wedge e_n$ has weight $L_1 + \dots + L_n = 0$

It is a trivial representation of $Sl_n(\mathbb{C})$ (but nontrivial $Gl_n(\mathbb{C})$ representation, which acts by the multiplication by $\det(A)$)

<u>Exercise</u> $\Lambda^{n-1}\mathbb{C}^n \simeq (\mathbb{C}^n)^*$ as $Sl_n(\mathbb{C})$ representation.

$\Lambda^i \mathbb{C}^n$ is an irreducible representation of $Sl_n(\mathbb{C})$, it is generated by $e_1 \wedge \dots e_i$. The highest weight $= L_1 + L_2 + \dots + L_i$.

<u>Claim</u> If v is a highest weight vector in V then $v^b = \underbrace{v \cdot \dots v}_{b} \in Sym^b V$ is a highest weight vector in $Sym^b V$ (but $Sym^b V$ does not have to be irreducible)

Weight of v^b $=$ b (weight of v)

7) <u>Claim</u> If $v \in V$ $w \in W$ are a highest weight vectors, then $v \otimes w$ is a highest weight vector in $V \otimes W$.

We constructed an irreducible representation of the highest weight

$$\omega = b_1 L_1 + b_2(L_1+L_2) + b_3(L_1+L_2+L_3) + \dots + b_{n-1}(L_1+\dots+L_{n-1})$$

$$V(\omega) \subset \mathrm{Sym}^{b_1}\Lambda^1\mathbb{C}^n \otimes \mathrm{Sym}^{b_2}\Lambda^2\mathbb{C}^n \otimes \dots \otimes \mathrm{Sym}^{b_{n-1}}\Lambda^{n-1}\mathbb{C}^n$$

$$V(\omega) = (\mathfrak{g}_-)^\infty\left(e_1^{b_1} \otimes (e_1\wedge e_2)^{b_2} \otimes \dots \otimes (e_1\wedge e_2 \dots \wedge e_{n-1})^{b_{n-1}}\right)$$

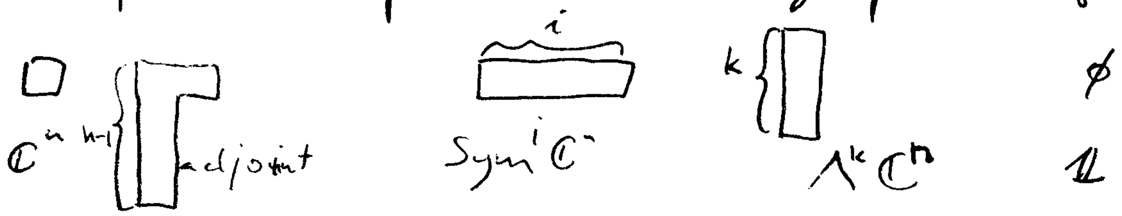
We have to take the subrepresentation, since full tensor may be not irreducible

eg. $$\Lambda^1\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3 = \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \simeq (\text{adjoint rep}) \oplus \text{trivial}$$

$$\mathrm{Sym}^2\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3 = \mathrm{Sym}^2\mathbb{C}^3 \otimes (\mathbb{C}^3)^* \simeq V_{(3L_1+L_2)} \otimes \underset{\mathbb{C}^3}{\underset{\parallel}{V_{L_1}}}$$

weight	
L_1	$\mathbb{C}^n$
$2L_1+L_2+L_3+\dots+L_{n-1}$	adjoint $\subset \mathbb{C}^n \otimes (\mathbb{C}^n)^* = \mathbb{C}^n \otimes \Lambda^{n-1}\mathbb{C}^n$
$L_1+L_2+\dots+L_i$	$\Lambda^i\mathbb{C}^n$
$k L_1$	$\mathrm{Sym}^k\mathbb{C}^n$
0	$\mathbb{C}$ with trivial action, denoted by $\mathbb{1}$

We present representations graphically



8)

Another method of constructing a representation with given weight ω.

- Universal enveloping algebra $U(\mathfrak{g})$

$$U(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \mathfrak{g}^{\otimes 3} \ldots \Big/ \sim$$

$\sim$ generated by

(*) $\mathfrak{g}^{\otimes 2} \ni X \otimes Y - Y \otimes X \sim [X, Y] \in \mathfrak{g}$

$\mathbb{1} \in \mathbb{C}$

Formally: we divide the tensor algebra

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

by the 2-sided ideal I generated by (*).

<u>Theorem:</u> As a vector space $\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})$

Representations of $\mathfrak{g}$ $\longleftrightarrow$ $U(\mathfrak{g})$-modules.

Let $\omega \in \mathfrak{h}^*$. Consider 1-sided ideal of $U(\mathfrak{g})$ generated by $X \in \mathfrak{g}_\alpha$ $\alpha > 0$ and $H - \omega(H)$, $H \in \mathfrak{h}$

$$J(\omega) = \sum_{\alpha > 0} U(\mathfrak{g})\, \mathfrak{g}_\alpha + \sum_{H \in \mathfrak{h}} U(\mathfrak{g}) \, (\underset{\mathfrak{g}}{\underset{\cap}{H}} - \omega(H) \cdot \underset{\mathbb{C}}{\underset{\cap}{\mathbb{1}}})$$

<u>Def</u> Verma modul associated to weight ω

$$M(\omega) = U(\mathfrak{g}) \big/ J(\omega)$$

<u>Example</u> $\mathfrak{g} = \mathfrak{sl}_2 \mathbb{C} = \mathrm{lin}(\underset{-}{Y}, \underset{0}{H}, \underset{+}{X})$, $\omega(H) = a$

Any element of $U(\mathfrak{g})$ is written uniquely as $\sum a_{k l n} Y^k H^l X^n$

In the quotient $Y^k H^l X^n = 0$ for $n > 0$, $Y^k H^l = Y^k a^l$

the highest weight vector $- \mathbb{1}$, $H \cdot \mathbb{1} = a$.

<u>But</u> M_ω is not finite dimensional!

<u>Theorem</u> There exists a maximal proper submodul $K(\omega) \subset M(\omega)$ such that $M(\omega)/K(\omega) \simeq V(\omega)$

9

Proof if $N \subset M(\omega)$, N is the sum of weight spaces for h. If $\mathbb{1} \in N$ then $N = M(\omega)$. So $K = \sum_{\substack{N \text{ proper} \\ \text{submodule} \\ \text{of } M(\omega)}} N \not\ni \mathbb{1}$.

We have a map $M(\omega) \twoheadrightarrow V(\omega)$.
The kernel is a proper submodule.
It has to be maximal (otherwise $V(\omega)$ would contain a proper submodule). ☺

In our example for $\boxed{a \in \mathbb{N}}$!

$$K = \langle Y^k \mathbb{1} \, , \, k > a \rangle$$

Note $X Y^k \mathbb{1} = k(a-k+1) Y^{k-1} \mathbb{1}$
so $X Y^{1+a} \mathbb{1} = 0$.

Remark $M(\omega)$ can be obtained in the following way:

let $b = h \oplus \bigoplus_{\alpha > 0} g_\alpha$ Lie algebra of the Borel group

$b \twoheadrightarrow h$ Lie algebra map

$\omega \rightsquigarrow U(h)$ acts on $\mathbb{C}$

$\rightsquigarrow U(b)$ acts on $\mathbb{C}$ via the map $U(b) \to U(h)$

$$\rightsquigarrow M(\omega) = U(g) \otimes_{U(b)} \mathbb{C} \, .$$

Note $M(\omega)/K$ also defined for nonintegral ω, but then it is infinite dimensional.