Wykład 8

1) <u>$SL_n(\mathbb{C})$ and its representations</u> (often $n=3$)

From the previous lecture:

- we fixed a maximal torus $T \subset SU(n)$
- we fixed an orientation in $it = \mathrm{lin}_{\mathbb{R}} H$
- we decomposed $sl_2(\mathbb{C})$ into 1-dimensional representations of H $\quad sl_2(\mathbb{C}) = \langle \underset{-2}{Y} \rangle \oplus \langle \underset{0}{H} \rangle \oplus \langle \underset{2}{X} \rangle$
- any representation of $SL_2(\mathbb{C})$ is decomposed into weight (or eigen-) spaces of H
$$V = V_{-n} \oplus \quad \cdots \quad \cdots \oplus V_n$$
- irreducible representations were characterized by the highest weight vector $v \in V_n$, $\dim V_n = 1$, $\quad V = sl_2(\mathbb{C}) \cdot V_n$

We repeat this construction for $SL_n(\mathbb{C})$ (with pictures for $n=3$.)

The maximal torus in $SU(n)$ – diagonal matrices

The corresponding subalgebra of $su(n)_{\mathbb{C}} = sl_n(\mathbb{C})$ also diagonal matrices called <u>Cartan algebra</u> and denoted by $\mathfrak{h} = t \otimes \mathbb{C}$

$$\mathfrak{h} = \{ \sum x_i E_{ii} \; : \; \sum x_i = 0 \}$$

where E_{ij} the elementary matrix $\begin{array}{c|ccc} & & j & \\ \hline i & 0 & 1 & 0 \\ & \bullet & & 0 \\ & 0 & 0 & 0 \end{array}$

$\mathfrak{h}$ acts on $sl_n(\mathbb{C}) = \mathrm{lin}(E_{kl} : k \neq l) \oplus \underset{\uparrow \text{trivially}}{\mathfrak{h}}$

$$\mathrm{ad}_{E_{ii}} E_{kl} = [E_{ii}, E_{kl}] = \begin{cases} E_{kl} & \text{if } k = i \\ -E_{kl} & \text{if } l = i \\ 0 & k \neq i \neq l \end{cases}$$

(this is the action of the Cartan alg. $\tilde{\mathfrak{h}} \subset gl_n(\mathbb{C})$)

2) Therefore the weight of the action of h on $\mathrm{lin}\, E_{kl}$ is equal to

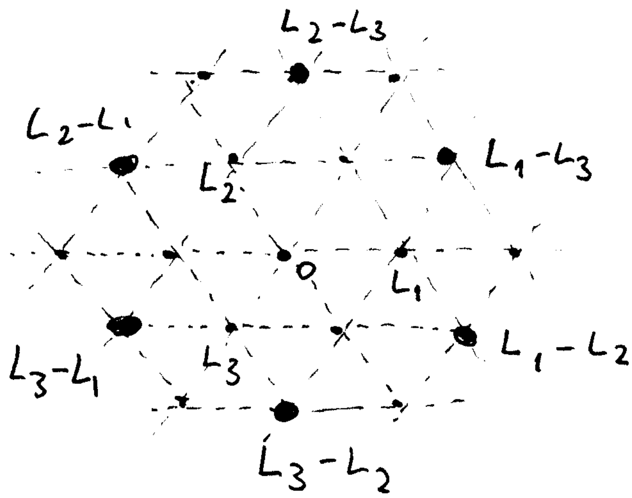
$$E_{kk}^* - E_{ll}^* \in \mathrm{Hom}(\tilde{h}, \mathbb{R}) \quad \text{weight vector } E_{kl}$$

We restrict this weight to $h \subset \tilde{h}$.
Note that this weight is integral:

$$\Lambda = \ker(\exp : \tilde{t} \to \tilde{T}) = \bigoplus_{i=1}^{n} \mathbb{Z} \cdot 2\pi i E_{ii} \xrightarrow{\text{weight}} 2\pi i \mathbb{Z}$$

Let $L_i = E_{ii}^*$

Picture of h^* for $sl_3(\mathbb{C})$ (its real part)



$$h^* = \mathrm{lin}(L_1, L_2, L_3) \big/ L_1 + L_2 + L_3$$

The weights of the representation ad are called <u>roots</u> of the Lie group

We have decomposed $sl_n(\mathbb{C})$ into weight spaces:

$$\mathfrak{g} = sl_n(\mathbb{C}) = \bigoplus_{\substack{\text{roots} \\ \alpha}} \mathfrak{g}_\alpha \oplus h$$

Where $\mathfrak{g}_\alpha = \mathrm{lin}(E_{k,l})$ for $\alpha = L_k - L_l$

③ Consider the scalar product in $\tilde{h}$ for which E_{ii} is the ortonormal basis. (Check that this product restricted to h is proportional to the Killing form). Then $\tilde{h} \simeq \tilde{h}^*$ and $h^* = (\sum E_{ii})^\perp$ has the induced scalar product (the identification commes from the scalar product)

<u>Claim</u> The group generated by S_α - the <u>reflections</u> in the hyperspaces $H_\alpha = \alpha^\perp$ for α - a root is isomorphic to the permutation group Σ_n = Weyl group $= NT/T$. Moreover, the action of the reflections on h^* may be identified with the action of the Weyl group.

<u>Proof</u> The Weyl group NT/T is generated by the transpositions, which correspond to ~~the~~ conjugations with

$$\sigma_{ij} = \begin{array}{c} \begin{array}{cc} j & i \end{array} \\ \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & 1 & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & -1 & & & 0 & \\ & & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix} \end{array}$$

Check for a diagonal matrix of dim 2

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

σ_{ij} exchanges $L_i \leftrightarrow L_j$

This is the same as the reflection in $(L_i - L_j)^\perp$

(4) The properties of the set of roots R - called <u>root system</u>

1) R is a finite set $\mathrm{lin}(R) = h^*$
2) if $\alpha \in R \Rightarrow -\alpha \in R$, but $k\alpha \notin R$ for any other k.
3) For any $\alpha \in R$ the reflection in $\alpha^\perp$ preserves R
4) For any α, β $n_{\beta\alpha} = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

<u>Property 3</u> can be checked directly, also we can argue the reflection is the same as the action of the Weyl group so it preserves the structure of h^* + roots.

<u>Property 4</u> Reflection in $\alpha^\perp$ $\beta \xmapsto{S_\alpha} \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$

4) Can be reformulated: $S_\alpha(\beta) - \beta$ is an integral multiplicity of α. In our case:

- $S_\alpha(\beta) = \beta$ if $\alpha = L_i - L_j$, $\beta = L_k - L_l$ with $\{i,j\} \cap \{k,l\} = \emptyset$ $n_{\beta\alpha} = 0$
- or $S_\alpha(\beta) = \beta \pm \alpha$ $\{k,l\} \cap \{i,j\} \neq \emptyset$, $\alpha \neq \beta$ $n_{\beta\alpha} = \pm 1$

$$n_{\beta\alpha} = \begin{cases} 1 & \text{if } k = i \text{ or } l = j \\ -1 & \text{if } k = j \text{ or } l = i \end{cases}$$

- $n_{\alpha\alpha} = 2$

This means that the angle $\measuredangle(\alpha, \beta) = 60^\circ$ or 120°.
The set of vectors in an arbitrary vector space is called a root system if it satisfies 1) - 4).

Importance of the root systems

(5) **Theorem** If G is a simple Lie group then the set of roots $R \subset \mathfrak{h}^*$ satisfies 1) - 4) i.e. it is a system of roots.

- It is possible to classify the systems of roots.
- If two simple Lie groups have isomorphic systems of roots, then their Lie algebras are isomorphic. Therefore their universal covers are isomorphic.

$$\begin{tikzcd} \tilde{G}_1 \arrow[r, "\sim"] \arrow[d] & \tilde{G}_2 \arrow[d] \\ G_1 & G_2 \end{tikzcd}$$

finite covers, $\deg = |\pi_1 G|$

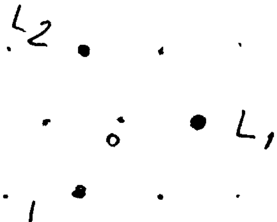
Note: For simple group $\pi_1 G$ is finite.

We will proove the above statements later, now we come back to $SL_n(\mathbb{C})$ and its representations

Let V be any representation of $Sl_n(\mathbb{C})$ we decompose V into weight spaces for $\mathfrak{h}$

Example $V = \mathbb{C}^3$ (natural representation of $Sl_3(\mathbb{C})$)

$$V = V_{L_1} \oplus V_{L_2} \oplus V_{L_3}$$



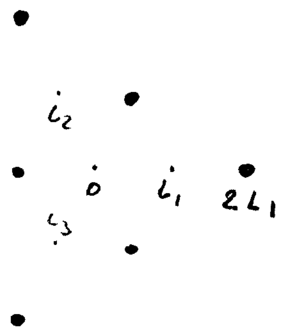
$$V_{L_i} = \mathrm{lin}\{e_i\}$$

for $h \in \mathfrak{h}$ $\quad h \cdot e_i = L_i(h) \cdot e_i = h_i \cdot e_1$

$h = (h_1, h_2, h_3)$

(6) $\mathrm{Sym}^2 \mathbb{C}^3$

	e_1^2	e_2^2	e_3^2	$e_1 \cdot e_2$	$e_1 \cdot e_3$	$e_2 \cdot e_3$
weight	$2L_1$	$2L_2$	$2L_3$	L_1+L_2	L_1+L_3	L_2+L_3



(the same for $\mathbb{C}^n$)

<u>Claim</u> If $v \in V_\omega$ for $\omega \in \Lambda^* \subset h^*$
and $X \in g_\alpha$ for α root
<u>then</u> $Xv \in V_{\omega+\alpha}$

<u>Proof</u> for $h \in \mathfrak{h}$

$$h(Xv) = Xhv + [h,X]v = X\underbrace{\omega(h)}_{\text{number}}v + \underbrace{\alpha(h)}_{\text{number}}Xv =$$

$$= \underbrace{(\omega+\alpha)(h)}_{\text{number}}\, Xv \qquad \square$$

<u>Observation:</u> $\mathrm{Sym}^2 \mathbb{C}^n$ is irreducible.

$E_{ij} : e_j \mapsto e_i$

$e_j^2 \mapsto 2e_ie_j \mapsto 2e_i^2$ any eigen-vector generates $\mathrm{Sym}^2\mathbb{C}^n$.

<u>Exterior power</u> $\Lambda^2 V$ $\qquad V \otimes V/\sim \qquad v\otimes w \sim -w \otimes v$

<u>Note</u> $\Lambda^2 \mathbb{C}^3 \simeq (\mathbb{C}^3)^*$ $\qquad v \wedge w$

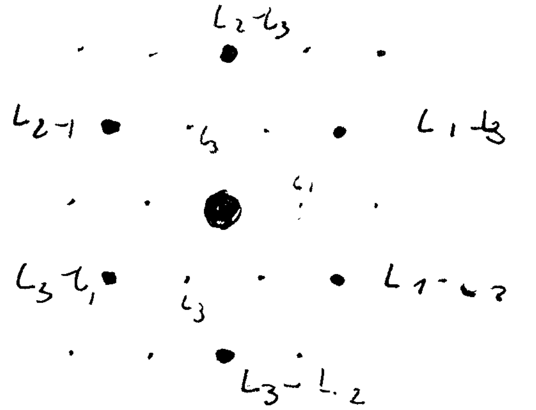
$\Lambda^2 \mathbb{C}^3$	$e_1 \wedge e_2$	$e_2 \wedge e_3$	$e_1 \wedge e_3$
	L_1+L_2	L_2+L_3	L_1+L_3

$\Lambda^2 \mathbb{C}^3$ is isomorphic as a subrepresentation of $\mathrm{Sym}^2 \mathbb{C}^3$ for the action of $\mathfrak{h}$, but not as a representation of $\mathfrak{sl}_3(\mathbb{C})$. Again it is irreducible.

7

$(\mathbb{C}^3)^* \otimes \mathbb{C}^3 = \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3)$ $\quad e_i^* \otimes e_j$

weight $L_j - L_i$



Picture in $\mathfrak{h}^*$

0 – is triple

$e_i^* - e_i \quad i = 1, 2, 3$

$\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3) = \{\mathrm{Tr}\, A = 0\} \oplus \mathbb{C} \cdot \mathrm{Id}$

splitting of the representation.

$$\{\mathrm{Tr}\, A = 0\}_0 = \mathrm{lin} \left\{ \begin{array}{ll} E_{11} - E_{22} & = e_1^* \otimes e_1 - e_2^* \otimes e_2 \\ E_{11} - E_{33} & E_{22} - E_{33} \end{array} \right\}$$

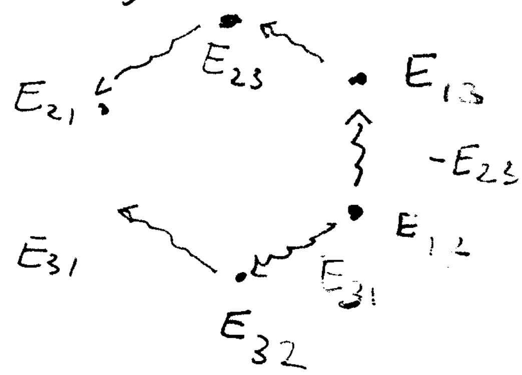
<u>Lemma</u> $\{\mathrm{Tr}\, A = 0\}$ is irreducible

Assume that $V \subset \{\mathrm{Tr}\, A = 0\}$.

V is a sum of the weight spaces

If $a E_{11} + b E_{22} + c E_{33} \in V \quad a \neq 0 \qquad a + b + c = 0, \; a \neq b$

$E_{21}(\;-\;) = (a - b) E_{21}$



$[E_{ij}, E_{jk}] = E_{ik}$

$i \neq j \neq k \neq i$

$[-E_{jk}, E_{ij}] = E_{ik}$

From any vertex of the hexagon we get the remaining vertices.

$[E_{ij}, E_{ji}] = E_{ii} - E_{jj} \qquad V_0 = \{\mathrm{Tr}\, A = 0\}$

(8)

<u>Corollary</u> $\{\operatorname{Tr} A = 0\}$ is irreducible representation of $sl_n(\mathbb{C})$.

<u>Corrolary</u> $\{\operatorname{Tr} A = 0\} = sl_n(\mathbb{C})$.

Subrepresentation = ideal in $sl_n(\mathbb{C})$

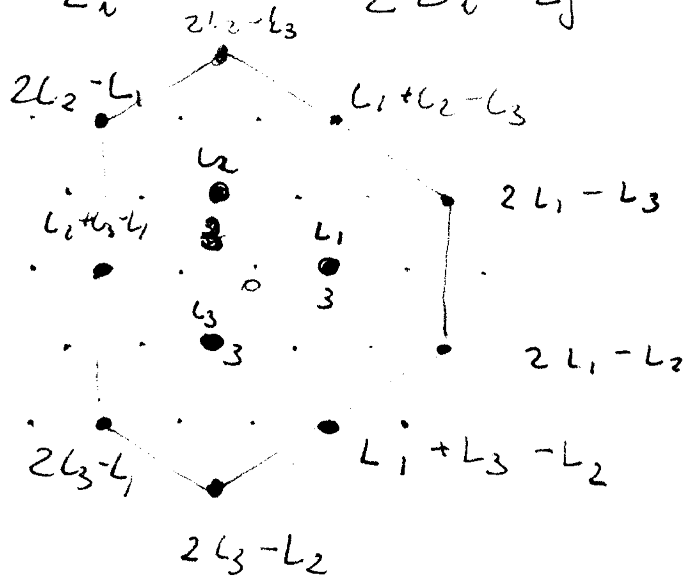
So $sl_n(\mathbb{C})$ is a simple algebra.

<u>Note</u> 0 is weight with $\dim V_0 > 1$.

This did not happen for $sl_2(\mathbb{C})$

$\mathrm{Sym}^2 \mathbb{C}^3 \otimes (\mathbb{C}^3)^*$

$e_i^2 \otimes e_i^*$	$e_i^2 \otimes e_j^*$	$e_i e_j \otimes e_i^*$	$e_i e_j \otimes e_k^*$
L_i	$2L_i - L_j$	L_j	$L_i + L_j - L_k$



But we have a map

$$\begin{tikzcd}
\mathrm{Sym}^2 \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \arrow[rr] \arrow[dr] & & \mathbb{C}^3 \\
& \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \arrow[ur] &
\end{tikzcd}$$

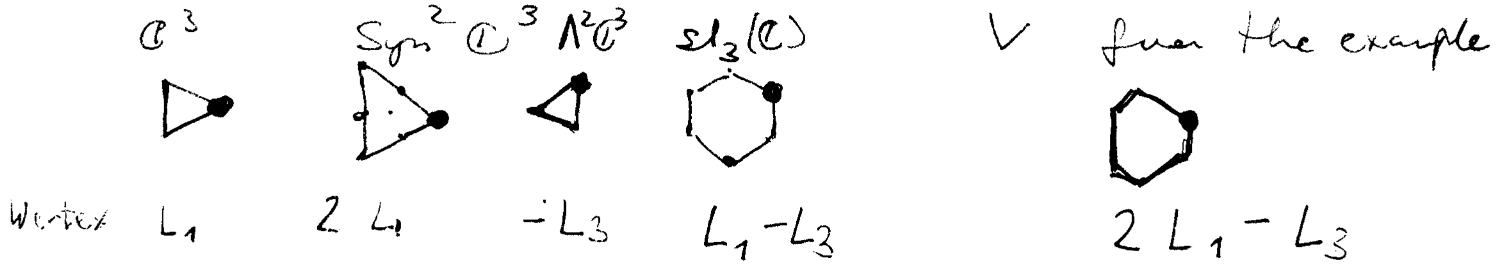
$$a \cdot b \otimes \ell \mapsto \tfrac{1}{2}\big(\ell(a) b + \ell(b) a\big)$$

Equivariant non zero map $\Longrightarrow$ surjective. We split one summand.

J) Exercise Show that

$$Sym^2 \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \simeq \mathbb{C}^3 \oplus \text{irreducible}$$

At most for $n=3$ we see that irreducible representation correspond to integral polygone generated by one extremal vertex, invariant with respect to 3 symmetries



To pick the extremal vertex in a unique way (for arbitrary n) we first have to make some choices.

Let us say which roots are positive:

We fix a generic hyperplane in h^* it divides h^* into positive and negative region.

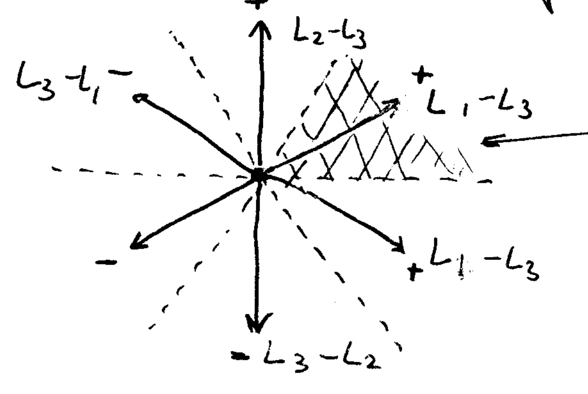
if α is positive root then $-\alpha$ is negative.

if α and β are positive then $\alpha+\beta$ is positive.

The standard choice for $sl_n(\mathbb{C})$:

$L_i - L_j$ is positive if $i<j$.

We consider only weights with the property $w \in h^*$: $\langle w, \alpha\rangle \geq 0$ for any root α.



Weyl chamber associated to the choice of positive roots

In fact for checking the inequality $\langle \omega, \alpha \rangle \geq 0$ we can take a smaller subset of roots: $\Delta \subset R_+$

Δ are called simple roots, they correspond to walls H_α of the Weyl chamber associated to positive roots
(In general Weyl chamber =
= closure of a component of $(h^* - \bigcup_{\alpha\text{-root}} H_\alpha)$)

With the standard choice of the positive roots for $sl_n(\mathbb{C})$ the simple roots are of the form $L_i - L_{i+1}$ for $i = 1, 2 \ldots n-1$.

The set of simple roots has the property:
- Δ is linearly independent
- any positive root is a sum of simple roots with nonnegative integer coefficients.

$$\beta = \sum_{\alpha \in \Delta} a_\alpha \alpha \qquad a_\alpha \in \mathbb{N}_{\geq 0}$$

With our choice the weights belonging to the Weyl chamber associated to the set of positive roots W

$$\omega = \sum_{i=1}^{n-1} a_i L_i \in W \iff a_1 \geq a_2 \geq a_3 \cdots \geq a_{n-1} \geq 0$$

Since it holds $\langle \sum a_i L_i, L_k - L_{k+1} \rangle = a_k - a_{k+1} \geq 0$

(11) We will show the following (all it is already clear):
Suppose that V is irreducible
1) any two weighs (occuring in V) differ by a sum of roots

2) The set of weights is symmetric with respect to H_α for α - root.

3) there exists an eigenvector $v \in V_\omega$ which is killed (i.e. $Xv=0$) for $X \in \mathfrak{g}_\alpha$ if α is positive a roots.
Such a weight ω has to be a vertex of conv(weights of V) and ω belongs to the Weyl chamber $\mathcal{W}$.
ω is unique, v is unique up to a scaler
$\boxed{\omega \text{ is called highest weight, } v \text{ - highest weight vector}}$

4) V is generated by the highest weight vector and the elemets of $\mathfrak{g}_\alpha$, for α - negative root.

5) For any weight $\nu \in \mathcal{W}$ there exists V_α with the highest weight ω.

6). If two representations have equal highest weights then they are isomorphic.