(1) $\boxed{\text{Lecture 7}}$

Representation of S^1

Irreducible representation (complex) $\iff \mathrm{Hom}(S^1, \mathbb{C}^*) = \mathrm{Hom}(S^1, S^1) = \mathbb{Z}$

are of the form $z \mapsto z^k$ $k \in \mathbb{Z}$

(all extend to the holomorphic representations of $\mathbb{C}^*$)

The distinguished (natural) representation

$\rho : z \mapsto z$

Any other representation is of the form

$$V = \bigoplus_{i=1}^{n} \rho^{k_i}$$

where $\rho^k = \begin{cases} \overbrace{\rho \otimes \rho \otimes \rho}^{k} & \text{if } k > 0 \\ \text{trivial action on } \mathbb{C} & \text{if } k = 0 \\ \underbrace{\rho^* \otimes \rho^* \otimes \ldots \otimes \rho^*}_{-k} & \text{if } k < 0 \end{cases}$

Note: $\rho^k \otimes \rho^{-k} = \mathbb{1}$ - trivial.

We also write $\sum a_i \rho^i$ where $a_i = \#$ of ρ^i in (any) decomposition of V into irreducible representations.

For the moment we will not discuss the theory of characters, but just in the case of S^1

The character of the representation

$\chi_V : S^1 \to \mathbb{C}$

$g \to \mathrm{tr}(g : V \to V)$

The character of $V = \sum a_i \rho^i$ is equal to $\sum a_i z^i$. It determines V. Moreover $\rho^i \perp \rho^j$ for $i \neq j$.
(Fourier analysis)

(2) Representations of Tori

If T is presented as $T = \overbrace{S^1 \times \dots \times S^1}^{n}$ then irreducible representations are identified with $\mathbb{Z}^n$

$$Hom(T, \mathbb{C}^*) = \prod_{i=1}^{n} Hom(S^1, \mathbb{C}^*) = \mathbb{Z}^n.$$

But there might be different presentations of T as the products of S^1's. Below I'll give a way to describe irreducible representations not using decompositions. Let Λ be the kernel of $\exp: t \to T$

$$T \simeq t/\Lambda.$$

Suppose we are given an irreducible representation, i.e. a map $T \xrightarrow{\phi} \mathbb{C}^*$ (the image $\phi(T) \subset S^1$).

Consider the map of Lie algebras

$$D\phi: t \longrightarrow i\mathbb{R}$$

The latice Λ is mapped into $2\pi i\mathbb{Z}$. If we normalise (i.e. remove $2\pi i$) $\omega := \frac{D\phi}{2\pi i}$ we obtain a map

$$\omega: t \to \mathbb{R} \qquad \omega(\Lambda) \subset \mathbb{Z}.$$

Since $t = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$

ω is determined by the values on Λ.

The element $\omega \in Hom(\Lambda, \mathbb{Z}) = \Lambda^*$ is called the <u>weight</u> of the representation V.

(3)

Study of SU(2)

The most important representation is Ad.

$su(2) = \{ A \in M_{2\times 2}(\mathbb{C}) : \overline{A}^T + A = 0, \ \mathrm{tr}\, A = 0 \}$

(since $\det \exp(A) = \exp(\mathrm{tr}\, A)$)

Fix a maximal torus in $SU(2)$

$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : |a| = 1 \right\}$

First we study the action of T on $su(2)$.

Fix the basis of $su(2)$

$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

Check

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mathbf{i} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = \mathbf{i}$$

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mathbf{j} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & a^2 \\ -a^{-2} & 0 \end{pmatrix} = \cos 2\varphi\, \mathbf{j} + \sin 2\varphi\, \mathbf{k}$$

where $a = \cos\varphi + i\sin\varphi$

This is a real representation and it decomposes into $\mathbb{1} \oplus \mathbb{R}^2$

$\mathbb{1}$: trivial 1-dimensional

$\mathbb{R}^2$: rotations.

Let us complexify $su(2) \otimes \mathbb{C} = sl_2(\mathbb{C})$

Better basis

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(4)

<u>Check</u>

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} H \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = H$$

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} X \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = a^2 X$$

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} Y \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = a^{-2} Y$$

<u>Therefore</u> H, X, Y is the basis of $sl_2(\mathbb{C})$ consisting of eigenvectors.

$sl_2(\mathbb{C}) = 1 \oplus \rho^2 \oplus \rho^{-2}$, i.e

the weights are $0, 2, -2$.

It is more convenient to look at the representation of the Lie algebra:

$t \otimes \mathbb{C} = \mathrm{lin}\langle H \rangle$ on $sl_2(\mathbb{C})$

$ad_H H = [H, H] = 0$

$ad_H X = [H, X] = 2X$

$ad_H Y = [H, Y] = -2Y$.

Let V be <u>any complex representation</u> of $SU(2)$, or equivalently a holomorphic representation of $SL_2(\mathbb{C})$.

Then V decomposes into eigenspaces for the action of S^1 (or $\mathbb{C}^*$)

$V = \bigoplus_{n=-\infty}^{+\infty} V_n$, where almost all $V_n = 0$, $n \in \mathbb{Z}$

$Hv = n \cdot v$ for $v \in V$.

(or equivalently $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v = a^n v$)

<u>Claim:</u> $Xv \in V_{n+2}$ and $Yv \in V_{n-2}$ for $v \in V_n$.

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Proof

$$HXv = [H,X]v + XHv = 2Xv + Xnv = (n+2)Xv \quad \square$$

Example: Natural representation $V = \mathbb{C}^2$

$$V = V_{-1} \oplus V_1 \qquad X\varepsilon_2 = \varepsilon_1, \quad Y\varepsilon_1 = \varepsilon_2$$

$$\varepsilon_2 \qquad \varepsilon_1$$

$$\mathrm{Sym}^2(V) = V \otimes V / \mathbb{Z}_2$$

$v \otimes w \sim w \otimes v$ we write $v \cdot w$

basis	$\varepsilon_2 \cdot \varepsilon_2 = \varepsilon_2^2$	$\varepsilon_1 \cdot \varepsilon_2$	$\varepsilon_1 \cdot \varepsilon_1 = \varepsilon_1^2$
weights	-2	0	2

$$H(v \cdot w) = Hv \cdot w + v \cdot Hw$$

$$H(\varepsilon_2 \cdot \varepsilon_2) = H\varepsilon_2 \cdot \varepsilon_2 + \varepsilon_2 \cdot H\varepsilon_2 = -2\varepsilon_1 \varepsilon_1$$

$$H(\varepsilon_1 \varepsilon_2) = H\varepsilon_1 \cdot \varepsilon_2 + \varepsilon_1 \cdot H\varepsilon_2 = \varepsilon_1 \cdot \varepsilon_2 - \varepsilon_1 \varepsilon_2 = 0$$

$$H(\varepsilon_1 \varepsilon_1) = \ldots 2\varepsilon_1 \varepsilon_1$$

The same weights appear in the adjoint representation

Y	H	X
-2	0	2

How X and Y act? $\quad \mathrm{ad}_X Y = [X,Y] = H$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathrm{ad}_X H \quad [X, H] = -[H, X] = -2X$$

$$\mathrm{ad}_Y X \quad [Y, X] = -H \qquad \mathrm{ad}_Y H = [Y, H] = -[HY] = 2Y$$

6) A way of constucting a good basis of any $sl_2(\mathbb{C})$ representation:

1. Find a vector of a highest weight v
2. Take images $Y^i v$

For the adjoint representation $v = X$

$$\begin{tikzcd}
-2Y \arrow[r, dotted, bend right] & -H \arrow[l, "Y"'] \arrow[r, dotted, bend right] & X \arrow[l, "Y"'] \\[-1em]
\rotatebox{90}{=} & \rotatebox{90}{=} & \rotatebox{90}{=} \\[-1em]
Y^2 v & Yv & v
\end{tikzcd}$$

Let's see how X acts:

$$X(Y^2 v) = -2[X, Y] = -2H = 2\, Yv$$
$$X(Yv) = -[X, H] = [H, X] = 2X = 2\, v$$

The same way for $Sym^2 V$

$$\begin{tikzcd}
2\varepsilon_2^2 \arrow[r, dotted, bend right, "X"'] & 2\varepsilon_1\varepsilon_2 \arrow[l, "Y"'] \arrow[r, dotted, bend right, "X"'] & \varepsilon_1^2 \arrow[l, "Y"'] \\[-1em]
\rotatebox{90}{=} & \rotatebox{90}{=} & \rotatebox{90}{=} \\[-1em]
Y^2 v & Yv & v
\end{tikzcd}$$

$$X(Y^2 v) = 2\, X(\varepsilon_2^2) = 4\varepsilon_1\varepsilon_2 = 2\, Yv$$
$$X(Yv) = X(2\varepsilon_1\varepsilon_2) = 2\varepsilon_1^2 = 2\, v.$$

<u>Corollary</u>: The adjoint representation is isomorphic to $Sym^2 \mathbb{C}^2$; the map
$$X \mapsto \varepsilon_1^2$$
$$-H \mapsto 2\varepsilon_1\varepsilon_2$$
$$-2Y \mapsto 2\varepsilon_2^2$$

The numbers 2 ~~can~~ be computed not knowing anything about representation:

7)

Lemma Let v be a highest weight vector (of the weight n)

then
$$X(Y^i v) = i(n-i+1) Y^{i-1} v$$

Proof $i=0$ $Xv=0$ (since Xv is of the weight $n+2$)

suppose $X(Y^{i-1}v) = (i-1)(n-i+2) Y^{i-2} v$.

? $XY^i v = YXY^{i-1}v + [XY]Y^{i-1}v =$

$= Y(i-1)(n-i+2)Y^{i-2}v + HY^{i-1}v =$

$= (i-1)(n-i+2)Y^{i-1}v + (n-2(i-1))Y^{i-1}v$

$= \left[\left(i(n-i+1)+i-\cancel{n}+i-2\right)+\left(n-\cancel{2i}+2\right)\right] Y^{i-1} v$ $\boxed{3}$

As always assume that $\dim V < \infty$.

Theorem Suppose V is an irreducible representation of $sl_2(\mathbb{C})$

1. The action of H is semisimple, all the weights of H are integers
2. If n is the highest weight then $n \geq 0$

$V = V_{-n} \oplus V_{-n+2} \oplus \dots \oplus V_{n-2} \oplus V_n$

($n+1$ summands)

V_{n-2i} is spanned by $Y^i v$, where v is the highest weight vector.

Proof 1) $SL_2(\mathbb{C})$ is simply connected. The map $sl_2\mathbb{C} \to gl(V)$ is induced by a map $SL_2(\mathbb{C}) \to GL_n(V)$, i.e. a representation. It is fully reducible.

2) If $n<0$ then $i(n-i+1) \neq 0$ for all $i>0$

$Y^{i-1}v \neq 0 \Rightarrow i(n-i+1)Y^{i-1}v \neq 0 \Rightarrow XY^i v \neq 0 \Rightarrow Y^i v \neq 0.$

contradicts finite dimensionality

8) Similarly if $n>0$ $\quad 0<i<n$ then $(n-i+1)i \neq 0$

$$Y^{i-1}v \neq 0 \implies Y^i v \neq 0.$$

The lowest weight nonzero vector is $Y^n v$ of the weight $-n$. If $Y^{n+k}v \neq 0$ (k biggest) then we interchange the role of X and Y and we show that $\underbrace{X^{n+2k}(\underbrace{Y^{n+k}v}_{\text{weight } -n-2k})}_{\text{weight } n+2k} \neq 0$,

this shows that $k=0$. $\square$

<u>Remark</u> For any representation of $sl_2(\mathbb{C})$ the maps $X^k : V_{-k} \to V_k$

and $Y^k : V_k \to V_{-k}$

are isomorphisms.

<u>Proof</u> We check for irreducible representations.

Now we know how the irreducible representations look like. We conclude:

<u>Theorem</u> Every irreducible representation is characterized by its highest weight $n \in \mathbb{Z}_{\geq 0}$

If $V(n)$ is a representation with the highest weight n then

$$V_{(n)} \simeq \mathrm{Sym}^n \mathbb{C}^2$$

and $\dim V(n) = n+1$.

)

Proof

The basis of $V_{(n)}$ consists of

$v, Yv, Y^2v, \dots, Y^n v$.

H acts diagonally - $Y^i v$ is of the weight $n-2i$

$X(Y^i v) = i(n-i+1) Y^{i-1} v$.

Symmetric powers: $Sym^n(\mathbb{C}^2) = \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2 / \Sigma_n$

basis $\varepsilon_1^k \cdot \varepsilon_2^{n-k}$ $k = 0, 1, \dots, n$

$$H(\varepsilon_1^k \cdot \varepsilon_2^{n-k}) = (H\varepsilon_1^k)\varepsilon_2^{n-k} + \varepsilon_1^k(H\varepsilon_2^{n-k})$$

$$= +k\varepsilon_1^k \varepsilon_2^{n-k} - \varepsilon_1^k (n-k)\varepsilon_2^{n-k} =$$

$$= (-n+2k)\,\varepsilon_1^k \varepsilon_2^{n-k}$$

Weights $n, n-2, \dots -n+2, -n$.

highest weight vector ε_1^n.

$$Y\varepsilon_1^k \varepsilon_2^{n-k} = k(Y\varepsilon_1)\varepsilon_1^{k-1}\varepsilon_2^{n-k} = k\,\varepsilon_1^{k-1}\varepsilon_2^{n-k+1}$$

The chain

$$\underset{-n}{n!\,\varepsilon_2^n} \leftarrow \dots \leftarrow n(n-1)\varepsilon_1^{n-2}\varepsilon_2^2 \leftarrow \underset{n-2}{n\,\varepsilon_1^{n-1}\varepsilon_2} \leftarrow \underset{n}{\varepsilon_1^n}$$

The action of $SU(2)$:

The torus $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ acts by $\varepsilon_1^k \varepsilon_2^{n-k} \longmapsto \xi^{2k-n}\, \varepsilon_1^k \varepsilon_2^{n-k}$

(10)

Corollary

The action of $so(3) = su(2)$ comes from the action of $SO(3) \iff$
$\iff$ n is even

Proof $SO(3) = SU(2)/\{\pm I\}$.

We have to check when $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $V = \mathrm{Sym}^n(\mathbb{C}^2)$.

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in$ maximal torus;

$$\varepsilon_1^k \varepsilon_2^{n-k} \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{2k-n} \varepsilon_1^k \varepsilon_2^{n-k}$$

$2k-n$ is even $\iff$ n even ∎.

- The same for $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}$

Products

Problem: decompose the tensor product of irreducible representations into irreducible summands:

$$V_\lambda \otimes V_\mu = \sum c_{\lambda\mu}^{\nu} V_\nu$$

$c_{\lambda\mu}^{\nu}$ natural number

V_λ, V_μ, V_ν — irreducible representations

$\boxed{\text{Find } c_{\lambda\mu}^{\nu}.}$ For $Sl_2(\mathbb{C})$ it is easy.

11)

Example. Here we give a decomposition of $\mathbb{C}^2 \otimes \mathbb{C}^2$ into the direct sum of subspaces

What are the possible weight spaces?

-2	0	2
$\varepsilon_2 \otimes \varepsilon_2$	$\varepsilon_2 \otimes \varepsilon_1$	$\varepsilon_1 \otimes \varepsilon_1$
	$\varepsilon_1 \otimes \varepsilon_2$	

Answer $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq V_{(2)} \oplus V_{(0)}$

$V_{(1)} \otimes V_{(1)}$

$\mathrm{Sym}^2 \mathbb{C}^2 \quad \mathrm{Sym}^0 \mathbb{C}^2$

generated by the highest weight vector

In fact $V_{(2)} = \mathrm{lin}\langle \varepsilon_1 \otimes \varepsilon_1 , Y(\varepsilon_1 \otimes \varepsilon_1), Y^2(\varepsilon_1 \otimes \varepsilon_1) \rangle$

$$= \mathrm{lin}\langle \varepsilon_1 \otimes \varepsilon_1 , \varepsilon_2 \otimes \varepsilon_1 + \varepsilon_1 \otimes \varepsilon_2 , \varepsilon_2 \otimes \varepsilon_2 \rangle$$

$$V_{(0)} = \left(V_{(0)}\right)_0 = \mathrm{Ker}\,(X : V_0 \to V_2)$$

$$= \langle \varepsilon_1 \otimes \varepsilon_2 - \varepsilon_2 \otimes \varepsilon_1 \rangle$$

Exercise Decompose $V_{(3)} \otimes V_{(2)}$ into irreducible subrepresentations.

Plethysm Decompose $\mathrm{Sym}^k \mathrm{Sym}^\ell \mathbb{C}^2$ into irreducible representations.

(For $SL_2(\mathbb{C})$ should not be hard.)

12)

What representations appear in $V_{(m)} \otimes V_{(n)}$? $m \geq n$

e_i f_j eigenvectors

$e_i \otimes f_j$.

We count dimensions via the diagram. Eg for $m = 6$ $n = 3$. the table of weights $e_i \otimes f_j$

f_j \ e_i	-6	-4	-2	0	2	4	6
-3	-9	-7	-5	-3	-1	1	3
-1	-7	-5	-3	-1	1	3	5
1	-5	-3	-1	1	3	5	7
3	-3	-1	1	3	5	7	9

$$V_{(6)} \otimes V_{(3)} \simeq V_{(9)} \oplus V_{(7)} \oplus V_{(5)} \oplus V_{(3)}$$

In general
$$V_{(m)} \otimes V_{(n)} \simeq \bigoplus_{\substack{k \in (m-n,\, m+n) \\ k \equiv m+n \bmod 2}} V_{(k)}$$

Method of finding a copy of $V_{(k)}$ inside $V_{(m)} \otimes V_{(n)}$:

The highest weight space
$$\left(V_{(k)}\right)_k = \operatorname{Ker}\left(X : V_k \to V_{k+2}\right).$$

Unitary trick

What are the representations of $SL_2(\mathbb{R})$?

1. $SL_2(\mathbb{R})$ - representations are semisimple

Proof

$SL_2(\mathbb{R})$ repr. on V (real vector space)

$\rightsquigarrow$

$SL_2(\mathbb{C})$ repr. on $V \otimes \mathbb{C}$

$\rightsquigarrow$

$SU(2)$ repr. on $V \otimes \mathbb{C}$

construct invariant Hermitian product

$\rightsquigarrow$ scalar product in V

If U is $SL_2(\mathbb{R})$ invariant space in V

$U \otimes \mathbb{C}$ $\subset$ $V \otimes \mathbb{C}$

$U^\perp$ is $SL_2(\mathbb{R})$ invariant. $U^\perp = (V \otimes \mathbb{C})^\perp \cap V$

2. $sl_2(\mathbb{R}) = \operatorname{lin}(X, Y, H)$

V irreducible, In $V \otimes \mathbb{C}$ H is diagonalizable

v highest weight vector of weight n $\bar{v}$ also of weight n

Then $v + \bar{v}$ is real or $v = i$(real vector).

So we can assume that v is real.

$\operatorname{lin}\{Y^i v\}$ inv. subspace $\Rightarrow V = \operatorname{lin}\{Y^i v\}$

$\Rightarrow V \cong \operatorname{Sym}^n(\mathbb{R}^2)$.