(1) # Representation theory

G topological group

V vector space over $\mathbb{R}$ or $\mathbb{C}$

<u>Representation of G in V</u> is a continuous homomorphism of topological groups

$$\rho : G \longrightarrow Gl(V), \qquad \rho(g)(v) =: gv$$

If G is a Lie group then ρ is automatically smooth (graph of ρ is a closed subgroup of $G \times GL(V)$, so it is a Lie group $\simeq G$).

$U \subset V$ is an <u>invariant subspace</u>, if for all $g \in G$ and all $u \in U$

$$\rho(g)(u) \in U.$$

V, U invariant $\rightsquigarrow$ representation of G in U

$\rightsquigarrow$ representation of G in V/U

<u>Example</u> $G = \mathbb{C}_+$ $V = \mathbb{C}^2$ $\rho(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in Gl_2(\mathbb{C})$

$U = \text{lin}\,(1,0)$

representation of G in U is trivial $\rho|_U$

V/U — " — $\tilde{\rho}$

Instead of (V, ρ) we write V and gv instead of $\rho(g)v$

V is <u>irreducible</u> if (definition)

U is invariant $\Rightarrow$ $U = \{0\}$ or $U = V$.

<u>Examples</u> - Trivial representation

- every 1-dimensional representation
- $\mathbb{C}^n$ for $Gl_n(\mathbb{C})$, $Sl_n(\mathbb{C})$, $SO_n(\mathbb{C})$
- $\mathbb{C}^{2n}$ for $Sp_n(\mathbb{C})$, $Sp(n)$

A representation V is <u>completely reducible</u> if any invariant subspace U admits a complement which is invariant.
i.e $\exists\, U' \subset V$ st. $U \oplus U' = V$ (action on coordinates)

<u>Then</u> $U' \simeq V/U$ as representations.
This means that there exist a isomorphism ϕ of vector spaces commuting with action of G

$$\forall g \qquad \begin{array}{ccc} U' & \xrightarrow{\phi} & V/U \\ {\scriptstyle \rho'(g)}\downarrow & & \downarrow {\scriptstyle \tilde\rho(g)} \\ U' & \xrightarrow{\phi} & V/U \end{array}$$

Here the isomorphism ϕ is given by $U' \hookrightarrow V \twoheadrightarrow V/U$.

<u>Theorem</u> If G is a compact group then any (finite dimensional) representation is completely reducible.
<u>Proof</u> We can choose an invariant scalar product $\langle\,,\,\rangle_G$ (in the previous lecture we took any and we averaged it using Haar measure). Then $U' := U^\perp$ is an invariant subspace

$$v \in U^\perp \iff \forall_{u \in U}\, v \perp u \iff \forall_{u \in U}\, v \perp g^{-1}u \iff \forall_{u \in U}\, gv \perp u$$
$$\iff gv \in U^\perp$$
$\square$

<u>Prop</u> Any completely reducible representation (of finite dimension) is a direct sum of irreducible representations:

$$V = \bigoplus_{i=1}^{n} U_i$$

If U minimal invariant subspace U is irreducible,
$V = U \oplus U'$ and $U' \simeq V/U$ comp. reducible

(3) Conversely: Let $V = \bigoplus_{i=1}^{n} V_i$, V_i - irreducible. Then V is completely reducible.

Proof Let $U \subset V$. Suppose $i_1 \dots, i_k$ a maximal sequence such that $U \cap (V_{i_1} \oplus \dots \oplus V_{i_k}) = 0$. We want to show that $V = U \oplus V_{i_1} \oplus \dots \oplus V_{i_k}$.

Suppose $v \notin U \oplus V_{i_1} \oplus \dots \oplus V_{i_k}$. This means that some component $0 \neq v_j \in V_j$ $j \notin \{i_1 \dots i_k\}$.

Then $V_j \cap (U \oplus V_{i_1} \oplus \dots \oplus V_{i_k}) \neq V_j$.

Since V_j is irreducible

$$V_j \cap (U \oplus V_{i_1} \oplus \dots \oplus V_{i_k}) = \{0\}.$$

Contradiction with maximality of $i_1 \dots i_k$ ∎.

Basic operations on representations

Any "natural" operation on vector spaces leads to an operation on representations

- direct sum (product)
- $Hom(V, W) \ni \phi$

$$(g \cdot \phi)(v) := g(\phi(g^{-1}(v)))$$

Note the subspace of invariants $Hom(V, W)^G$ is the same as the space of G-invariant maps i.e.

$$g \cdot \phi = \phi \iff \forall_{v \in V, g \in G}\ g\,\phi(v) = \phi\, g(v)$$

$$\left(\text{pf. } \phi(v) = g \cdot \phi(v) = g\phi(g^{-1}v) \underset{v := gv}{\iff} \phi(gv) = g\phi(g^{-1}gv) = g\phi(v)\right).$$

(4) – dual representation (conjugate)

$V^* = \mathrm{Hom}(V, \mathbb{C})$ ($\mathbb{C}$ – trivial repr.)

$(g\cdot\phi)(v) = \phi(g^{-1}v)$

In coordinates $\rho^*(g) = (\rho(g)^{-1})^T$.

- tensor product $V \otimes W$
- bilinear forms $V^* \otimes V^*$

Does everybody know?

- symmetric forms $S^2(V^*)$
- antisymmetric forms $\Lambda^2(V^*)$

$$V^* \otimes V^* = S^2(V^*) \oplus \Lambda^2(V^*)$$

Exercise Show that this is a decomposition into irreducible representations of $Gl(V)$.

Example $G = \mathbb{C}^*$ $V = \mathbb{C}^2$, $t\cdot(x,y) = (t^5 x, t^{-2} y)$? $V \otimes V$

Field extension:

Given a real representation

$\rho : G \to Gl(V)$, where V is a real vector space $V \cong \mathbb{R}^n$

$\rho_{\mathbb{C}} : G \to Gl(V \otimes_{\mathbb{R}} \mathbb{C})$

(in a basis $\rho(g)$ is a matrix of real entries, the same matrix defines a complex map $V \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n$)

Conjugation $\overline{V}$, in coordinates $\bar\rho(g) = \overline{\rho(g)}$.

If the representation is unitary: $\rho : G \to U(n)$

then $\overline{\rho(g)} = (\rho(g)^{-1})^T$, therefore $\overline{V} = V^*$.

Note, that if ρ is holomorphic, then (for groups of dim >0) $\bar\rho$ is not.

5) Morphism of irreducible representations

Notation $\{\varphi : V \to W \mid \varphi \text{ is } G \text{ invariant}\}$

$= Hom_G(V, W)$.

Representations of a fixed group G form a category.

Proposition Let V_1 V_2 - irreducible representations, $\varphi \in Hom_G(V_1, V_2)$ then either $\varphi = 0$ or φ is an isomorphism.

Proof If $\varphi \neq 0$ then $\ker \varphi = 0$ and $\operatorname{im} \varphi = V_2$. $\square$

Shur Lemma: Let V be a complex irred. representation of G. $\varphi \in Hom_G(V, V)$. Then there exists $a \in \mathbb{C}$ s.t.

$\varphi(v) = a \cdot v$.

Proof Let a be an eigenvalue of φ. Then $K = \ker(\varphi - aI) \neq 0$.

This space is G-invariant

$(\varphi(v) = a \cdot v \Rightarrow \varphi(gv) = g\varphi(v) = a\, g(v).)$

Hence $K = V$, i.e. $\forall v \in V$ $\varphi(v) = a \cdot v$. $\square$

Corollary If V is a sum of irreducible representations then for V_α - irreducible

$\dim(Hom(V_\alpha, V))$

is equal to the number of copies of V_α in V.

Proof Suppose $V = \overbrace{V_\alpha \oplus \cdots \oplus V_\alpha}^{n} \oplus V_{\beta_1} \oplus V_{\beta_n}$.
$\mathrm{Hom}(V_\alpha, V) = \mathrm{Hom}(V_\alpha, V_\alpha)^n \oplus \mathrm{Hom}(V_\alpha, V_{\beta_1}) \cdots$
$= \mathbb{C}^n$.

Corollary The decomposition into irreducible representation is unique up to a permutation of summands

Corollary: There exists a natural isomorphism

$$\bigoplus \mathrm{Hom}(V_\alpha, V) \otimes V_\alpha \to V$$

where the sum is taken over all isomorphism classes of representations.
The natural map
defined on the simple tensors: $\varphi \otimes v \mapsto \varphi(v)$
$\varphi \in \mathrm{Hom}(V_\alpha, V)$, $v \in V_\alpha$.

Abelian groups

G abelian V irreducible complex representation. For a fixed $h \in G$ define a selfmap of the representation V
$\Phi_h : v \mapsto hv$.
This is a map of representations:

$$g\Phi_h(v) = \Phi_h(gv) \quad \text{because } gh = hg.$$

Therefore $\Phi_h = a_h I$. So $\dim V = 1$.

(7)

Corollary Irreducible representations of an abelian groups are in bijection with $\mathrm{Hom}(G, \mathbb{C}^*)$.
(because $\Phi_{h_1} \circ \Phi_{h_2} = \Phi_{h_1 h_2} \in \mathrm{Hom}(V,V)$

$$\Rightarrow \quad a_{h_1} \cdot a_{h_2} = a_{h_1 h_2} \in \mathbb{C}^*.\ )$$

Corollary If G is compact and abelian, then the action is diagonal in some basis.

Representation of Lie groups

Every representation $\rho: G \to GL(V)$ leads to a representation of a Lie algebra $\mathfrak{g} \to \mathfrak{gl}(V)$

(**Note** that for the tensor product of representations: $X \in \mathfrak{g}$

$$X(V \otimes W) = XV \otimes W + V \otimes XW$$

Proof let $X = \dot{g}_t$ (g_t – one parameter subgroup)

$$X(V \otimes W) = \frac{d}{dt}\left(g_t v \otimes g_t w\right)_{t=0} =$$

$$= \left(\dot{g}_t v \otimes g_t w + g_t v \otimes \dot{g}_t w\right)_{t=0} =$$

$$= Xv \otimes w + v \otimes Xw.\ )$$

Fixed points

Suppose G is connected, V a representation

Theorem $V^G = \{v \in V : \forall g \in G \ \ gv = v\}$ is equal to $V^{\mathfrak{g}} = \{v \in V : \forall X \in \mathfrak{g} \ Xv = 0\}$

Proof $X \in \mathfrak{g}$ defines a vector field on V $\vec{X}(v) = \frac{d}{dt} g_t v \big|_{t=0} = Xv$. The flow of $\vec{X}$ is given by g_t.

(8) v is a fixed point of the flow g_t if and only if $\vec{X}(v) = 0$.

- The fixed points of G of course are fixed by any $X \in \mathfrak{g}$
- <u>Conversely</u> if $Xv = 0$ for all $X \in \mathfrak{g}$ then v is fixed by all $g_t \in \operatorname{im}(\exp : \mathfrak{g} \to G)$. The image of exp generates G (since it contains an open neighbourhood of 1).

(Note that for G compact and connected – as in the case of $SO(n)$ or $U(n)$ im $\exp = G$).

Hence $V^G = V^{\mathfrak{g}}$ for G connected ☐

<u>Application</u> Let G be a complex reductive group. It is a complexification of a compact group K. Let V be a holomorphic representation of G, ie $G \to GL(V)$ is holomorphic (V is a complex space) V as a representation of K decomposes

$$V = V_1 \oplus \cdots \oplus V_n .$$

Let π_i be the projection of V on V_i.

9. $\pi_i \in \mathrm{Hom}_K(V, V)$ (i.e. is K invariant)

We will show that π_i is G invariant.

Proof. Let $W = \mathrm{Hom}(V, V)$.

$\pi_i \in W^{K \leftarrow \text{group}} = W^{k \leftarrow \text{Lie algebra}}$.

$\mathfrak{g} = k + ik$.

Assumption of holomorphicity means that

$$iXv = Xiv \quad \text{for } X \in \mathfrak{g},\ v \in V.$$

It follows that

$$\mathfrak{g} \to \mathfrak{gl}(W)$$

commutes with i.

Hence if $w \in W$ is k-invariant then it is $\mathfrak{g}$-invariant.

So $W^K = W^G$.

So π_i is G invariant.

Conclusion Holomorphic representations are completely reducible.

Note that non holomorphic representations do not have to be reducible

e.g. $\mathbb{C}^* = S^1 \times \mathbb{R}_+ \twoheadrightarrow \mathbb{R}_+ \to \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right\}$.