(1)

For $G = Gl_n(\mathbb{C}$ or $\mathbb{R})$ I will describe various theorems which are valid also for other connected Lie groups

(1.) Polar decomposition

$$G \underset{\text{diffeo}}{\simeq} K \times P$$

K - maximal compact $U(n)$ $(O(n))$

P = eigenspace for Cartan involution $\lambda = 1$ symmetric matrices

(2) KAN, Iwasawa decomposition (Gram - Schmidt process)

(3) Maximal compact subgroups

(4) Bruhat decomposition (Gauss Method)

(5) Maximal tori in compact groups (presentation of an isometry in a cannonical form)

<u>Ad 2</u> Given a basis of $\mathbb{R}^n$

$$\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$$

we define

$$\beta_1 = \frac{\alpha_1}{\|\alpha_1\|} = \alpha_{11}\alpha_1$$

$$\beta_2 = \frac{\beta_2 - \langle \beta_2 \cancel{\beta_1} \rangle \cdot \cancel{\beta_1}}{\| \quad \|} = \lambda_{12}\alpha_1 + \lambda_{22}\alpha_2$$

$$\beta_3 = \lambda_{13}\alpha_1 + \lambda_{23}\alpha_2 + \lambda_{33}\alpha_3$$

$\vdots$ get ortonormal basis

In the matrix notation:

X = matrix with α_i as columns
U = — " — with β_i as colums

U is orthonormal $U \in O(n)$ (or $U(n)$ in the complex case)

$$X \cdot \begin{bmatrix} \lambda_{ij} & & \\ & \ddots & \\ 0 & & \end{bmatrix} = U$$

let $$b = \begin{bmatrix} \lambda_{ij} & & \\ & \ddots & \\ 0 & & \end{bmatrix}^{-1} \in B = \text{group of upper triangular matrices.}$$

We have proved that any matrix $X \in Gl_n(\mathbb{R})$ can be presented as

$X = U \cdot b$

B is called Borel subgroup. For any connected G it is defined as a maximal solveble connected subgroup of G.

Solveble means that in the sequence $B_0 > B_1 > B_2 \dots$, where $B_{i+1} = [B_i, B_i]$ is equal $B_i = \{1\}$ for large i.

For upper triangular matrices:

$$B_i = \begin{bmatrix} 1 & \overbrace{\cdots\, 0}^{i} & * \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix}.$$

A finer decomposition we get decomposing

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$b = a \cdot m$, where

$$a = \begin{pmatrix} x_{11}^{-1} & & 0 \\ & \ddots & \\ 0 & & x_{nn}^{-1} \end{pmatrix} \qquad m = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ 0 & & 1 & \\ & & & 1 \end{pmatrix}.$$

Note that the diagonal entries x_{ii}^{-1} (are real) and > 0.

The presentation

$$X = u \cdot a \cdot m$$

where $u \in O(n)$ (or $U(n)$), $a \in \mathbb{R}_{>0}^n = A$ and m is strictly upper triangular "nilpotent group" N

We have proved that

$$Gl_n(\underset{\mathbb{C}}{\mathbb{R}}) = KAN \quad \text{where}$$

$K = O(n)$ or $U(n)$ is the maximal compact subgroup.

This is Iwasawa decomposition

it is valid for an arbitrary group with finite many components

k - maximal compact subgroup
$\mathfrak{a}$ - abelian $[x, y] = 0$
$\mathfrak{n}$ - nilpotent $\mathfrak{n}^0 = \mathfrak{n}$ $\mathfrak{n}^{i+1} = [\mathfrak{n}, \mathfrak{n}^i]$ $\mathfrak{n}^i = 0$ for $i \gg 0$

Moreover $\exp: \mathfrak{a} \to A$ and $\exp: \mathfrak{n} \to N$ are isomorphisms. Hence $G \underset{\text{diff}}{\cong} K \times \mathfrak{a} \times \mathfrak{n}$.

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Ad 3 Maximal compact subgroups

Let K be a compact subgroup of $Gl_n(\mathbb{C})$

There exists (allways for locally compact groups) an invariant measure (Haar measure) on K.

invariant means

$$\int_G f(g)\, d\mu(g) = \int_G f(gh)\, d\mu(g) = \int_G f(hg)\, d\mu(g)$$

For compact groups can assume that $\mu(K) = 1$.

Using this measure we average a scalar / unitary product in $\mathbb{R}^n$ (or $\mathbb{C}^n$)

$$\langle v, w \rangle_K = \int_K \langle gv, gw \rangle \, d\mu(g)$$

This new scalar product is K-invariant

$$\langle hv, hw \rangle_K = \int_K \langle ghv, ghw \rangle \, d\mu(g) = \int_K \langle gv, gw \rangle \, d\mu(g) = \langle v, w \rangle_K$$

Let $\alpha_1, \dots \alpha_n$ be an orthonormal basis of $\mathbb{R}^n / \mathbb{C}^n$ with respect $\langle\ \rangle_K$, and C be the matrix of the change of the basis $C = [\alpha_1 \dots \alpha_n]$.

Then CKC^{-1} preserves the standard scalar product. Hence $CKC^{-1} \subset O(n)$

(5) We have shown that each compact subgroup up to a conjugation is contained in the maximal compact subgroups $O(n)$ (or $U(n)$). It follows that any two maximal compact subgroups are conjugate.
This is true in general for any connected Lie group.

Let $\mathcal{K}$ be the space of all maximal compact subgroups.

<u>Theorem</u> $\mathcal{K} \cong \mathbb{R}^d$ for some d.
(This is also true for any connected Lie group.)

<u>Pf</u> $Gl_n(\mathbb{C})$ acts on $\mathcal{K}$ transitively (via conjugation).
Therefore $\mathcal{K} = Gl_n(\mathbb{C}) \cdot [U(n)]$ (orbit of $U(n)$),
so $\mathcal{K} = Gl_n(\mathbb{C}) / \text{normalizer of } U(n)$. What is $N(U(n))$?
The adjoint representation stabilizes the Lie algebra $u(n)$. Let $X \in u(n)$ (i.e $X^* = -X$). Then
$$(gXg^{-1})^* = -gXg^{-1} \text{ for } g \in N(U(n))$$
Hence $g^{*-1} X^* g^* = -g^{*-1} X g^* = -gXg^{-1}$.
Therefore $Ad\, g^*g(X) = X$. But $u(n) + i\,u(n) = gl_n(\mathbb{C})$.
so $Ad\, g^*g = I$. Therefore $g \in U(n) \cdot Z(Gl_n(\mathbb{C}))$

$$\mathcal{K} = \frac{Gl_n(\mathbb{C})}{U(n)\cdot Z(Gl_n\mathbb{C})} = \frac{A\cdot N}{\mathbb{R}^*_{\geq 0}} = \mathbb{R}^d$$

$d = (n-1) + n(n-1) =$
$= n^2-1$

6 Reduced echelon form – Gauss process (without transpositions)

Operations on columns:

- multiplication of a column by a scalar
- subtracting a column which is on the left

Get a unique "reduced echelon form"

$$W=\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- each column ends with 1
- the entries to the right of 1 is 0.

(algorythm:
- Take first column, find the lowest nonzero entry and normalize to get 1.
- kill all the entries to the right of 1.
- take the second column and find the lowest nonzero entry ...)

All the 1's define a permutation

here $\Pi=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

⑦

Permuting columns get $n = \begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

In matrix notation $w = n \cdot \pi$

<u>Theorem</u> Any element of $g \in Gl_n(\mathbb{C})$ can be factorized $g = n \pi b$, where $n \in N$, π is a permutation matrix $b \in B$. $Gl_n = \bigsqcup_{\pi \in \Sigma_n} B \pi B$.

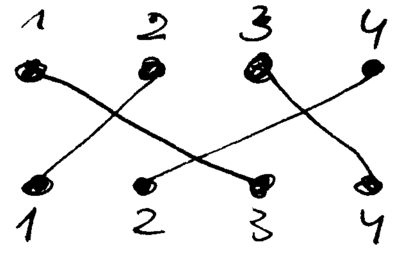
π and b are unique.

n can be made unique if we assume that $n \in N \cap \pi N_- \pi^{-1}$ where N_- is the group of lower triangular matrices.

The homogenous Space

$$Gl_n / B = \bigsqcup_{\pi} B \pi B / B = \bigsqcup_{\pi} \underbrace{N \cap \pi N_- \pi^{-1}}_{\mathbb{C}^{l_n(\pi)}}$$

where $l_n(\pi)$ = lenght of the permutation



$l(\pi) = 3$

We show that $Gl_n(\mathbb{C}) / B = U(n) / T$

where $T = B \cap U(n)$ = maximal compact abelian connected = maximal torus

$$T = \left\{ \begin{bmatrix} e^{i\varphi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\varphi_n} \end{bmatrix} \varphi_i \in [0, 2\pi] \right\} = (S^1)^n$$

(8) The natural map:

$$U(n)/T \longrightarrow Gl_n(\mathbb{C})/B$$

(1.) is surjective
(any basis can be transformed to orthonormal via Gramm-Schmidt $Gl_n(\mathbb{C}) = U(n)B$)

(2) To check injectivity: for $g_1, g_2 \in U(n)$

$$g_1 \cdot B = g_2 \cdot B \Leftrightarrow g_1^{-1} g_2 \in B \underset{?}{\Rightarrow} g_1^{-1} g_2 \in T$$

i.e $B \cap U(n) = T$. ok.

The space $U(n)/T = Gl_n(\mathbb{C})/B$ is a compact topological space. It has a structure of a complex manifold (as a quotient of a complex group by a complex group). It can be identified with the flag variety $Fl(\mathbb{C}^n)$

$$Fl(\mathbb{C}^n) = \{ V_1 \subset V_2 \subset V_3 \subset \ldots \subset V_n = \mathbb{C}^n \mid V_i \text{ is a linear vector space of dimension } i \}$$

$$Fl(\mathbb{C}^n) \subset \prod_{i=1}^{n} Grass_i(\mathbb{C}^n) \quad \text{(Grassmanians)}$$

<u>Proof</u> $Gl_n(\mathbb{C})$ acts on $Fl(\mathbb{C}^n)$.

- the action is transitive
(every flag can be translated to any other)

- stabilizer of the standard flag
$\mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$
is equal to B. Hence $Fl(\mathbb{C}^n) = Gl_n(\mathbb{C})/B$.
(no problem with topology, since the spaces are compact)

(9) General case. G reductive

- B - maximal connected solvable group (in general it)

- T maximal abelian connected group in K.

$T \cong (S^1)^n$, therefore it is called the maximal torus.

- $N(T)$ normalizer of T ie $\{g \in K : gTg^{-1} = T\}$

Then the space

$$G/B = K/T$$

is called the generalized flag manifold. It is a compact complex manifold. Consider the action of B.

$$(b, gB) \to bgB.$$

<u>Theorem</u> (Bruhat decomposition)

$$G/B = \bigsqcup_{\pi \in W} B\pi B/B$$

the decomposition into B-orbits the indexing set $W = N(T)/T$ is the Weyl group of K.

Each orbit

$$B\pi B/B = B/(\pi^{-1}B\pi) \cap B \text{ is homeo } \mathbb{C}^{\ell(\pi)}$$

<u>Remark</u> Weyl group can be computed using the complex torus $W = N T(\mathbb{C}) / T(\mathbb{C})$

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<u>Where</u> $T(\mathbb{C})$ is the maximal connected abelian reductive subgroup of G,
$T(\mathbb{C}) = (\mathbb{C}^*)^n$

It is a complexification of $T \subset K$.

<u>Exercises:</u> For $G = Gl_n(\mathbb{C})$

- Show ~~that~~ indeed the group of diagonal matrices is the maximal connected abelian subgroup of $U(n)$,
- blocks of rotations $\begin{pmatrix} e_\alpha & & \\ & e_\beta & \\ & & \ddots \end{pmatrix}$ is a maximal connected abelian subgroup in $SO(n)$
- compute normalizers, show that for $U(n)$ $W = NT/T = \Sigma_n$ group of permutations.
- What is $W = NT/T$ for $SO(n)$?
- compute T, NT, $W = NT/T$ for $Sp(n)$.
- Show that in general NT/T is finite

(11) Ad 5: Maximal tori in compact groups

Let $g \in SO(n)$.

Theorem GAL: there exists an orthonormal basis of $\mathbb{R}^n$ such that in that basis g has a block form

$$\begin{pmatrix} \boxed{R_{\alpha_1}} & & & 0 \\ & \boxed{R_{\alpha_2}} & & \\ & & \boxed{R_{\alpha_3}} & \\ 0 & & & \ddots 1 \end{pmatrix}$$

where R_{α_i} is a rotation

Interpretation: choice of a basis $\Longleftrightarrow$ conjugation.

Theorem' Every element of $SO(n)$ belongs to a maximal torus hTh^{-1}.

Even more is true: A-connected commutative subgroup, Then there is an orthonormal basis which is good for each $g \in A$.

Conclusion: All maximal tori are conjugate.

Complex version: Every isometry is diagonalizable.

General theorem for compact groups: Every element belongs to some maximal torus. All tori are conjugate.