Wykład 4

The assignment

$$L: \text{Lie groups} \to \text{Lie algebras}$$

is a funtor, i.e

for a map of Lie groups $f: G \to H$ we obtain a map $L(f): L(G) \to L(H)$ of their Lie algebras, and

$$L(f \circ g) = L(f) \circ L(g).$$

Proof

$f(1) = 1$ Therefore $L(f) := Df_1 : T_1 G \to T_1 H$.

We have to show that $L(f)$ preserves the bracket. Use the definition

$$[X, Y] = \lim_{t \to 0} \frac{\gamma_X(t)\gamma_Y(t) - \gamma_Y(t)\gamma_X(t)}{t^2} \Big| \circ \gamma_X^{-1}\gamma_Y^{-1} =$$

$$= \lim_{t \to 0} \frac{\gamma_X(t)\gamma_Y(t)\gamma_X^{-1}(t)\gamma_Y^{-1}(t) - 1}{t^2}$$

$$= \frac{d^2}{dt^2}\left(\gamma_X \gamma_Y \gamma_X^{-1} \gamma_Y^{-1}\right)(0)$$

(here γ_X, γ_Y are the 1-parameter subgroups $\dot\gamma_X(0) = X$ $\dot\gamma_Y(0) = Y$.

Since the map f preserves the multiplication it has to preserve $[\,,\,]$.

The study of Lie groups is partially reduced to the study of Lie algebras due to the following very important theorem.

<u>Theorem</u>
The function L restricted to connected and simply-connected Lie groups (ie $\pi_1(G)=0$)

$$L : \underset{\substack{\text{connected}\\ +\text{ simply-connected}}}{\text{Lie groups}} \longrightarrow \underset{\text{finite dimensional}}{\text{Lie algebras}}$$

is an <u>equivalence of categories</u>.

<u>This means:</u>

(1) for any Lie algebra g there exists (a unique up to isomorphism) group G (simply connected & connected) such that $L(G) = g$

it will follow from (2)

(2) for a map of Lie algebras
$$f : L(G) \to L(H)$$
there exists a unique map $F : G \to H$ such that $L(F) = f$.

(here one needs the assumption of simply-connectness: there is an ~~is~~ isomorphism $L(S^1) \xrightarrow{\sim} L(\mathbb{R}_+)$ but there are no nontrivial maps $S^1 \to \mathbb{R}_+$.

<u>The proof</u> of this theorem starts with the <u>theorem of Ado</u>

<u>Theorem</u> Every finite dimensional algebra can be embedded in $gl(V)$ for some vector space V.

- To construct a Lie group with the prescribed $\mathfrak{g} \subset \mathfrak{gl}(V)$ consider the distribution in $GL(V)$ constructed by (left) translating $\mathfrak{g} \subset \mathfrak{gl}(V) = T_1 GL(V)$. This distribution is integrable. Leaves are translations of $\exp(U)$ (for $U \subset \mathfrak{g}$ s.t. $\exp(U)$ is a immersion).
  Let G_0 be the leaf passing through 1, and $G := \widetilde{G_0}$ be the universal covering of G_0. Since $L(G) = L(\widetilde{G_0}) = \mathfrak{g}$, we are done.

- To construct a map $G \to H$.

  Consider the product $G \times H$
  $T_1(G \times H) = \mathfrak{g} \times \mathfrak{h}$
  let $\mathfrak{k} = \mathrm{graph}(f : \mathfrak{g} \to \mathfrak{h})$
  this is a Lie subalgebra.
  Let K be the Lie group constructed as before

$$
\begin{tikzcd}
K \arrow[r] \arrow[d, dashed, "\cong"'] & G \times H \arrow[dl, "\pi_1"] \arrow[r, "\pi_2"] & H \\
G & &
\end{tikzcd}
$$

  The map $K \to G \times H \xrightarrow{\pi_1} G$ is a covering. Since G is simply connected, this map is an isomorphism.

- If two maps induce the same on Lie algebras then by Baker-Campbell-Hausdorff formula they are equal on the neighbourhood of 1. Every neighbourhood of 1 generates the identity component. so the maps have to be equal.

Conclusion: up to a covering everything is encoded in the Lie algebra.

Dictionary

subgroup	subalgebra
normal subgroup $H \trianglelefteq G$ $ghg^{-1} \in H$ for $g \in G$, $h \in H$	ideal $I \subset \mathfrak{g}$ $[X,Y] \in I$ for $X \in \mathfrak{g}$, $Y \in I$
quotient G/H	quotient $\mathfrak{g}/I$
simple group (i.e. no normal subgroups $\neq \{1\}$ or G)	simple Lie algebra no ideals $\neq \{0\}, \mathfrak{g}$ exclusion $\mathfrak{g} \neq \mathbb{C}_+$
abelian group $ghg^{-1}h^{-1} = 1$	abelian algebras $[X,Y] = 0$
Soluble groups: exists a sequence $G = G_0 \trianglerighteq G_1 \trianglerighteq \dots \trianglerighteq G_n \trianglerighteq G_{n+1} = \{1\}$ s.t. G_i/G_{i+1} abelian	Solvable algebras $\mathfrak{g} = I_0 \supset I_1 \supset \dots \supset I_n \supset I_{n+1} = 0$ s.t. I_i/I_{i+1} abelian
equivalently: $G^{(i)} = \{1\}$ for $i \gg 0$ where $G^{(0)} = G$ $G^{(i+1)} = [G^{(i)}, G^{(i)}]$	equivalently $\mathfrak{g}^{(i)} = \{0\}$ where $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$

The main example of a solvable algebra is the upper-triangular algebra

$$\mathfrak{b} = \left\{ \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix} \right\}$$

The related group

$$\mathbb{B} = \left\{ \begin{pmatrix} \mathbb{C}^* & & & & * \\ & \mathbb{C}^* & & & \\ & & \mathbb{C}^* & & \\ & & & \mathbb{C}^* & \\ 0 & & & & \mathbb{C}^* \end{pmatrix} \right\}$$ i.e. invertible elements on the diagonal.

<u>Very important theorem</u>, but we will not use it.

<u>Lie Theorem</u>: Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a solvable subalgebra. Then there exists a basis of V s.t. $\mathfrak{g}$ consists of upper triangular matrices i.e. $\mathfrak{g} \subset \mathfrak{b}$.

For further study of Lie algebras we need some tools.

We will define Killing form B – certain invariant bilinear symmetric form, which for compact groups is negative definite (i.e. $(-B)$ is a scalar product in $\mathfrak{g}$.

Let V be a complex linear space, $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie subalgebra (i.e. $\mathfrak{g}$ acts on V).
Recall:
$$\operatorname{tr}(X) = \sum_{i=1}^{\dim V} X_{ii}$$
does not depend on the choice of coordinates. Moreover $\operatorname{tr}(XY) = \operatorname{tr}(YX)$. Therefore

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$$B_0^{\mathbb{C}}(X, Y) := \operatorname{tr}(X \cdot Y),$$

is a complex bilinear symmetric form
$$\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

Also we have a real form
$$B_0(X, Y) = \operatorname{Re} B(X, Y).$$

Both forms are nondegenerate for $\mathfrak{g}$ reductive: $B_0^{\mathbb{C}}(X, X^*) = \sum_{i,j} \|x_{ij}\|^2$

where $X = (X_{ij})_{ij}$

Pf $(X \cdot \overline{X}^T)_{kk} = \sum_i X_{ki} \overline{X}_{ki}$ ← diagonal entry.

- This form is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$; $\mathfrak{p} \perp \mathfrak{k}$

Pf $B_0(X, X) = \operatorname{tr}(X \cdot X) = \operatorname{tr}(X \cdot X^*) > 0$ for $X \in \mathfrak{p}$ $X \neq 0$
(similarly for $X \in \mathfrak{k}$).

- The form B_0 is G invariant.

Pf Compute the derivative

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$$\frac{d}{dt} B_0 \left(Ad_{\gamma(t)} X , Ad_{\gamma(t)} Y \right) =$$

for $\gamma(t)$ a curve in G

$\gamma(0) = 1 \quad \dot\gamma(0) = Z$

$$= B_0 (ad_Z X , Y) + B_0 (X , ad_Z Y)$$

by Leibniz formula

$$= B_0 ([Z,X] , Y) + B_0 (X , [Z,Y])$$

$$= tr ((ZX - XZ) Y) + tr (X (ZY - YZ))$$

$$= tr (ZXY) - tr (XYZ) = 0$$

$$(tr (AB) = tr (BA))$$

The form B_0 generalises to a bilinear form B_ρ defined for any map $\rho : G \to Gl(V)$ where V is a vect space (i.e ρ is a representation in V)

In particular we have the adjoint representation $Ad : G \to Gl(\mathfrak{g})$.

The image is isomorphic to $G/Z(G)$.

The resulting form

$$B = B_{Ad}$$

is called the <u>Killing form</u>.

<u>Note that</u> If G is reductive, then $\mathfrak{g}$ is equipped with a hermitian product $\langle\langle X,Y \rangle\rangle = B_0(X, Y^*)$

<u>Lemma</u> $Ad\, G \subset Gl(\mathfrak{g})$ is a reductive subgroup, ie invariant with respect to $*$ in $Gl(\mathfrak{g})$.

<u>Proof</u> We have to show that $(\mathrm{ad}_X)^* \in \mathrm{ad}_{\mathfrak{g}}$ for $X \in \mathfrak{g}$. We check that

$$(\mathrm{ad}_X)^* = \mathrm{ad}_{X^*}$$

i.e. $\forall\, Y, Z \in \mathfrak{g}$

$$\langle\!\langle \mathrm{ad}_X Y, Z \rangle\!\rangle \overset{?}{=} \langle\!\langle Y, \mathrm{ad}_{X^*} Z \rangle\!\rangle$$

$$LHS = B_0([X,Y], Z^*) = \mathrm{tr}([X,Y] \circ Z^*)$$

$$RHS = B_0(Y, [X^*, Z]^*) = B_0(Y, [Z^*, X])$$

$$= \mathrm{tr}(Y \circ [Z^*, X])$$

$$LHS = \mathrm{tr}(XYZ^*) - \mathrm{tr}(YXZ^*)$$

$$RHS = \mathrm{tr}(YZ^*X) - \mathrm{tr}(YXZ^*)$$

$$LHS = RHS$$

<u>Exercise</u> Show that $B(\ ,\ )$ for $G = SL_n(\mathbb{C})$ is equal up to a constant to $B_0(\ ,\ )$.

(For $G = GL_n(\mathbb{C})$ the Killing form vanishes on $z(G)$ = diagonal matrices with equal entries.)

Since $G/z(G)$ is a subgroup of $GL(\mathfrak{g})$ the Killing form is nondegenerate on $\mathfrak{g}/z(G)$

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<u>Proposition</u> Suppose $\mathfrak{g}$ is a simple Lie algebra. Then Killing from of $\mathfrak{g}$ is nondegenerate.

<u>Pf</u> Let $\mathfrak{g}^\perp = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{g} \ B(X,Y) = 0 \}$

<u>Claim</u> $\mathfrak{g}^\perp$ is an ideal, ie

$$B([X, Y], Z) = 0$$

for all $X, Z \in \mathfrak{g}$ $Y \in \mathfrak{g}^\perp$

This is so : $B([X,Y],Z) = -B(X,[Y,Z]) = 0$

($\mathfrak{g}^\perp \neq \mathfrak{g}$ otherwise by Cartan criterion below $\mathfrak{g}$ would be solvable.)

On the other extreme: if $\mathfrak{g}$ is solvable

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \dots \ \mathfrak{g}^{(n)} \supset 0$$

then $\mathfrak{g}^{(1)} \subset \mathfrak{g}^\perp$

We need the following [proof Humphreys: Intr. to Lie alg...]

<u>Cartan criterion</u> $\mathfrak{g}$ is solvable $\Longleftrightarrow$

$$B(X, Y) = 0 \quad \forall X \in \mathfrak{g}, \ Y \in [\mathfrak{g}, \mathfrak{g}]$$

<u>Example</u> $\mathfrak{g} = \mathfrak{b} \subset \mathfrak{gl}_2(\mathbb{C})$ $\mathfrak{b} = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}$

basis $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$[a,b] = 0$ $[a,c] = c$ $[b,c] = -c$

$ad_a : \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $ad_b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $ad_c = 0$ (in the basis a, b, c)

$\mathfrak{g}^{(1)} = \mathrm{lin}(c)$ $\mathrm{tr}(ad_X \cdot ad_c) = \mathrm{tr}(0) = 0.$

<u>Exercise</u> Compute $B(-,-)$ for $\mathfrak{b} \subset \mathfrak{gl}_n(\mathbb{C})$.

5)

Radical of the Lie algebra

(in group theory: maximal solvable normal subgroup)

Rad($\mathfrak{g}$) = maximal solvable ideal.

Theorem The radical is unique.

Pf if J_1 J_2 are normal solvable, then $J_1 + J_2$ is normal solvable

$$J_1 + J_2 \supset \underbrace{\qquad\qquad}_{\substack{(\text{sequence resolving} \\ J_1) + I_2}} \supset \underbrace{J_2 \supset \dots \supset 0}_{\text{sequence resolving } I_2}$$

We use the epimorphism $\mathfrak{g}_i / \mathfrak{g}_{i+1} \twoheadrightarrow \dfrac{\mathfrak{g}_i + I_2}{\mathfrak{g}_{i+1} + I_2}$ •

Theorem The following are equivalent

(A) • $\mathfrak{g}$ is semisimple (ie $\oplus$ simple)

(B) • the Killing form is nondegenerate

(C) • Rad($\mathfrak{g}$) = 0.

(A) => (B) clear (direct sum of nondegenerate forms)

(B) => (C) r = Rad $\mathfrak{g}$ $\quad r \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/r$

$\mathfrak{g}$ acts on that sequence $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

$B = B_r + B_{\mathfrak{g}/r}$

if r is abelian $[r, r] = 0$ then $r = \mathfrak{g}^\perp$

in general if $r^{(n)} \neq 0$ $r^{(n+1)} = 0$ then $r^{(n)} \subset \mathfrak{g}^\perp$.

C $\Rightarrow$ A

Suppose $\mathfrak{h}$ is a normal subalgebra.
We have to decompose $\mathfrak{g} = \mathfrak{h} \oplus$ something
Something $= \mathfrak{h}^\perp$

Claim $\mathfrak{h}^\perp$ is a normal subalgebra
i.e. for $X \in \mathfrak{g}$ $Y \in \mathfrak{h}^\perp$ $[X, Y] \in \mathfrak{h}^\perp$

i.e. $\forall Z \in \mathfrak{h}$ $B([X,Y], Z) \overset{?}{=} 0$

but $B([X,Y],Z) = B(Y, \underset{\mathfrak{h}}{\underset{\Uparrow}{[X,Z]}}) = 0$.

We have to show $\mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$.

Claim $\mathfrak{h} \cap \mathfrak{h}^\perp$ is a solvable ideal

Cartan criterion: $[X,Y] = 0$
(here for any $X, Y \in \mathfrak{h} \cap \mathfrak{h}^\perp$)
$\Rightarrow$ solvable.
Hence $\mathrm{rad}(\mathfrak{g}) \supset \mathfrak{h} \cap \mathfrak{h}^\perp$.
by C $\mathrm{rad}(\mathfrak{g}) = 0$.

For any (possibly degenerate forms)

$\dim \mathfrak{h} + \dim \mathfrak{h}^\perp \geq \dim \mathfrak{g}$
but since $\mathfrak{h} \cap \mathfrak{h}^\perp = 0$ there is an equality and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$.

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A similar criterion for reductive groups:

<u>Theorem</u> The following are equivalent:

- $\mathfrak{g}$ = semisimple $\oplus$ abelian
- rad($\mathfrak{g}$) is abelian
- there exists an embedding $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ s.t. $\mathfrak{g}^* = \mathfrak{g}$
- $\mathfrak{g}$ is a complex group, which is the complexification of a compact group

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- Later we will see for groups every representation of G decomposes into irreducible components.

Reductive groups and (semi)simple groups will be the most interesting for us.