(Wykład 3)

1 Example 1 $G = O(n)$

What is $L(O(n)) = \mathfrak{o}(n)$?

Given by the equation $A^T A = I$

Take $\gamma(t)$ a curve in $O(n)$

$$\gamma(0) = 1 \qquad \dot\gamma(0) = X \in \mathfrak{o}(n)$$

$$\forall t \qquad \gamma(t)^T \gamma(t) = I \quad \Big| \frac{d}{dt}$$

$$\dot\gamma(t)^T \gamma(t) + \gamma(t)^T \dot\gamma(t) = 0 \quad \Big|_{t=0}$$

$$\dot\gamma(0)^T + \dot\gamma(0) = 0$$

$$X^T + X = 0 \quad \text{i.e. } X \text{ is real antisymmetric}$$

What is the complexification $O(n)_{\mathbb{C}}$?

$O(n)_{\mathbb{C}} = O(n, \mathbb{C})$ is the group of complex matrices s.t. $A^T A = I$

(this is a polynomial equation for entries of the matrix)

In another words A preserves the quadratic nondegenerate form $z_1^2 + z_2^2 + \dots + z_n^2$.

The Lie algebra $\mathfrak{o}(n, \mathbb{C})$ is given by $X^T + X = 0$ $X \in M_{n\times n}(\mathbb{C})$

(complex antisymmetric matrices)

Example 2 $G = U(n)$.

The equation $A^T \bar{A} = I$

Lie algebra : $\gamma^T(t) \cdot \bar\gamma(t) = I \quad \Big| \frac{d}{dt}$

$$\dot\gamma(t)^T \bar\gamma(t) + \gamma^T(t) \cdot \dot{\bar\gamma}(t) = 0 \quad \Big|_{t=0}$$

$$\dot\gamma(0)^T + \overline{\dot\gamma}(0) = 0$$

$$X^T + \bar{X} = 0$$

in another form $\bar{X}^T = -X$.

What is the complexification?

$U(n) \subset GL_n(\mathbb{C})$

$u(n) \subset gl_n(\mathbb{C}) = M_{n\times n}(\mathbb{C})$

Claim $\quad u(n) + iu(n) = M_{n\times n}(\mathbb{C})$

<u>Cartan involution:</u> $X \longmapsto \bar{X}^T$
decomposes $M_{n\times n}(\mathbb{C})$ into eigenspaces for the eigenvalues 1 and -1

$$M_{n\times n}(\mathbb{C}) = \{\bar{X}^T = X\} \oplus \{\bar{X}^T = -X\}$$
$$gl_n(\mathbb{C}) = \quad i\cdot u(n) \quad \oplus \quad u(n)$$

we check: $\quad \overline{(iX)}^T = -i\,\bar{X}^T = iX$.

<u>Conclusion</u> $\quad u(n)_{\mathbb{C}} = gl_n(\mathbb{C})$
$\qquad U(n)_{\mathbb{C}} = Gl_n(\mathbb{C})$.

<u>Remark</u> $Gl_n(\mathbb{C})$ can be embedded as a closed set in $M_{(n+1)\times(n+1)}(\mathbb{C})$

$$A \longmapsto \begin{bmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{bmatrix}.$$

In fact it is embedded in $Sl_{n+1}(\mathbb{C})$. Again it is given by polynomial equations.

The list of algebras, complexifications, equations

Compact		Complexification	
$so = o(n)$	$X + X^T = 0$, X real	$o(n, \mathbb{C})$	$X + X^T = 0$, X complex
$u(n)$	$\bar{X} + X^T = 0$	$\mathfrak{gl}_n(\mathbb{C})$	no equation
$su(n)$	$\begin{cases} \bar{X} + X^T = 0 \\ \operatorname{tr} X = 0 \end{cases}$	$\mathfrak{sl}_n(\mathbb{C})$	$\operatorname{tr} X = 0$
$sp(n)$	$\begin{cases} XJ + JX = 0 \\ \bar{X} + X^T = 0 \end{cases}$	$sp(n, \mathbb{C})$	$XJ + JX = 0$

Exercise: Check the remaining cases.

Denote by X^* the Cartan involution $\bar{X}^T$

Let $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ be a subalgebra. We call $\mathfrak{g}$ <u>reductive</u> if

$X^* \in \mathfrak{g}$ for $X \in \mathfrak{g}$. As for $\mathfrak{gl}_n(\mathbb{C})$ $\mathfrak{g}$ decomposes into eigen-spaces of $*$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$$

let $\mathfrak{k} = \mathfrak{g}_{-1}$ $\quad \mathfrak{p} = \mathfrak{g}_1$.

Since $[X, Y]^* = -[X^*, Y^*]$ we have: $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ (i.e. $\mathfrak{k}$ is a subalgebra) in fact $\mathfrak{k} = \mathfrak{g} \cap u(n)$

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$$
$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

Polar Decomposition

(4) Let G be a subgroup of $Gl_n(\mathbb{C})$ with the reductive Lie algebra

Theorem: The map

$$K \times \mathfrak{p} \longrightarrow G, \quad (B, P) \mapsto B \cdot \exp(P)$$

is a diffeomorphism. Here

$K = U(n) \cap G$, $L(K) = k = \mathfrak{g}_{-1}$.

Proof For $G = Gl_n(\mathbb{C})$.

hermitian symetric metices = self adjoint maps $\mathbb{C}^n \to \mathbb{C}^n$
$A = A^*$

for $P \in \mathfrak{p}$ there exists an orthonormal basis in which it is diagonal

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ $\quad P = CDC^T \quad C^T = C^{-1}$

Then $\exp(P) = C \exp(D) C^T = C \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} C^T$

$\exp(P)$ is a symmetric matrix with real positive eigenvalues.

$$\exp : \mathfrak{p} \longrightarrow \mathfrak{p}_+$$

There exists an inverse map which for diagonal matrices is of the form $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \xmapsto{\log} \begin{bmatrix} \ln \lambda_1 & & \\ & \ddots & \\ & & \ln \lambda_n \end{bmatrix}$.

Claim log is C^∞ map.

J) Proof: The sequence

$$\log(I+A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \dots$$

converges if all the eigenvalues of A are in $(-1,1)$.

Therefore $\log(P)$ defined and smooth if eigenvalues $\in (0,2)$.

Moreover for all $t \in \mathbb{R}$ $t>0$

$$\log(tP) = \log P + \log(t \cdot I) \quad \otimes$$

(because $\log(AB) = \log(A) + \log(B)$ if A and B commute)

For arbitrary P we define $\log P$ by the formula $\otimes$ for sufficiently small t:

$$\log(P) := \log(tP) - \ln t \cdot I$$

Now let $A \in Gl_n(\mathbb{C})$ $\bar{A}^T A$ is a positive definite symmetric matrix.

let $P = \frac{1}{2}\log(\bar{A}^T A)$ $Q = \exp(P)$

then $\bar{A}^T A = Q^2$.

let $B = A \cdot Q^{-1}$, $(\overline{AQ^{-1}})^T A Q^{-1} = Q^{-1} \bar{A}^T A Q^{-1} = I$

hence $B \in U(n)$.

We get $A = B \cdot Q = A \exp(P)$

The map $A \longmapsto (B, P)$ is C^∞ map inverse to $B \cdot \exp(P)$.

(6) Other classical groups:
The general case: $G \subset Gl_n(\mathbb{C})$ closed subgroup described by polynomials
We have to show that for $A \in G$ constructed B and P belong to $G \cap U(n)$ and $\mathfrak{g} \cap$ Symmetric matrices.

$\bar{A}^T \in G$ since G is reductive
$\bar{A}^T A \in G$ and equal to $(\exp(P))^2$

Changing coordinates can assume that P is diagonal. $Q = \exp(P)$.
$Q^t = \begin{pmatrix} a_1^t & & 0 \\ & \ddots & \\ 0 & & a_n^t \end{pmatrix}$ satisfy the equation of G for t even natural $t = 2n$
$\left(Q^{2n} = (\bar{A}^T A)^n \in G \right)$.

<u>Therefore</u> (this is a small ~~lemma~~-exercise*)
$Q^t = \exp(tP) \in G$ for all t.

Hence $B = A Q^{-1} \in G$.

* use that G is given by polynomial equations

(7) (Baker) – Campbell – Hausdorff formulas

$\exp : \mathfrak{g} \to G$ is diffeo on $\mathcal{U} \subset \mathfrak{g}$

The inverse is denoted by log.

For matrix groups

$$\log(1+A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \cdots$$

The logarythm allows to define a multiplication in $\mathfrak{g}$ (defined locally)

$$X * Y = \log(\exp(X) \cdot \exp(Y)) =$$

$$\log\left(\left(1 + A + \frac{A^2}{2} + \ldots\right)\left(1 + B + \frac{B^2}{2} + \ldots\right)\right) = \log\left(1 + A + B + \frac{A^2}{2} + AB + \frac{B^2}{2} + \ldots\right)$$

$$= A + B + \frac{A^2}{2} + AB + \frac{B^2}{2} - \frac{(A+B)^2}{2} + \ldots = A + B + AB - \frac{1}{2}AB - \frac{1}{2}BA + \ldots$$

$$= A + B + \frac{1}{2}[A, B] + \ldots$$

> **Theorem** (we will not prove it) All the higher terms can be expressed using $[\ ,\ ]$ only.
> e.g. 3-rd term: $\frac{1}{12}\left([A, [A, B]] + [B, [B, A]]\right)$

(exercise: compute few terms)

Easy: if $[A, B] = 0$, then $\exp(A)$, $\exp(B)$ commute.

Conclusion The multiplication in G is determined by $[\ ,\ ]$.

Proof Given locally $\Longrightarrow$ at the component of 1.

For example we deduce that the multiplication formulas are analytic.

(8) Theorem The function

see Segal's proof

Lie Groups $\longrightarrow$ Lie Algebras

Connected + simplyconnected
$\pi_1(G)=1$

is an equivalence of categories.

This means:
- for any Lie algebra $\mathfrak{g}$ there exists a Lie group G s.t. $L(G) \cong \mathfrak{g}$
- for any map of Lie algebras $\mathfrak{g} \to \mathfrak{h}$ there exists a unique map of corresponding Lie groups (in particullar if $L(G) \simeq L(H)$ then $G \simeq H$).

For non simplyconneted groups might be a problem with existance of maps
eg $L(S^1) \simeq \mathbb{R} = L(\mathbb{R}^1)$ but there is no map $S^1 \longrightarrow \mathbb{R}$ except the constant.

Abelian Lie Groups

One of corollaries: if $[X,Y]=0$ for all X, Y, then G is commutative (if G is connected)

Pf for matrix groups it follows from $\exp(X) \cdot \exp(Y) = \exp(X \cdot Y)$ whenever $XY=YX$.

Or for general group it follows from B-C-H

Ten $\exp : \mathfrak{g} \to G$ is a homomorphism.

hence $G = \mathfrak{g} /$ discrete subgroup $\simeq \mathbb{R}^n / \mathbb{Z}^k$ for some $k \leq n$

• (ex. show that exp is epi)

(9) Adjoint representation

Let $\phi_g : G \to G$
$h \mapsto ghg^{-1}$
be the conjugation.
$\phi_g(1) = 1$ and $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$
Therefore G acts on $T_1 G = \mathfrak{g}$.

$Ad : G \longrightarrow GL(\mathfrak{g})$

Action of a Lie group on a vector space is called a <u>representation</u>.
This one is called the <u>adjoint representation</u>

If G is commutative, then $Ad(g) = 1$ for any g. In general for connected groups $\ker(Ad) = Z(G)$ where $Z(G)$ denotes the center:
$Z(G) = \{ g \in G : \forall h \in G \ ghg^{-1} = g \}$

<u>Corollary</u> We get for free an embedding $G/Z(G) \hookrightarrow GL(\mathfrak{g})$

For example all simple (in the algebraic sense) groups (i.e. those which have no normal subgroups) embed in $GL(\mathfrak{g})$.

For example $PGL_n(\mathbb{C}) = GL_n(\mathbb{C}) / \mathbb{C}^* \cdot Id$
$PGL_n(\mathbb{C}) \subset GL_{n^2}(\mathbb{C})$

(10) <u>Small</u> $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the derivative at the map $G \to Gl(\mathfrak{g})$.

<u>For matrix groups</u>

$\phi_A(B) = ABA^{-1}$

It is linear with respect to B.

Hence $Ad_A(X) = (D\phi_A)_1(X) = AXA^{-1}$.

The derivative of $Ad_A(X)$ (with respect to A at 1 in the direction Y):

$$\left(D\, Ad_{\_}(X)\right)_1 (Y) = \frac{d}{dt}\, \gamma(t) X \gamma(t)^{-1}\Big|_{t=0}$$

where $\gamma(t)$ is a curve s.t.

$\gamma(0) = 1$

$\dot\gamma(0) = Y$ - can assume it is 1-parameter subgroup $\gamma(t)^{-1} = \gamma(-t)$

$$= \dot\gamma(0) X \gamma(0)^{-1} + \gamma(0) X \dot\gamma^{-1}(0)$$

$$= YX - XY = [Y, X].$$

<u>Notation</u> $ad_Y(X) = [Y, X]$.

<u>Theorem</u> Every complex connected compact group is iso to $\mathbb{C}^n/\Lambda$, where $\Lambda \cong \mathbb{Z}^{2n}$

<u>Proof</u>: There are no linear compact complex groups (no nonconstant maps $G \to M_{n\times n}(\mathbb{C})$).

Consider $Ad : G \to Gl(\mathfrak{g})$. This map is constant. It follows that $\forall X, Y \in \mathfrak{g}$ $ad_Y(X) = [Y, X] = 0$. Hence G is commutative.