(1) Division algebras and related Lie groups

Algebra	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
Group of units	$\{\pm 1\}$	S^1	S^3	—
Automorphisms	1	$\mathbb{Z}_2$	$S^3/\{\pm 1\}$	G_2
General linear group	$GL_n(\mathbb{R})$	$GL_n(\mathbb{C})$	$GL_n(\mathbb{H})$	5 exceptional groups
Maximal compact subgroup	$O(n)$	$U(n)$	$Sp(n)$	— " —
Special groups	$SO(n)$	$SU(n)$	—	—

<u>Theorem</u> Every compact group is of the form: $G_1 \times G_2 \times \dots \times G_m \times (S^1)^k / A$

Where A is a finite abelian group

G_i is a simple group, one of the list: $\widetilde{SO}(n)$, $SU(n)$, $Sp(n)$ or one of the 5 exceptional groups

Complex Lie groups: G is a complex manifold, multiplication and inverse are holomorphic

e.g. $GL(n)$, $SL(n)$, not $U(n)$!

"There are no complex, compact, noncommutative groups"

(2) <u>Theorem</u> If G is compact complex connected group, then G is abelian. It is of the form

$G \simeq \mathbb{C}^n / \Lambda$ where Λ is a maximal latice in $\mathbb{C}^n$, $\Lambda \simeq \mathbb{Z}^{2n}$. As a real Lie group $G \simeq (S^1)^{2n}$.

[proof will be later]

- Instead of compact groups one considers „reductive groups" (definition and properties will be later). They can be classified

$$G \simeq G_{1\mathbb{C}} \times \dots \times G_{m\mathbb{C}} \times (\mathbb{C}^*)^l / A$$

where $G_{i\mathbb{C}}$ is a complexification of a simple Lie group, A is finite abelian

<u>Complexifications</u> $G \subset G_{\mathbb{C}}$ $(T_1 G) \otimes_{\mathbb{R}} \mathbb{C} = T_1 G_{\mathbb{C}}$

$(SU(n))_{\mathbb{C}} = SL_n(\mathbb{C})$ $\det A = 1$

$(SO(n))_{\mathbb{C}} = SO(n, \mathbb{C})$ complex matrices preserving a complex nondegenerate quadratic form $A^T A = I$ and additionally $\det A = 1$

What is $Sp(n)_{\mathbb{C}}$?

We defined $Sp(n)_{\mathbb{C}}$ as

$$Sp(n) = Gl_n(\mathbb{H}) \cap U(2n)$$

i.e $jA = Aj$ and $A^T \bar{A} = I$

(3) 1° condition:

$$cJA = AcJ \qquad c \text{ - conjugation}$$

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

$$cJA = c\bar{A}J$$

$$JA = \bar{A}J$$

2° condition $\quad \bar{A}^{-1} = A^T$

$$\Rightarrow \quad A^T J A = J$$

this means that A preserves the bilinear form given by the matrix J.

$$\omega(x,y) := x^T J y \quad \text{for } x,y \in \mathbb{C}^{2n}$$

ω is antisymmetric $\quad \omega(x,y) = -\omega(y,x)$ and nondegenerate

Conclusion:

$Sp(n)$ is the group of unitary matrices preserving the form ω.

ω is called the symplectic form

We define $Sp_n(\mathbb{C}) = \{A \in GL_{2n}(\mathbb{C}) : A^T J A = J\}$ i.e. complex maps preserving ω.

Clearly $\quad Sp(n) = Sp_n(\mathbb{C}) \cap U(2n)$

<u>Exercise</u> Show that $Sp(n)$ is a maximal compact subgroup in $Sp(n, \mathbb{C})$ and the dimensions satisfy $\dim_{\mathbb{C}} Sp(n,\mathbb{C}) = \dim_{\mathbb{R}} Sp(n)$

In the end show that $Sp(n)_{\mathbb{C}} = Sp(n, \mathbb{C})$

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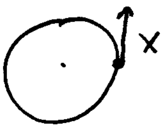
The exponential map

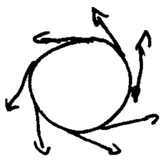
Example $\quad G = S^1 \ni 1 \qquad S^1 \subset \mathbb{C}$



$T_1 S^1 = i\mathbb{R}$ tangent space at 1 (linear space)

$x = r \cdot i$

1° x can be extended to a vector field tangent to S^1, which is invariant with respect to rotations; for $g \in S^1$



$$\vec{V}(g) = \underbrace{D(L_g)_0}_{\text{differential of the left translation}}(x) = g \cdot x = gri$$

2° There exists a homomorphism of groups $\gamma : \mathbb{R}_+ \to S^1$ s.t. $\dot{\gamma}(0) = x$

$$\gamma(t) := e^{t \cdot x} = e^{tr \cdot i}$$

γ satisfies the differential equation

$$\dot{\gamma}(t) = \vec{V}(\gamma(t))$$

$$\frac{d}{dt}\left(e^{tri}\right) = e^{tri}\, ri$$

We have identified

$T_1 S^1 = i\mathbb{R} \leftrightarrow$ invariant vector fields $\leftrightarrow$ homomorphisms $\mathbb{R}_+ \to S^1$

The same construction works for any Lie group

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Example 2: Matrices $G = Gl_n(\mathbb{R})$

G is open in $\mathbb{R}^{n^2} = M_{n\times n}(\mathbb{R})$

(condition $\det A \neq 0$)

For any $A \in Gl_n(\mathbb{R})$ $\boxed{T_A\, Gl_n(\mathbb{R}) \simeq M_{n\times n}(\mathbb{R})}$

Left translation: $Gl_n(\mathbb{R}) \to Gl_n(\mathbb{R})$

$L_A \qquad B \longmapsto AB$

This is a linear map.

The derivative

$$D(L_A)_I : T_I\, Gl_n(\mathbb{R}) \longrightarrow T_A\, Gl_n(\mathbb{R})$$

$$M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$$

$$X \longmapsto AX$$

1° We can extend the vector X tangent at I to a left invariant vector field $\vec{v}(A) := A\cdot X$

(Check

$$T_{A_1}\, Gl_n(\mathbb{R}) \xrightarrow{(DL_{(A_2A_1^{-1})})_{A_1}} T_{A_2}\, Gl_n(\mathbb{R})$$

$$A_1X \longmapsto (A_2A_1^{-1})(A_1X)$$

$$\vec{v}(A_1) \longrightarrow \vec{v}(A_2) \;)$$

2° homomorphism: $\mathbb{R}_+ \to Gl_n(\mathbb{R})$

$$\gamma : t \longmapsto \exp(tX) := I + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \dots$$

- $\dot{\gamma}(0) = X$
- γ is homomorphism $\exp(sX)\cdot\exp(tX) = \exp((s+t)X)$
- $\dot{\gamma}(t) = \gamma(t)\cdot X = \vec{v}(\gamma(t))$

Pf $\gamma(t+s) = \gamma(t)\cdot\gamma(s) \;\Big|\; \frac{d}{ds}$

$\dot{\gamma}(t+s) = \gamma(t)\cdot\dot{\gamma}(s) \qquad \Big|\; s=0$

$\dot{\gamma}(t) = \gamma(t)\cdot X$

(6) The map $\exp: M_{n\times n}(\mathbb{R}) \to Gl_n(\mathbb{R})$

The differential at 0

$$D(\exp): T_0(M_{n\times n}(\mathbb{R})) \to T_I(Gl_n(\mathbb{R}))$$

$$\quad\quad\quad\quad\quad\quad \| \quad\quad\quad\quad\quad\quad\quad \|$$

$$M_{n\times n}(\mathbb{R}) \quad\quad\quad M_{n\times n}(\mathbb{R})$$

$$D(\exp)_0(X) = \frac{d}{dt}\exp(tX)\Big|_{t=0} = X$$

$$D(\exp)_0 = id_{M_{n\times n}(\mathbb{R})} \quad \text{identity}$$

Conclusion: there exists a neighbourhood U of 0 such that

$$\exp: U \to \exp(U)$$

is a diffeomorphism.

(The same holds for $Gl_n(\mathbb{C})$)

<u>Further property</u>

Suppose $G \subset Gl_n(\mathbb{R})$ is a subgroup

$$X \in T_I G \subset T_I Gl_n(\mathbb{R})$$

Claim: Then $\exp(X) \in G$

<u>Proof</u> $X \rightsquigarrow$ vector field L_g-invariant $\rightsquigarrow$ integral curve

$$\vec{v}(g) = D(L_g)_0(X) \quad\quad\quad \gamma(t)$$

if $g \in G$ then L_g preserves G

$\vec{v}(g) = D(L_g)_0(X)$ is tangent to G

$\dot{\gamma} = \vec{v} \implies \gamma$ contained in G.

We get $\exp: T_I G \to G$

We will show that this map does not depend on the embedding.

(7)

Abstract setting:

G - a Lie group $g \in G$

$L_g : G \to G$

$h \mapsto gh$ left translation

For any vector $X \in T_1 G$ we define a G-invariant vector field

$\vec{V}(g) = D(L_g)_1 (X)$

(This vector field is left invariant

$D(L_g)_h(\vec{V}(h)) = D(L_g)_h \circ D(L_h)_1(X) =$

$= D(L_g \circ L_h)_1 (X) = D(L_{gh})_1 (X)$)

Proposition There is a bijection

$T_1 G \longleftrightarrow$ left invariant fields

Proposition For any left invariant field $\vec{V}$ the solution of the differential equation

$$(*) \begin{cases} \dot{\gamma}(t) = \vec{V}(\gamma(t)) \\ \gamma(0) = 1 \end{cases}$$ exists for all t

and $\gamma : \mathbb{R}_+ \to G$ is a homomorphism.

Proof A solution exists locally. (In fact $\gamma(t) = \varphi_t(1)$ where φ_t is the flow of $\vec{V}$.)

We want to show that $\gamma(t+s) = \gamma(s) \circ \gamma(t)$ whenever $\gamma(t+s)$ is defined. We will show that $\gamma(s)^{-1} \cdot \gamma(t+s)$ is a solution of

(8) the differential equation (*)

$$\frac{d}{dt}\left(\gamma(s)^{-1}\gamma(t+s)\right) = \left(DL_{\gamma(s)^{-1}}\right)_{\gamma(t+s)} \dot\gamma(t+s)$$

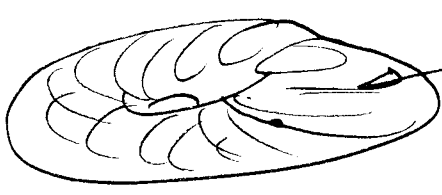
$$= \left(DL_{\gamma(s)^{-1}}\right)_{\gamma(t+s)} \vec{V}(\gamma(t+s)) = \vec{V}\left(\gamma(s)^{-1}\gamma(t+s)\right).$$

By uniqueness of solutions we get the claim.

Now we extend the solution:
if given on a interval $\gamma : (-\varepsilon, \varepsilon) \to G$
we define $\gamma(t) = \gamma(t_1)\cdot\gamma(t_2)\cdots\gamma(t_n)$
where $t_1 + \dots + t_n = t$ and $|t_i| < \varepsilon$.
By the previous calculation the result does not depend on the partition and it satisfies the differential equation. $\boxed{}$

Remark 1 $\gamma : \mathbb{R}_+ \to G$ is called
"1-parameter subgroup" but
it does not have to be a subgroup
- e.g. $\mathbb{R}_+ \to S^1$

it can be a subgroup which is not closed
- eg $\mathbb{R}_+ \to (S^1)^2 = \mathbb{R}^2/\mathbb{Z}^2$
$t \mapsto (t, \lambda t) \bmod \mathbb{Z}^2$
where $\lambda \notin \mathbb{Q}$



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Lie algebras

Lie algebra is a vector space V equipped with a bilinear, antisymmetric operation $[\ ,\ ] : V \times V \to V$

$$[a,b] = -[b,a]$$

satisfying the Jacobi identity

$$[a[b,c]] + [b,[c,a]] + [c[a,b]] = 0$$

The Jacobi identity is equivalent to the Leibniz rule of the operation $[a,-] : V \to V$ with respect to $[\ ,\ ]$, i.e.

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

$$\qquad\qquad\ \ \| \qquad\qquad\qquad \|$$

$$\qquad -[c,[a,b]] \qquad -[b,[c,a]]$$

Example 1   Associative algebra A

$$[\ ,\ ] : A \times A \to A$$

$$a,b \mapsto ab - ba \quad \text{(check the Jacobi identity)}$$

(special case $A = M_{n\times n}(\mathbb{R})$)

Example 2 M - differential manifold (or open set $U \subset \mathbb{R}^n$)

$V =$ C^∞-vector fields

the commutator $[X,Y]$ is a vector field satisfying $\vec{X}(\vec{Y}f) - \vec{Y}(\vec{X}f) = [\vec{X},\vec{Y}]f$

for any $f \in C^\infty(M)$ (C^∞-function)

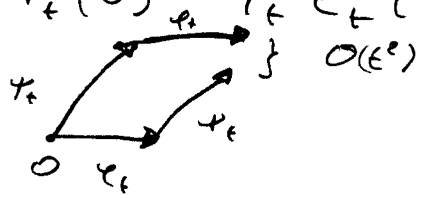
Geometric description of $[X,Y]$:

φ_t flow of $\vec{X}$ $\quad\psi_t$ flow of $\vec{Y}$

assume that $M = U \subset \mathbb{R}^n$ open set

(i.e. fix coordinates)

then $\varphi_t \psi_t(0) - \psi_t \varphi_t(0) = \mathcal{O}(t^2)$



$$[X,Y](0) = \lim_{t\to 0} \frac{\varphi_t \psi_t(0) - \psi_t \varphi_t(0)}{t^2}$$

How to compute commutator in coordinates: $x_1 \dots x_j$? Use the rules

- $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$
- $[X, Y+Z] = [X,Y] + [X,Z]$
- $[X, fY] = X(f)\cdot Y + f[X,Y]$ another Leibniz rule

and (the equivalent one)

- $[fX, Y] = f[X,Y] - Y(f)X.$

Another Leibniz rule

$$X(f\cdot g) = X(f)\cdot g + f\cdot X(g)$$

serves as a definition for derivations in an abstract algebra

$$\mathrm{Der}(A) = \{X \in \mathrm{Hom}(A,A) : \forall a,b \in A \quad X(ab) = X(a)\cdot b + a X(b)\}$$

$$\mathrm{Der}(C^\infty(M)) = \text{vector fields}.$$

Show that if A has a unit then $X(1) = 0$ for all $X \in \mathrm{Der}(A)$

<u>Prop</u> ⊗ if $X, Y \in \mathrm{Der}(A)$ $X, Y : A \to A$
then $X \circ Y - Y \circ X$ is a derivation ☐

<u>Proof</u> Compute $X \circ Y(f \cdot g) - YX(fg)$ by definition ☐

<u>Corollary</u> $\mathrm{Der}(A)$ is a Lie algebra
(sub Lie-algebra of $\mathrm{Hom}(A, A)$)

<u>Remark</u> To define $\mathrm{Der}(A)$ one does not need associativity. The proposition (⊗) also holds.

Example: $\mathrm{Der}(\mathbb{O}) =: \mathfrak{g}_2$
it is an exceptional Lie algebra.

<u>Lie Algebra of a Lie group.</u>

We have identified $T_1 G$ with left invariant vectors. $\lambda \in T_1 G \rightsquigarrow V_\lambda$

- commutator of left invariant vector fields is left invariant (because $[\ ,\ ]$ does not depend on choice of coordinates)
- for a homomorphism $G \xrightarrow{\varphi} H$
  we have a map of tangent spaces $T_1 G \xrightarrow{(D\varphi)_1} T_1 H$

$\varphi_* := (D\varphi)_1$ is a map of Lie algebras

$$\varphi_* [\lambda, \mu] = [\varphi_* \lambda, \varphi_* \mu] \quad \text{for } \lambda, \mu \in T_1 G.$$

Notation

Lie algebra associated to G
is denoted by $L(G)$ or $\mathfrak{g}$ (gothic)

Proposition $\mathfrak{gl}_n(\mathbb{R}) = M_{n\times n}(\mathbb{R})$
$\mathfrak{gl}_n(\mathbb{C}) = M_{n\times n}(\mathbb{C})$ with usual commutator

Proof $\exp(tA)\cdot\exp(tB) - \exp(tB)(\exp tB) =$
$\left(1 + tA + \frac{t^2A^2}{2} + \dots\right)\left(1 + tB + \frac{t^2B^2}{2}\right) - \left(1 + tB + \frac{t^2B^2}{2} + \dots\right)\left(1 + tA + \frac{t^2A^2}{2} + \dots\right)$
$= \left(1 + tA + \frac{t^2A^2}{2} + \dots\right) + tB + t^2AB + \dots - \left(1 + tB + \frac{t^2B^2}{2} + \dots + tA + t^2BA + \dots\right) =$
$= t^2(AB - BA) + \dots$ ☐

Corollary For any matrix group
$\exp(X)$ is the usual $\exp(X) = 1 + X + \frac{X^2}{2} + \dots$