We study complex algebraic varieties, possibly singular.
\[ \dim(X) = n \]
Maps = algebraic maps.

We will define subspaces
\[ \text{IM}_k(X) \subset H_k(X) \]
Name: Image homology.

Homology has coefficients in \( \mathbb{Q} \).

Theorem (AW, Topology 43)
(a) If \( X \) is smooth,
then \( \text{IM}_k(X) = H_k(X) \),
(b) If \( f : X \to Y \) algebraic map,
then \( f_*\text{IM}_k(X) \subset \text{IM}_k(Y) \),
(c) If \( f : X \to Y \) proper, surjective
then \( f_*\text{IM}_k(X) = \text{IM}_k(Y) \),
(d) If \( X \) is complete,
then \( \text{IM}_k(X) = W^kH_k(X) \),
(pure weight subspace)
(e) If \( X \) is equidimensional,
then \( \text{IM}_k(X) = \text{im}(H_k(X) \to H_k(X)) \).

The essential content of the theorem:
\( X \) equidimensional,
\( \tilde{X} \) - a resolution of \( X \),
then \( \text{im}(H_k(\tilde{X}) \to H_k(X)) = \text{im}(H_k(X) \to H_k(X)) \)

Another approach:
Generalization for Borel-Moore homology for noncomplete \( X \).
(Hanamura-M. Saito)

Remarks
\( \text{IM}_*(X) \) well defined for:
- integer coefficients,
- noncomplete varieties,
- complex analytic varieties,
- real varieties*, coefficients \( \mathbb{Z}/2 \)

\[ \text{IM}_k(X) = \text{im}(H^k(\tilde{X}) \to H^k(X)) \]
for any resolution of singularities
\[ \xymatrix{ \text{surj} & H_k(\tilde{X}_3) \ar[r]^{\text{surj}} & H_k(\tilde{X}_2) \ar[r] & H_k(X) } \]
\* the set of real points

Problem:
Give a topological description/estimation of weight filtration.

a relation between weight and Zeeman filtrations
\[ \downarrow \]
only lower bound
\[ \text{IM}_k(X) \supset \text{im}(H^{2n-k}(X) \to H_k(X)) \]
(image of Poincaré map \( [X] \cap - \))

INTERSECTION (CO)HOMOLOGY
Goresky-MacPherson,
Beilinson, Bernstein, Deligne, Gabber
\( \text{IH}_k(X) = H^{2n-k}(X, IC_X) \)
\( IC_X \) constructible sheaf
an object of the derived category
\( IC_{X_{\text{reg}}} = Q_{X_{\text{reg}}} \)
Verdier duality
\( D[IC_X] \cong IC_X[2n] \)
Definition is purely topological:

Stratification $X = \bigsqcup S_n$

$k$-cycle $\xi$ defines a class in $\text{IH}_k(X)$ if

$\dim_{R}(\xi \cap S_n) < k - \text{codim}(S_n)$

for singular strata.

$\text{IH}_k(X)$ is a topological invariant

For $X$ complete:

Finite characteristic analogue

$\text{IH}_k(X_{\mathbb{F}_q})$ is pure

$\text{IH}_k(X)$ can be equipped with a pure

Hodge structure

M. Saito, De Cataldo - Migliorini.

Purity on the level of sheaves

Theorem $\Rightarrow$ Factorization of Poincaré map

$H^k(X) \leq k$

$\downarrow$

$\text{IH}_k(X)/W_{k-1}H^k(X)$

$\text{pure inj.}$

$\downarrow$

$\text{IH}_k^h(X) = \text{IH}_{2n-k}(X)$

$\text{pure surj.}$

$\downarrow$

$W^{2n-k}H_{2n-k}(X)(-n)$

$\text{pure}$

$\downarrow$

$H_{2n-k}(X)(-n) \geq k$

Have to show

$\text{im}(H_k(\tilde{X}) \to H_k(X)) = \text{im}(\text{IH}_k(X) \to H_k(X))$

Proof. $\supset$ for complete $X$ follows from

weight consideration.

In general

$\pi : \tilde{X} \to X$ resolution

$H_k(\tilde{X})$

$\text{IH}_k(\tilde{X})$

$\downarrow$

$\downarrow$

$\text{IH}_k(X) \to H_k(X)$

Level of sheaves on $X$

$R\pi_*Q_X[2n]$

$\text{IC}_X \to \text{R}^\pi_*Q_X$

may be constructed by decomposition theorem.

The map $IC_X \to R\pi_*Q_X$ may be constructed by decomposition theorem.

$IC_X$ is a direct summand of $R\pi_*Q_X$


Any map of varieties can be covered by a map $IC_Y \to \text{Rf}_IC_X$.

Their proof based on a local topological property + reduction to the codimension 1 inclusions.

Recent proof by Hanamura-M. Saito

TOPOLOGICAL LOCAL PROPERTY

$X \subset Y$ a pair of varieties,

codim(X) = 1

a stratum $S$ of codimension $k$ in $X$,

$L_X, L_Y$ links of $S$

$IC_Y \otimes \mathbb{F} \to IC_X \otimes \mathbb{F}$

the (unique) sheaf morphism.

Then the induced map

$IH^k(L_Y) \to IH^k(L_X)$

vanishes.

This property allows to extend the sheaf morphism along $S$.

Weights of link cohomology

$IH^i(L_X)$ has weights $\{ \leq i \text{ for } i < k \}

> i \text{ for } i \geq k$,

$IH^i(L_Y)$ has weights $\{ \leq i \text{ for } i < k + 1 \}

> i \text{ for } i \geq k + 1$.

In the crucial degree

$IH^k(L_X)$ has weights $> k$

$IH^k(L_Y)$ has weights $\leq k$

$IH^k(L_Y) \to IH^k(L_X)$ vanishes
Local topology of analytic varieties is the same as of algebraic varieties:

**THEOREM** (Mostowski (1984)): Every germ of an analytic set is homeomorphic to a germ of an algebraic set. The result can be generalized for pairs. Remains the problem of factorizing a map through a projection and codimension one inclusions.

The opposite inclusion
\[
\text{im}(H_k(\bar{X}) \to H_k(X)) \subset \text{im}(IH_k(X) \to H_k(X))
\]
\[
H_k(\bar{X}) \quad ? \quad \downarrow
\]
\[
IH_k(X) \quad \to \quad H_k(X)
\]

**Level of sheaves on X**
\[
\text{R}^\pi \text{Q}X[2n] \quad \quad ? \quad \downarrow
\]
\[
\text{IC}_X[2n] \quad \to \quad D_X
\]

**Commutativity of the triangle:**
\[
\text{Hom}(\text{R}^\pi \text{Q}X[2n], D_X) = \text{Hom}(\text{Q}X, \text{R}^\pi \text{Q}X) = H^n(\bar{X}).
\]

is determined by the restriction to an open-dense subset.

(topological property, without weight argument)

**Example 1.**
Residue classes
M smooth of dim = n + 1,
X hypersurface.
\[
H^*(M \setminus X) \to H^{*+1}(M, M \setminus X)
\]
\[
\text{res} \quad \cap[M] \quad \downarrow
\]
\[
H_{BM}^{2n+1-*}(X)
\]

Remark. For real varieties IM_*(\ldots, Z/2) is not a topological invariant. Is it invariant with respect to "arc-symmetric maps"?

For smooth X target \(\simeq H^{*-1}(X)\).
holomorphic n+1-form on M\X with 1st order pole along X

$$\text{res}[\omega] \in H_n(X)$$

Also we have a form

$$\text{res}[\omega] \in \Omega^2_{\text{reg}}$$

Also we have a form

$$\pi: \hat{M} \to M \text{ embedded resolution of } X$$

**Proposition:** Suppose

$$\pi^*\omega \in W_{n+2}\Omega^{n+1}_\hat{M}(\log(\pi^{-1}X))$$

(no higher residues). Then \(\pi^*\omega\) has no other poles except those on \(\hat{X}\) and

$$\text{res}[\omega] = \text{im}(\text{res}[\pi^*\omega]) \in \text{IM}_n(X).$$

(e.g. \(X\) has canonical singularities)

Suppose \(X\) is complete.

\(\text{IH}^*_G(X)\) is always pure.

For homology we have Eilenberg-Moore spectral sequence

$$E_2^{p,q} = \text{Tor}^H_{p,q}(H^{BM}_{G,*}(X), Q)$$

converging to \(H_*^*(X)\).

**THEOREM** (M. Franz, AW)

The Eilenberg-Moore spectral sequence preserves weights.

**COROLLARY**

If \(H^{BM}_{G,*}(X)\) is pure, then

$$H_i(X) \simeq \bigoplus_{p+q=i} \text{Tor}^H_{p,q}(H^{BM}_{G,*}(X), Q)$$

and the \(W^k\) space coincides with \(q \geq k\).

**Example 2.**

An algebraic group \(G\) acts on \(X\).

**Equivariant Borel-Moore homology**

$$H^{BM}_{G,*}(X) = H^{-*}(E_G \times_G X, \mathcal{D}X)$$

In many cases \(H^{BM}_{G,*}(X)\) has pure Hodge structure, e.g. when

- \(X\) finitely many orbits
- \((\text{toric varieties, spherical varieties})\)
- there is a stratification of \(X\) admitting fibrations

\(G/H \times A^{d_\alpha} \subset S_n \to X_{\alpha}\)

with \(X_{\alpha}\) smooth, complete.

**How to construct a lift of \(\text{res}[\omega]\) to \(\text{IH}_n(X)\)?**

Construct an action of \(\text{res}[\omega]\) on semialgebraic chains not contained in \(X_{\text{sing}}\)

$$\xi \subset X$$

proper transform \(\xi \subset \bar{X}$$

$$\int_{\xi} \text{res} \pi^*\omega$$

If \(\xi = \partial \eta\) then

$$\bar{\xi} - \bar{\partial} \eta \subset \pi^{-1}X_{\text{sing}}$$

Since \(\dim(X_{\text{sing}}) \leq n-1\)

$$\int_{\xi} \text{res} \pi^*\omega = 0$$

The same: a way lift to \(H^n(X)\)