TOPOLOGICAL PROPERTIES

OF THE WEIGHT FILTRATION

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We study complex algebraic varieties, possibly singular.

$\dim(X) = n$

Maps = algebraic maps.

We will define subspaces

$$IM_k(X) \subset H_k(X)$$

Name: Image homology.

Homology has coefficients in $\mathbb{Q}$*.*

Theorem (AW, Topology 43)

(a) If X *is smooth,*
then $IM_k(X) = H_k(X)$,

(b) If $f : X \to Y$ *algebraic map,*
then $f_* IM_k(X) \subset IM_k(Y)$,

(c) If $f : X \to Y$ *proper, surjective*
then $f_* IM_k(X) = IM_k(Y)$,

(d) If X *is complete,*
then $IM_k(X) = W^k H_k(X)$,
(pure weight subspace)

(e) If X *is equidimensional,*
then $IM_k(X) = \mathrm{im}(IH_k(X) \to H_k(X))$.

The essential content of the theorem:

X *equidimensional,*
$\widetilde{X}$ *- a resolution of* X*,*
<u>then</u>

$$\mathrm{im}(H_k(\widetilde{X}) \to H_k(X)) = \mathrm{im}(IH_k(X) \to H_k(X))$$

Annother approach:
Generalization for Borel-Moore homology for noncomplete X*.*
(Hanamura-M. Saito)

WEIGHT FILTRATION IN HOMOLOGY

X *complete*

$W^k H_k(X) \subset W^{k-1} H_k(X) \subset \ldots \subset H^k(X)$

$W^i H_k(X) =$ *annihilator of* $W_{i-1} H^k(X)$

$W^k H_k(X)$ *pure Hodge structure dual to* $H^k(X)/W_{k-1} H^k(X)$*.*

$W^k H_k(X) = \mathrm{im}(H^k(\widetilde{X}) \to H^k(X))$
for any resolution of singularities

$$\begin{array}{ccccc} & \text{surj} & H_k(\widetilde{X}_3) & \text{surj} & \\ & \swarrow & & \searrow & \\ H_k(\widetilde{X}_1) & & & & H_k(\widetilde{X}_2) \\ & \searrow & & \swarrow & \\ & & H_k(X) & & \end{array}$$

Remarks

$IM_*(X)$ *well defined for:*

- *integer coefficients,*
- *noncomplete varieties,*
- *complex analytic varieties,*
- *real varieties*, coefficients* $Z/2$

* *the set of real points*

Problem:

Give a topological description/
/estimation of weight filtration.

McCrory (1984), F. Guillen (1987):
a relation between weight and Zeeman filtrations

$\Downarrow$

only lower bound

$IM_k(X) \supset \mathrm{im}(H^{2n-k}(X) \to H_k(X))$

(image of Poincaré map $[X] \cap -$*)*

INTERSECTION (CO)HOMOLOGY

Goresky-MacPherson,
Beilinson, Bernstein, Deligne, Gabber

$IH_k(X) = \mathbb{H}^{2n-k}(X, IC_X)$

IC_X *constructible sheaf*
an object of the derived category

$IC_{X_{reg}} = \mathbb{Q}_{X_{reg}}$

Verdier duality

$D(IC_X) \simeq IC_X[2n]$

Definition is purely topological:

Stratification $X = \coprod S_\alpha$

k-*cycle* ξ *defines a class in* $IH_k(X)$ *if*
$\dim_{\mathbf{R}}(\xi \cap S_\alpha) < k - \mathrm{codim}_{\mathbf{C}}(S_\alpha)$
for singular strata.

$IH_*(X)$ *is a topological invariant*

For X *complete:*

Finite characteristic analogue
$IH_*(X_{\overline{\mathbf{F}}_q})$ *is pure*

$IH_*(X)$ *can be equipped with a pure Hodge structure*
M. Saito,
De Cataldo - Migliorini.

Purity on the level of sheaves

Theorem⇒Factorization of Poincaré map

$$\begin{array}{lcc}
 & H^k(X) & \le k \\
 & \downarrow & \\
 & H^k(X)/W_{k-1}H^k(X) & \text{pure} \\
\text{inj.} & \downarrow & \\
 & IH^k(X) = IH_{2n-k}(X) & \text{pure} \\
\text{surj.} & \downarrow & \\
 & W^{2n-k}H_{2n-k}(X)(-n) & \text{pure} \\
 & \downarrow & \\
 & H_{2n-k}(X)(-n) & \ge k
\end{array}$$

Have to show
$\mathrm{im}(H_k(\widetilde{X}) \to H_k(X)) =$
$\mathrm{im}(IH_k(X) \to H_k(X))$

Proof. $\supset$ *for complete* X *follows from weight consideration.*

In general
$\pi : \widetilde{X} \to X$ *resolution*

$$\begin{array}{ccc}
 & & H_k(\widetilde{X}) \\
 & ?\nearrow & \downarrow \\
IH_k(X) & \longrightarrow & H_k(X)
\end{array}$$

Level of sheaves on X

$$\begin{array}{ccc}
 & & R\pi_*\mathbf{Q}_{\widetilde{X}}[2n] \\
 & ?\nearrow & \downarrow \\
IC_X[2n] & \longrightarrow & D_X
\end{array}$$

Commutativity of the triangle:
$\mathrm{Hom}(IC_X[2n], D_X) =$
$\mathrm{Hom}(\mathbf{Q}_X, IC_X) = \mathbf{Q}$
for irreducible X.

(topological property, without weight argument)

The map $IC_X \to R\pi_*\mathbf{Q}_{\widetilde{X}}$ *may be constructed by decomposition theorem.*

IC_X *is a direct summand of* $R\pi_*\mathbf{Q}_{\widetilde{X}}$

Another argument: Barthel-Brasselet-Fieseler-Gabber-Kaup (1995).
Any map of varieties can be covered by a map $IC_Y \to Rf_*IC_X$.

Their proof based on a local topological property + reduction to the codimension 1 inclusions.

Recent proof by Hanamura-M. Saito

TOPOLOGICAL LOCAL PROPERTY

$X \subset Y$ *a pair of varieties,*
$\mathrm{codim}(X) = 1$
a stratum S *of codimension* k *in* X,
L_X, L_Y *links of* S

$$IC_{Y \setminus \overline{S}} \to IC_{X \setminus \overline{S}}$$

the (unique) sheaf morphism.

<u>Then</u> *the induced map*

$$IH^k(L_Y) \to IH^k(L_X)$$

vanishes.

This property allows to extend the sheaf morphism along S.

Weights of link cohomology

$IH^i(L_X)$ has weights $\begin{cases} \le i \text{ for } i < k \\ > i \text{ for } i \ge k\,, \end{cases}$

$IH^i(L_Y)$ has weighs $\begin{cases} \le i \text{ for } i < k+1 \\ > i \text{ for } i \ge k+1\,. \end{cases}$

In the crucial degree
$IH^k(L_X)$ *has weights* $> k$
$IH^k(L_Y)$ *has weights* $\le k$

$$IH^k(L_Y) \to IH^k(L_X)$$

vanishes

Local topology of analytic varieties is the same as of algebraic varieties:

THEOREM (Mostowski (1984)):
Every germ of an analytic set is homeomorphic to a germ of an algebraic set.

The result can be generalized for pairs.

Remains the problem of factorizing a map through a projection and codimension one inclusions.

The opposite inclusion

$$\mathbf{im(H_k(\widetilde{X}) \to H_k(X)) \subset}$$
$$\mathbf{\subset im(IH_k(X) \to H_k(X))}$$

$$\begin{array}{ccc} & & \mathbf{H_k(\widetilde{X})} \\ & ?\swarrow & \downarrow \\ \mathbf{IH_k(X)} & \longrightarrow & \mathbf{H_k(X)} \end{array}$$

Level of sheaves on $\mathbf{X}$

$$\begin{array}{ccc} & & \mathbf{R\pi_* Q_{\widetilde{X}}[2n]} \\ & ?\swarrow & \downarrow \\ \mathbf{IC_X[2n]} & \longrightarrow & \mathbf{D_X} \end{array}$$

Commutativity of the triangle:

$$\mathbf{Hom(R\pi_* Q_{\widetilde{X}}[2n], D_X) =}$$
$$\mathbf{= Hom(Q_X, R\pi_* Q_{\widetilde{X}})}$$
$$\mathbf{= H^0(\widetilde{X}).}$$

is determined by the restriction to an open-dense subset.

(topological property, without weight argument)

$$\mathbf{X \to Y}$$

The sheaf map $\mathbf{IC_Y \to IC_X}$ *is not unique, but can be made canonical.*

$$\begin{array}{ccc} \mathbf{E} & \overset{\text{codim }1}{\subset} & \mathbf{Bl_{graph(f)} X \times Y} \\ {\scriptstyle \text{projective}}\downarrow & & {\scriptstyle \text{projective}}\downarrow \\ \mathbf{graph(f)} & \subset & \mathbf{X \times Y} \\ \| & & \downarrow \\ \mathbf{X} & & \mathbf{Y} \end{array}$$

COROLLARY

The bottom term of the weight filtration for complete variety

$$\mathbf{W^k H_k(X)}$$

is topologically invariant.

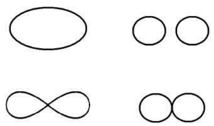
OPEN QUESTIONS

- *For analytic varieties both* IH_* *and* IM_* *(the image of* $\mathrm{H}^*(\widetilde{\mathrm{X}})$*) well defined.* *From the factorization on p.20 and the decomposition theorem for projective morphism [M. Saito] it follows** *that*

  IM_* *is the image of* IH_*

* *Added after the conference.*

- *For real varieties* $\mathrm{IM}_*(-, \mathbf{Z/2})$ *well defined. How can be characterized?*



Remark. *For real varieties* $\mathrm{IM}_*(-, \mathrm{Z}/2)$ *is not a topological invariant.*
Is it invariant with respect to "arc-symmetric maps"?
$\sim$ *maps preserving analytic arcs.*
(Parusiński)

Example 1.
Residue classes

$\mathbf{M}$ *smooth of* $\dim = \mathbf{n+1}$,
$\mathbf{X}$ *hypersurface.*

$$\begin{array}{ccc} \mathbf{H^*(M \setminus X)} & \to & \mathbf{H^{*+1}(M, M \setminus X)} \\ & {\scriptstyle \mathbf{res}}\searrow & {\scriptstyle \cap[\mathbf{M}]}\downarrow \\ & & \mathbf{H^{BM}_{2n+1-*}(X)} \end{array}$$

For smooth $\mathbf{X}$ *target* $\simeq \mathbf{H^{*-1}(X)}$

ω holomorphic $n+1$-form on $M \setminus X$ with 1^{st} order pole along X

$$\mathrm{res}[\omega] \in H_n(X)$$

Also we have a form

$$\mathrm{res}\,\omega \in \Omega^n_{X_{reg}}.$$

$\pi : \widetilde{M} \to M$ embedded resolution of X

Proposition: *Suppose*

$$\pi^*\omega \in W_{n+2}\Omega^{n+1}_{\widetilde{M}}(\log(\pi^{-1}X))$$

(no higher residues). Then $\pi^*\omega$ has no other poles except those on $\widetilde{X}$ and

$$\mathrm{res}[\omega] = \mathrm{im}(\mathrm{res}[\pi^*\omega]) \in IM_n(X).$$

(e.g. X has canonical singularities)

Note: $\mathrm{res}\,\omega \in L^2\Omega^n_{X_{reg}}$

$$\|\omega\| = C \int_{X_{reg}} \omega \wedge \overline{\omega}$$

does not depend on the metric

How to construct a lift of $\mathrm{res}[\omega]$ to $IH_n(X)$?

Construct an action of $\mathrm{res}\,\omega$ on semialgebraic chains not contained in X_{sing}

$$\xi \subset X$$
$$\downarrow$$
$$\text{proper transform } \tilde{\xi} \subset \widetilde{X}$$
$$\downarrow$$
$$\int_{\tilde{\xi}} \mathrm{res}\,\pi^*\omega$$

If $\xi = \partial\eta$ then

$$\tilde{\xi} - \partial\tilde{\eta} \subset \pi^{-1}X_{sing}$$

Since $\dim(X_{sing}) \le n-1$

$$\int_{\tilde{\xi}} \mathrm{res}\,\pi^*\omega = 0$$

The same: a way lift to $H^n(X)$

Example 2.

An algebraic group G acts on X.

Equivariant Borel-Moore homology

$$H^{BM}_{G,q}(X) = \mathbb{H}^{-q}(EG \times_G X; D_X)$$

In many cases $H^{BM}_{G,*}(X)$ has pure Hodge structure, e.g. when

- *X finitely many orbits (toric varieties, spherical varieties)*
- *there is a stratification of X admitting fibrations*

$$G/H \times A^{d_\alpha} \subset S_\alpha \twoheadrightarrow X_\alpha$$

with X_α smooth, complete.

Suppose X is complete.

$IH^*_G(X)$ is always pure.

It is a free module over $H^*(BG)$

$$IH^*_G(X) \simeq H^*(BG) \otimes IH^*(X)$$

Then

$$IH^*(X) = IH^*_G(X) \otimes_{H^*(BG)} Q$$

For homology we have Eilenberg-Moore spectral sequence

$$E_2^{p,q} = \mathrm{Tor}^{H^*(BG)}_{p,q}(H^{BM}_{G,*}(X), Q)$$

converging to $H_*(X)$.

THEOREM *(M. Franz, AW)*

The Eilenberg-Moore spectral sequence preserves weights.

COROLLARY

If $H^{BM}_{G,*}(X)$ is pure, then

$$H_i(X) \simeq \bigoplus_{p+q=i} \mathrm{Tor}^{H^*(BG)}_{p,q}(H^{BM}_{G,*}(X), Q)$$

and the W^k space coincides with

$$q \ge k.$$

Eilenberg-Moore spectral sequence for IH reduces to the first column.

Transformation of spectral sequences

$\mathrm{Tor}(IH^*_G(X), Q)$

*	0	0	0	0	0
*	0	0	0	0	0
*	0	0	0	0	0
*	0	0	0	0	0
*	0	0	0	0	0
*	0	0	0	0	0
*	0	0	0	0	0

$\rightarrow$

$\mathrm{Tor}(H^{BM}_{G,*}(X), Q)$

*	0	0	0	0	0
*	0	0	0	0	0
*	*	0	0	0	0
*	*	0	0	0	0
*	*	*	0	0	0
*	*	*	0	0	0
*	*	*	*	0	0

Surjection on the first column.

The weight filtration is the limit of the Eilenberg-Moore spectral sequence.

COROLLARY

The weight filtration is determined by topology of X and the action of G.

More results for integral homology of toric varieties.