Leray spectral sequence for arrangements

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Spectral sequence argument quite often appears in study the hyperplane arrangements or configuration spaces (see for example [2, 3, 4, 9, 10, 15]). Sometimes the spectral sequence can be identified with the Leray spectral sequence of the embedding $j : U \hookrightarrow X$ of an open subset to a compact complex variety

$$E_2^{p,q} = H^p(X; R^q j_* \mathcal{L}) \Rightarrow H^{p+q}(U; \mathcal{L}).$$

(1)

When the coefficient system has geometric nature, then the entries of the spectral sequence are equipped with the weight filtration. For the constant coefficients $\mathbb{Q}_U$ it happens that the entries of $E_2$ table are of pure weight. It implies the degeneration of the spectral sequence on $E_3$. The argument of purity can be applied in a much more general setup. It was already noticed by Totaro [15] that if $X = Y^n$ for $Y$ a smooth compact algebraic variety and $U$ is the set of tuples consisting of pairwise distinct points, i.e. when $U$ is the configuration space of $n$ points in $Y$, then the Leray spectral sequence has only one nontrivial differential.

The concept of weight is central for this paper. Talking about weight filtration in the cohomology of complex algebraic variety one obviously uses Deligne construction [5, 6]. While dealing with sheaves there are (at least) two approaches possible. It is natural to work in the category of mixed Hodge modules developed by M. Saito [12]. Nevertheless we feel more comfortable using the reduction to a $\mathbb{Q}_\ell$-sheaf over a variety over a finite characteristic field, [1]. If we say that a constructible sheaf of $\mathbb{Q}$ vector spaces is pure of certain weight we mean that after tensoring with $\mathbb{Q}_\ell$ and after a good reduction to a finite characteristic $p$ with $(p, \ell) = 1$, the sheaf is pure in the sense of [1, §5.1] (i.e. in the sense of the action of the Frobenius automorphism on stalks and costalks).

Many results about arrangements of hyperplanes can be obtained purely topologically, one can say ”elementarily”. Sometimes the weight argument

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is hidden somewhere in the references. Our goal is to remind some basic mechanism and extend a bit the situation it can be applied. The weight argument works for cohomology with rational coefficients. Some of the results might hold for integer homology, but the weight argument is too weak to handle torsion.

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1 The results

We will consider the following situation. Let $X$ be a smooth compact algebraic variety. Let $r$ be a natural number. We define an admissible collection of codimension $c$ subvarieties inductively.

**Definition 2** A collection of distinct varieties $\mathcal{Z} = \{Z_i\}_{i=1,2,\ldots,s}$ is admissible of codimension $c$ in $X$ if

- $\dim(X) = c$, each $Z_i$ is a point or
- each $Z_i$ is smooth of codimension $c$ and for any set of indices $I \subset \{1,2,\ldots,s\}$ and $j \notin I$ the union of the set-theoretic intersections
  \[ \bigcup_{i \in I} Z_i \cap Z_j \]
  is the union of an admissible codimension $c$ collection in each $Z_j$

Each collection of smooth subvarieties satisfying the following condition is admissible:

- for any set of indices $I \subset \{1,2,\ldots,s\}$ the set-theoretic intersection
  \[ \bigcap_{i \in I} Z_i \]
  is a smooth subvariety of codimension $c\ell$ for some $\ell \in \mathbb{N}$.

Note that in [13, §3] there were considered hyperplanelike divisors, i.e. collection of hyperplanes which locally look like hyperplane arrangements. The hyperplanelike divisors satisfy the above condition with $c = 1$. There are more complicated examples, such as the collection of three curves $x = 0$, $x = y^2$ and $y = 0$ on the plane. Another example of admissible divisors is the collection of two surfaces in $\mathbb{C}^3$ given by the equations $z = 0$ and $z = xy$.

We will prove the theorem:

**Theorem 3** Let $\mathcal{Z} = \{Z_i\}_{i=1,2,\ldots,s}$ be an admissible collection of codimension $c$ cycles in $X$. Then the Leray spectral sequence of the inclusion

\[ j : U = X \setminus \bigcup_{i \in \{1,2,\ldots,s\}} Z_i \rightarrow X \]
\[ E_2^{p,q} = H^p(X; R^q j_* \mathbb{Q}_U) \Rightarrow H^{p+q}(U; \mathbb{Q}) \, . \]

has all differentials \( d_r, r \geq 2 \) vanishing except from \( d_{2c} \).

The result follows from the local computation

**Theorem 4** With the notation as above the sheaves \( R^k j_* \mathbb{Q}_U = 0 \) for \( k \) not divisible by \( 2c - 1 \) and for \( k = (2c - 1)\ell \) the sheaf \( R^k j_* \mathbb{Q}_U \) is pure of weight \( 2c\ell = \frac{2c}{2c-1} k \).

With further work one can show

**Theorem 5** With the notation as above the sheaf \( R^{(2c-1)\ell} j_* \mathbb{Q}_U = 0 \) is isomorphic to a direct sum of constant sheaves supported by smooth subvarieties of codimensions \( c\ell \).

## 2 Ideal situation: normal crossing divisor

Let us start with recalling the Deligne spectral sequence defining the weight filtration in the cohomology of the normal crossing divisor complement \( X \setminus D \).

The the sheaf of meromorphic forms on \( X \) with logarithmic poles along \( D \) admits the weight filtration defined by the number of factors in denominators. The iterated residue defines an isomorphism from the graded pieces of the filtration to the sheaves of forms on the intersections of the divisors. One obtains a spectral sequence allowing to compute the cohomology of \( X \setminus D \) from the cohomologies of the intersections. The main theorem says that the spectral sequence degenerates on \( E_2 \), hence the only thing one has to know is the first differential.

Let us be more precise. We fix the notation: Let \( X \) be a compact smooth complex algebraic variety. We assume that \( D \subset X \) is a smooth divisor with normal crossings i.e. the irreducible components of \( D = \bigcup_{i=1}^m D_i \) are smooth and locally \( D \) is given by the equation \( z_1 z_2 \ldots z_k = 0 \) in a certain coordinate system. For a multiindex \( I = \{i_1, i_2, \ldots, i_\ell\} \) define \( X_I = D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_\ell} \) and \( X^\ell = \bigsqcup_{|I| = \ell} X_I \) with \( X^0 = X \). Deligne considers the spectral sequence associated to the weight filtration in the complex of logarithmic forms [5, (3.2.4.1)]. It converges to \( H^*(X \setminus D) \). The first table of the weight spectral sequence lies in the second quarter and has the form

\[ W E_1^{p,q} = H^{q-2p}(X^p) \, , \]

[5, (3.2.7)]. Here is the example for the surface case:

\[
\begin{array}{cccccc} 
0 & \rightarrow & H^0(X^2) & \rightarrow & H^2(X^1) & \rightarrow & H^4(X^0) \, ^4 \\
0 & \rightarrow & H^1(X^1) & \rightarrow & H^3(X^0) \, ^3 \\
0 & \rightarrow & H^0(X^1) & \rightarrow & H^2(X^0) \, ^2 \\
0 & \rightarrow & H^1(X^0) \, ^1 \\
0 & \rightarrow & H^0(X^0) \, ^0 \\
\end{array} 
\]

\[
\begin{array}{c}
-3 \\
-2 \\
-1 \\
0 \\
\end{array}
\]
The differential $Wd_1$ is the alternating sum of Gysin maps

$$H^*(X_I) \to H^{*+2}(X_{I \setminus \{i\}})$$

induced by the inclusions $X_I \hookrightarrow X_{I \setminus \{i\}}$. Since we care about the Hodge structure we should rather write $H^*(X^I)(-1) \to H^{*+2}(X^{I \setminus \{i\}})$, where $V(-1)$ denotes the Tate twist rising the weight by 2. The differential

$$Wd_1 : H^*(X^p)(-p) \to H^{*+2}(X^{p-1})(1-p)$$

might be nontrivial. The weight principle says: the maps between Hodge structures of distinct weights vanish. Therefore the second differential of the spectral sequence is the zero map and the weight spectral sequence degenerates on $W^2E_2$, [5, (3.2.10)]. Therefore

$$Gr^W_kH^k(U) = H^k(0 \leftarrow H^k(X^0) \leftarrow H^k(X^1)(-1) \leftarrow H^k(X^2)(-2) \leftarrow \ldots ) .$$

As a matter of fact the weight spectral sequence after a reorganization of indices coincides with the Leray spectral sequence of the inclusion $j : X \setminus D \hookrightarrow X$. We have

$$R^qj_*\mathbb{Q}_U = \bigoplus_{|I|=q} \mathbb{Q}_{X_I}(-q) .$$

The Leray spectral sequence in this case has the following second table

$$E_{2}^{p,q} = H^p(X^q)(-q) .$$

Here is the picture for the surface case

<table>
<thead>
<tr>
<th></th>
<th>$H^0(X^2)(-2)$</th>
<th>$H^1(X^1)(-1)$</th>
<th>$H^2(X^0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
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<td>$H^0(X^0)$</td>
<td>$H^1(X^0)$</td>
<td>$H^2(X^0)$</td>
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<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

The second differential

$$d_2 : E_2^{p,q} = H^p(X^q)(-q) \to E_2^{p+2,q-1} = H^{p+2}(X^{q-1})(1-q)$$

preserves the weight: both entries have weight $p + 2q$. Checking directly the definitions of both differentials we find, that $d_2 = Wd_1$. According to the weight principle the Leray spectral sequence degenerates on $E_3$, [5, (3.2.13)].

**Remark 6** Let $\mathcal{L}$ be an unitary local system on a normal crossing divisor complement. The degeneration of the Leray spectral sequence (1) was proven in [14].
The goal of this paper is to show that a similar degeneration happens for an admissible collection of subvarieties and cohomology are taken with constant coefficients, i.e. Theorem 3. In the final section we will briefly discuss a possible degeneration for nontrivial local systems, but we are bounded by the category in which our object live. We will assume that \( L \) is of geometric origin to be able to talk about purity.

3 Proofs

Proof of Theorem 4. We will prove a statement a bit stronger: the sheaves \( R^k j_\ast Q_U \) are pure and pointwise pure (compare the definition of purity [1, 5.1.8]). We proceed inductively with respect to the number of elements of the collection \( Z \) and the dimension of \( X \). If \( Z = \{Z\} \), then there is a long exact sequence

\[
\to H^{k-1}(U) \to H^{k-2c}(Z)(-c) \to H^k(X) \to H^k(U) \to ,
\]

which localizes to an exact sequence of sheaves

\[
\to R^{k-1} j_\ast Q_U \to \mathcal{H}^{k-2c}(Q)_Z(-c) \to \mathcal{H}^k(Q)_X \to R^k j_\ast Q_U \to ,
\]

Therefore \( R^0 j_\ast Q_U = Q_X \) and \( R^{2c-1} j_\ast Q_U = Q_Z(-c) \). The derived images vanish in the remaining degrees. In particular the theorem holds when the dimension of \( Z_i \) is zero. Assume now that the theorem holds for all families in varieties of dimension smaller then \( n = \dim(X) \). We also assume that the theorem holds for families consisting of \( s - 1 \) elements. Consider the Mayer-Vietoris exact sequence for the pair \( U_1 = X \setminus Z_1, U_2 = X \setminus \bigcup_{i > 1} Z_i \):

\[
\to H^{k-1}(U) \to H^k(U_1 \cup U_2) \to H^k(U_1) \oplus H^k(U_2) \to H^k(U) \to
\]

which can be localized to the long exact sequence of sheaves

\[
\to R^{k-1} j_\ast Q_U \to R^{k} j'_\ast Q_{U_1 \cup U_2} \to R^k j_1 \ast Q_{U_1} \oplus R^k j_2 \ast Q_{U_2} \to R^k j_\ast Q_U \to ,
\]

where \( j_1, j_2, j' \) are inclusions of the sets \( U_1, U_2 \) and \( U_1 \cup U_2 \) into \( X \). Note that

\[
R^k j'_\ast Q_{U_1 \cup U_2} = \begin{cases} Q_X & \text{for } k = 0, \\ 0 & \text{for } 0 < k \leq 2c, \\ R^{k-2c} j''_\ast Q_{Z_1 \cap U_2}(-c) & \text{for } k > 2c, \end{cases}
\]

where \( j'' : Z_1 \cap U_2 \to X \) is the inclusion. Topologically this formula can be justified by the following: locally, i.e. if \( X \) was a ball, then \( U_1 \cup U_2 \) would be the \( 2c \)-fold suspension of \( Z_1 \cap U_2 \). To compute the weight we notice that for \( k > 2c, x \in Y = Z_1 \cap \bigcup_{i > 1} Z_i \) by Poincaré duality in \( X \)

\[
\left( R^k j'_\ast Q_{U_1 \cup U_2} \right)_x = \mathcal{H}_{2n-k}(Y)_x(n),
\]

5
where \( \mathcal{H} \) is the sheaf of local homology. Again by Poincaré duality in \( Z_1 \), which is of dimension \( n - c \),

\[
\mathcal{H}_{2n-k}(Y)_x(n) = \left( R^k j''_* \mathbb{Q}_{Z_1 \cap U_2} \right)_x (-c).
\]

Using the inductive hypothesis we find that \( R^k j'_* \mathbb{Q}_{U_1 \cup U_2} \) is nonzero only if \( k - 1 \) is divisible by \( 2c - 1 \) and the weight of \( R^{(2c-1)\ell+1} j'_* \mathbb{Q}_{U_1 \cup U_2} \) is equal to \( 2c\ell \). If \( c > 1 \) the map \( R^k j'_* \mathbb{Q}_{U_1 \cup U_2} \to R^k j_{1*} \mathbb{Q}_{U_1} \oplus R^k j_{2*} \mathbb{Q}_{U_2} \) vanishes by dimensional reasons. For \( c = 1 \) we use the weight argument: the source is of weight \( 2k - 1 \), the target is of weight \( 2k \). We conclude that we obtain a short exact sequence of sheaves of weight \( \frac{2c}{2c-1} k \),

\[
0 \to R^k j_{1*} \mathbb{Q}_{U_1} \oplus R^k j_{2*} \mathbb{Q}_{U_2} \to R^k j_* \mathbb{Q}_U \to R^{k+1} j'_* \mathbb{Q}_{U_1 \cup U_2} \to 0.
\]

By inductive assumption the edge sheaves are pure and pointwise pure. Therefore the middle one is pure and pointwise pure. \( \square \)

**Proof of Theorem 5.** Again by induction we find that the sheaf \( R^{(2c-1)\ell} j_* \mathbb{Q}_U \) fits to an exact sequence

\[
0 \to \bigoplus_{\alpha} \mathbb{Q}_{A_\alpha}(c\ell) \to R^{(2c-1)\ell} j_* \mathbb{Q}_U \to \bigoplus_{\beta} \mathbb{Q}_{B_\beta}(c\ell) \to 0.
\]

Here \( A_\alpha \) and \( B_\beta \) are families of smooth varieties of codimensions \( c\ell \). By [1, 5.4.6] \( R^{(2c-1)\ell} j_* \mathbb{Q}_U \) decomposes into the sum of constant sheaves. (In the case of collection homeomorphic to a vector space arrangement see [8, III §3.7] for a topological proof.) \( \square \)

**Proof of Theorem 3.** We follow the Totaro argument [15, §4] (see also his remark at the beginning of p.1062). If a sheaf \( S \) is pure of weight \( w \), the base space is compact, then the cohomology \( H^p(X; S) \) is pure of weight \( p + w \) ([1, 5.1.13]). By Theorem 4 the entries of the Leray spectral sequence \( E^p_q = H^p(X; R^q j_* \mathbb{Q}_U) \) are pure of weight \( p + 2c \frac{q}{2c-1} \) for \( q \) divisible by \( 2c - 1 \). The remaining rows of the \( E_2 \) table are zero. The first possible differential is

\[
d_{2c} : E^{p, (2c-1)\ell}_2 \longrightarrow E^{p+2c, (2c-1)\ell-2c+1}_2,
\]

that is

\[
d_{2c} : H^p(X; R^{(2c-1)\ell} j_* \mathbb{Q}_U) \longrightarrow H^{p+2c}(X; R^{(2c-1)(\ell-1)} j_* \mathbb{Q}_U). \]

The weights of both entries are equal to \( p + 2c\ell = (p + 2c) + 2c(\ell - 1) \) and the differential \( d_{2c} \) can be nontrivial. Further differentials hit the entries \( E^{p+k,(2c-1)\ell-k+1}_p \) which are subquotients of \( H^{p+k}(X; R^{(2c-1)\ell-k+1} j_* \mathbb{Q}_U) \) and have weights

\[
(p + k) + \frac{2c}{2c-1}((2c - 1)\ell - k + 1) = p + 2c\ell + \frac{2c - k}{2c - 1} \leq p + 2c\ell
\]

for \( k > 2c \). Therefore the higher differentials vanish. \( \square \)
4 Twisted coefficients

We would like to conclude with some remarks concerning the cohomology with coefficients in a local system. We restrict our attention to the collection of divisors \((c = 1)\), when the local monodromy might be nontrivial. When \(L\) is a local system on \(U = X \setminus D\) the Leray spectral sequence relates the local invariants of intersections of divisors with the global cohomology. Consider the case of a collection of hyperplanelike divisors, i.e. we assume as in [13] that in some local analytic coordinates the divisors are given by linear equation. Under the condition that:

- \((\text{Mon})\) \: \text{dim}(\mathcal{L}) = 1 \text{ and for each } x \in D \text{ the product of the monodromies associated to the divisors passing through } x \text{ is not equal to one.}

one obtains the vanishing of local cohomology

\[
(R^k j_* \mathcal{L})_x = H^k(U \cap B_{x,\epsilon}; \mathcal{L}) = 0
\]

for any \(k\) at any point belonging to \(D\), see [3, Lemma 5.2]. (Here \(B_{x,\epsilon}\) is a sufficiently small ball centered in \(x\).) For abelian local systems of higher rank the generalization of \((\text{Mon})\) would demand that the products of all possible eigenvalues are not equal to one. If \((\text{Mon})\) or an appropriate analogue for higher rank systems is satisfied then the full derived push forward reduces to the push forward with compact supports

\[
Rj_* \mathcal{L} = j!* \mathcal{L}.
\]

It follows that the Leray spectral sequence reduces to the bottom row, which is

\[
E^{p,0}_2 = H^p(X; j!* \mathcal{L}) = H^p(X, D; j_* \mathcal{L}).
\]

Hence

\[
H^p(U; \mathcal{L}) = H^p(X, D; j_* \mathcal{L}).
\]

This is essentially the argument e.g. of [4, Theorem 5.1].

Now suppose that \(\mathcal{L}\) is enriched, so that it belongs to the category studied in [1] (e.g. is of geometric origin, \([\text{loc.cit. 6.2.4}]\)). Furthermore suppose that \(\mathcal{L}\) is pure of weight \(w\). Then \([\text{loc.cit. 5.1.13}]\) \(Rj_* \mathcal{L}\) is of weight \(\geq w\) and \(Rj! \mathcal{L}\) is of weight \(\leq w\). Therefore if the condition \((\text{Mon})\) is satisfied, then \(Rj_* \mathcal{L} = j!* \mathcal{L}\) is pure. Under assumption that \(X\) is compact the cohomology group \(H^*(X; Rj_* \mathcal{L})\) are pure. The Hodge decomposition of these groups were studied in [11] for a collection of planes in \(\mathbb{P}^3\).

If a different condition is satisfied \((\text{Mon})^*\) of [13, Theorem 4.1] (see also [7]), then the local cohomology of \((R^k j_* \mathcal{L})_x = H^k(U \cap B_{x,\epsilon}; \mathcal{L})\) is generated by logarithmic forms \(df_i/f_i\), where \(f_i\) is an equation of a divisor component. Then we can only conclude that
Claim 7 Under the assumptions as above \((R^k j_* \mathcal{L})\) is pointwise pure of weight \(w + 2k\).

Since it is a sum locally constant sheaves supported by the intersections of divisor components it is also pure. We do not prove the claim here, since it would demand different techniques. Having that for granted we proceed as in the proof of Theorem 4. The Leray spectral sequence can have a nontrivial second differential, but the higher differentials vanish. Therefore

Corollary 8 Let \(D \subset X\) be a hyperplanelike divisor in a compact algebraic variety and \(\mathcal{L}\) be a locally constant sheaf on \(U = X \setminus D\). Suppose that \(\mathcal{L}\) is of geometric origin. Further assume that the sheaves \(R^k j_* \mathcal{L}\) are pure of weight \(w + 2k\). Then

\[
Gr^W_{w+k+\ell} H^k(U; \mathcal{L}) = H^k(X; R^0 j_* \mathcal{L}) \leftarrow H^{k-2}(X; R^1 j_* \mathcal{L}) \leftarrow H^{k-4}(X; R^2 j_* \mathcal{L}) \leftarrow \ldots.
\]

Possibly the result can be extended to the case of unitary local systems considered in [14].

References


