Equivariant cohomology in algebraic geometry: notes 2023

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1.1 Prehistory: Poincaré-Hopf theorem. Suppose M is a manifold, v a vector field with isolated zeros, then

$$\chi(M) = \sum_{p \in Zeros} Ind_p(v)\,,$$

where $Ind_p(v)$ is the index of the vector field, i.e. the degree of the map from a small sphere around p $S(p,\epsilon)$ to the unit sphere in T_pM given by $v(p)/||v(p)||$.

1.2 Suppose a circle S^1 acts smoothly on M with isolated fixed points. Let v be the fundamental field of the action, i.e.

$$v(x) = \tfrac{d}{dt}(t \cdot x)_{|t=0}\,.$$

Then if $p \in M^{S^1}$ the index $Ind_p(v) = 1$. Hence

$$\chi(M) = |M^{S^1}|\,.$$

This statement is true in a much greater generality.

1.3 Let X be a simplicial complex (or any decent compact topological space, e.g. a manifold). Suppose p is a prime number. Let P be a p-group acting on X. Then the Euler characteristic of fixed points $\chi(X^P) \equiv \chi(X) \mod p$.
Proof: We assume that P acts simplicially and the relation follows from the property of p groups acting on finite sets: $|X^P| \equiv |X| \mod p$.

1.4 Exercise: give a proof for compact manifolds, not using triangulations.
[Sören Illman, Smooth equivariant triangulations of G-manifolds for G a finite group. Math. Ann.233(1978), no.3, 199–220.]
See a far-reaching generalization: Dwyer–Wilkerson Smith theory revisited. Ann. of Math. (2) 127 (1988), no. 1, 191–198.

1.5 Corollary: no decent compact contractible space admits a finite group action without fixed points.

1.6 Theorem does not hold for infinite dimensional spaces, e.g. $\mathbb{Z}_2$ acts on $S^\infty \sim pt$ without fixed points (action via antipodism).

1.7 Theorem: Let X be a compact (decent) compact topological space (e.g. a manifold). Suppose $\mathbb{T} = (S^1)^r$ acts on X. Then $\chi(X) = \chi(X^{\mathbb{T}})$.
Proof: $X^{S^1} = X^{\mathbb{Z}_{p^\infty}} = X^{\mathbb{Z}_{p^n}}$ for $n >> 0$.

Examples of the spaces with torus action.

1.8 $X = S^{2n+1} \subset \mathbb{C}^{n+1}$ with $S^1 \subset \mathbb{C}$ action via scalar multiplication. (No fixed points, $\chi(X) = 0$.)

1.9 The projective space $\mathbb{P}^n = \mathbb{CP}^n = \mathbb{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ can be presented as S^{2n+1}/S^1.

1.10 $X = S^{2n} \subset \mathbb{C}^n \times \mathbb{R}$ with $S^1 \subset \mathbb{C}$ acting on the factor $\mathbb{C}^n$. ($\chi(X) = 2$, two fixed points.)

1.11 Projective space $\mathbb{P}^n$ (in particular $\mathbb{P}^1 = S^2$) admits the action of $\mathbb{T}_\mathbb{C} = (\mathbb{C}^*)^{n+1}$. There are $n+1$ fixed points. Also the small torus consisting of the sequences $(1, t, t^2, \dots, t^n)$ has the same fixed points. We check directly that $\chi(\mathbb{P}^n) = n+1$.
[For holomorphic actions does not matter whether we take compact torus S^1 or $\mathbb{C}^*$. The fixed points are the same.]

Białynicki-Birula decomposition by examples.

1.12 Let $X = \mathbb{P}^n$,

$$T = \{(1, t, t^2, \dots, t^n) \in \mathbb{T}_\mathbb{C} \mid t \in \mathbb{C}^*\}$$

acting as above. For $p \in X^T$ let

$$X_p^+ = \{z \in X \mid \lim_{t \to 0} t \cdot z = p\}.$$

The sets $X_p^\pm$ are homeomorphic (isomorphic as algebraic varieties) with affine spaces. We obtain the well known decomposition of the projective space

$$\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \mathbb{C}^{n-2} \sqcup \dots \sqcup \mathbb{C}^0 .$$

$$X^+_{[0:0:\dots:\overset{k}{1}:0:\dots:0]} = \{z_k \neq 0,\ z_\ell = 0 \text{ for } \ell < k\} \simeq \mathbb{C}^{n-k}$$

1.13 The quadric $z_0 z_3 - z_1 z_2 = 0$ in $\mathbb{P}^3$ with the $T = \mathbb{C}^*$ action as above.

$$Q_{[1,0,0,0]} = \{[1 : z_1 : z_2 : z_1 z_2] \mid z_1, z_2 \in \mathbb{C}\} \simeq \mathbb{C}^2$$

$$Q_{[0,1,0,0]} = \{[0 : 1 : 0 : z_3] \mid z_3 \in \mathbb{C}\} \simeq \mathbb{C}$$

$$Q_{[0,0,1,0]} = \{[0 : 0 : 1 : z_3] \mid z_3 \in \mathbb{C}\} \simeq \mathbb{C}$$

$$Q_{[0,0,0,1]} = \{[0 : 0 : 0 : 1]\} \simeq pt$$

1.14 Theorem [Białynicki-Birula 1973] Let X be a complex projective algebraic variety with algebraic $T = \mathbb{C}^*$ action. For a component $F \subset X^T$ let

$$X_p^+ = \{z \in X \mid \lim_{t \to 0} t \cdot z \in F\}.$$

(1) Then

$$X = \bigsqcup_F X_F^+$$

(the sum over connected components) is a decomposition into locally closed algebraic subsets.
(2) The limit map

$$p_F = \lim_{t \to 0} : X_F^+ \to F$$

is an algebraic map. If X is smooth then p_F is a Zariski-locally trivial fibration with the fiber isomorphic to $\mathbb{C}^{n_F}$.

(3) The number n_F is the rank of $\nu_F^+ \subset \nu_F$, the subbundle of the normal bundle on which T acts with positive weights.

• The field $\mathbb{C}$ can be replaced by any algebraically closed field.

1.15 Note that existence of the limit $\lim_{t\to 0} t\cdot z$ follows from the fact that the closure of the orbit is an algebraic curve. The map

$$\alpha_z : \mathbb{C}^* \to \mathbb{P}^1 \times X$$

$$t \mapsto (t, t\cdot z)$$

extends to a map from $\mathbb{P}^1$. To see that one can note that the image of $\mathbb{C}^*$ is a constructible algebraic set (by Tarski-Seidenberg theorem), hence the closure is an algebraic curve, dominated by $\mathbb{P}^1$. Hence we have a unique extension of α_z

$$\bar\alpha_z : \mathbb{P}^1 \to \mathbb{P}^1 \times X \stackrel{\pi}{\to} X$$

and

$$\lim_{t\to 0} t\cdot z := \pi(\bar\alpha_z(0))\,.$$

• If the action is not algebraic, the above argument does not work: $\mathbb{C}^*$ acts transitively on any elliptic curve, there are no fixed points.

2 Basics about actions of compact groups

2.1 Let $\mathbb{T} = (S^1)^r \subset \mathbb{C}^r$ and $\mathfrak{t} = i\mathbb{R}^r \subset \mathbb{C}^r$. The map exp coordinatewise induces the exact sequence

$$0 \longrightarrow \mathfrak{t}_\mathbb{Z} \longrightarrow \mathfrak{t} \stackrel{\exp}{\longrightarrow} \mathbb{T} \longrightarrow 0,$$

where $\mathfrak{t}_\mathbb{Z} = 2\pi i\mathbb{Z}^r \subset i\mathbb{R}^r = \mathfrak{t}$ is the kernel, also denoted by N

2.2 Weights and characters. See [Anderson-Fulton, Ch. 3,§1]

• Homomorphisms $\mathrm{Hom}(\mathbb{T}, S^1)$ are called „characters". This set is a group with respect to multiplication pointwise. It is isomorphic to $\mathbb{Z}^r$. In toric geometry denoted by M.

• any character in coordinates is of the form

$$(t_1, t_2, \dots, t_r) \mapsto t_1^{w_1} t_2^{w_1} \dots t_r^{w_r} \qquad \text{denoted by } t^w.$$

• the sequence $(w_1, w_2, \dots, w_r) \in \mathbb{Z}^r$ is the called weight.

2.3 Without coordinates:

$$\text{Weights } = \mathrm{Hom}(N, \mathbb{Z})$$

In toric geometry $\mathrm{Hom}(N, \mathbb{Z})$ is denoted by M, in representation theory $\mathfrak{t}^*_\mathbb{Z}$.

$$\begin{array}{ccccc} & \mathfrak{t}_\mathbb{Z} & \stackrel{weight}{\longrightarrow} & 2\pi i\mathbb{Z} & \simeq \; \mathbb{Z} \\ & \cap & & \cap & \\ & \mathfrak{t} & \longrightarrow & i\mathbb{R} & \\ \exp & \downarrow & & \downarrow & \\ & \mathbb{T} & \stackrel{character}{\longrightarrow} & S^1 & \end{array}$$

For a weight $w \in \mathfrak{t}_\mathbb{Z}$ the corresponding character is denoted by e^w.

2.4 For the complex torus $\mathbb{T}_\mathbb{C} \simeq (\mathbb{C}^*)^r$ any polynomial map is determined by the values on $\mathbb{T} \simeq (S^1)^r$

$$\mathrm{Hom}_{alg}(\mathbb{T}_\mathbb{C}, \mathbb{C}^*) = \mathrm{Hom}(\mathbb{T}, S^1)\,.$$

2.5 Linear actions of $\mathbb{T}$ one a vector space $\mathbb{C}^n$ can be diagonalized
(Commuting linear maps of finite order have a common diagonalization.)

2.6 Exercise: for any field $\mathbb{F} = \overline{\mathbb{F}}$ any linear action of $\mathbb{T}_\mathbb{F} = (\mathbb{F}^*)^r$ on $\mathbb{F}^n$ can be diagonalized.

2.7 Up to an isomorphism any linear action of $\mathbb{T}$ on a complex vector space is determined by the multi-set of weights.

- Let $\mathbb{C}_w$ be equal to $\mathbb{C}$ as a vector space with the action of $\mathbb{T}$ via $e^w : \mathbb{T} \to S^1 \subset \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C})$
- If $\mathbb{T}$ has fixed coordinates, i.e. it is identified with $(S^1)^r$ and $w = (w_1, w_2, \dots, w_r)$ then for $t \in \mathbb{T}$ the linear map $e^w(t) : \mathbb{C}_w \to \mathbb{C}_w$ is the multiplication by $t_1^{w_1} t_2^{w_2} \dots t_r^{w_r}$.
- We have a canonical decomposition

$$V = \bigoplus_{w \in M} V_w,$$

where $V_w = \{v \in V \mid \forall t \in \mathbb{T}\ t \cdot v = e^w(t)v\} \simeq \mathrm{Hom}_\mathbb{T}(\mathbb{C}_w, V)$ is the eigenspace (called *weight space*) corresponding to the weight w.

- For a vector bundle $E \to B$, with torus action such that $\mathbb{T}$ acts on B trivially and on the fiber the action is linear we have a decomposition into a direct sum of subbundles $E = \bigoplus_w E_w$.
- The decomposition into weight subspaces can be noncannonically refined

$$V = \bigoplus_{k=1}^{\dim V} \mathbb{C}_{w_k}.$$

(Note: If we have fixed coordinates of $\mathbb{T}$, then each w_k is a sequence of numbers $(w_{k,1}, w_{k,2}, \dots, w_{k,r})$.)

- The element

$$e(V) = \prod_{k=1}^{\dim V} w_k = \prod_w w^{\dim V_w} \in Sym^{\dim V}(\mathfrak{t}^*_\mathbb{Z})$$

does not depend on the above decomposition and it is called the Euler class of the representation.

- The product

$$c(V) = \prod_{k=1}^{\dim V} (1 + w_k) = \prod_w (1 + w)^{\dim V_w} \in Sym(\mathfrak{t}^*_\mathbb{Z})$$

is also well defined. It is called the Chern class of the representation

- After tensoring with $\mathbb{R}$ (or $\mathbb{Q}$) we can identify $Sym(\mathfrak{t}^*_\mathbb{Z}) \otimes \mathbb{R}$ with polynomial functions on $\mathfrak{t}$.

2.8 Exercise: for a representation V of $\mathbb{T}$ consider an action of $\tilde{\mathbb{T}} = \mathbb{T} \times S^1$ on $\tilde{V} = V$, where S^1 acts by the scalar multiplication. Denote by $\hbar$ the weight corresponding to the character $\tilde{T} \to S^1$, which is the projection. Show that

$$c(V) = e(\tilde{V})_{|\hbar=1}\,.$$

Action of a compact group (in particular torus) on a manifold

2.9 Exercise: (algebraic geometry) Let A be an algebra over a field $\mathbb{F}$ and $X = Spec(A)$. Defining an action of $\mathbb{G}_m = Spec(\mathbb{F}[t, t^{-1}])$ on X is equivalent to defining a $\mathbb{Z}$-gradation of A. Prove this correspondence and generalize it to an action of the algebraic torus $\mathbb{G}_m^r$.

2.10 Let X be a manifold with a smooth action of $\mathbb{T}$. Suppose $x \in X^{\mathbb{T}}$ is a fixed point. Then $\mathbb{T}$ acts on T_xX. If x is an isolated fixed point, then the weight space $(T_xX)_0$ corresponding to the weight $w = 0$ is trivial.

2.11 Proposition. There exists a neighbourhood $x \in U \subset T_xX$ and an equivariant map $f : U \to X$, which is an isomorphism on the image.
Proof: Fix an S^1 invariant metric, take U to be the ball of a sufficiently small radius, $f = \exp$ in the sense of the differential geometry.

2.12 Reminder: Orbit, stabilizer(=isotopy group): Suppose a group G acts on X, $x \in X$
- the stabilizer $= G_x = \{g \in G \mid gx = x\}$.
- if $y = gx$ then $G_y = gG_xg^{-1}$
- the orbit $= G \cdot x \simeq G/G_x$.
- the isotropy group G_x acts on the tangent space T_xX and the fiber of the normal bundle $(\nu_{G\cdot x})_x$

2.13 Construction of the associated bundle: Suppose V be a representation of a group H, and suppose P be a H-principal bundle. Let us define

$$P \times^H V = P \times V / \{(ph, v) \sim (p, hv)\} .$$

The projection $P \times^H V \to P/H = Y$ is a vector bundle.
For the definition and basic facts about principal bundles [Anderson-Fulton, Ch.2.1]

2.14 Slice theorem for manifolds: Assume that X is a smooth manifold, G a compact Lie group (can assume a torus) acting smoothly. Let $V = (\nu_{G\cdot x})_x$. There exist an equivariant neighbourhood of $0 \in S \subset V$, such that the map $G \times^{G_x} S \to X$ induced by $\exp : G \times^{G_x} V \to X$ is an equivariant diffeomorphism onto the image. This image is a neighbourhood of $G \times^{G_x} \{0\} \simeq G \cdot x$. The set S or its image is called the slice, whole neighbourhood is called the tube. See [Anderson-Fulton, Ch.5 Th.1.4].
- In other words: any orbit has a neighbourhood isomorphic to the disk bundle of the associated vector bundle over the orbit.
- Proof. The map $\exp : \mathbb{T} \times V \to X$ induces

$$(g, v) \mapsto g \cdot \exp(v) .$$

Exp isG_x-invariant, i.e. $\exp(g\cdot v) = g\cdot\exp(v)$ for $g \in G_x$. Hence the above map factorizes $G \times^{G_x} V \to X$.
- Exercise: Show that the above map is well defined.

2.15 Exercise: Let G be a group, H a subgroup, $E \to G/H$ be a vector bundle with G-action, such that for any $g \in G$, $x \in G/H$ the map $g : E_x \to E_{gx}$ is linear. Show that $E \simeq G \times^H E_{[e]}$. Here $[e]$ denotes the coset eH.

2.16 There is a more general theorem for topological spaces:
– If X is a topological space (completely regular), G a compact Lie group, then a slice V is a certain subspace of X, invariant with respect to G_x. [Bredon, Introduction to Compact Transformation Groups. Section II.5]

– In algebraic geometry [Luna slice theorem] we assume that G is reductive ($(\mathbb{C}^*)^r$ is fine, $\mathrm{GL}_n(\mathbb{C})$ too) X is an affine variety, and the orbit is closed. The neighbourhood is in the étal topology. [Luna, Domingo (1973), *Slices étales*, Sur les groupes algébriques, Bull. Soc. Math. France, Paris, Mémoire, vol. 33]

3 Classifying spaces

3.1 It is convenient to introduce a notion of G-CW-complex. By definition, we assume that X admits a filtration

$$X_{-1} = \emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_N$$

such that

$$X_i = X_{i-1} \cup_\phi (G \times^H D^{n_i}),$$

where D^{n_i} is the unit disk of a linear orthogonal representation of $H \to Aut(\mathbb{R}^{n_i})$,

$$\phi : G \times^H S^{n_i - 1} \to X_{i-1}.$$

(with weak topology.)

3.2 Any smooth action of a compact Lie group G on a compact manifold admits a G-CW-decomposition.

3.3 Example: S^2 with the standard S^1 action has 3 cells 0, ∞ and $S^1 \times D^1$.

3.4 Exercise: find a CW-decomposition of $\mathbb{P}^n$ with the standard action of $(S^1)^{n+1}$

3.5 The topological spaces we study will be assumed to admit a G-CW-decomposition

3.6 Equivariant cohomology of a G space:

• topological model $H^*_G(X) = H^*(EG \times^G X)$, where EG is a contractible free G-space (unfortunately in almost all cases EG is of infinite dimension)

• differential model if X is a G-manifold $H^*_{G,dR}(X) = H^*(\Omega^*(X, G))$

• de Rham theorem $H^*_G(X; \mathbb{R}) \simeq H^*_{G,dR}(X)$

3.7 We will assume, that G is compact (or linear algebraic reductive, e.g. $(\mathbb{C}^*)^r$).

3.8 A G bundle $P \to B = E/G$ is universal if for any G bundle $P' \to B'$ there exist a map $f : B' \to B$ such that $F^*(P) = P'$. Moreover f is unique up to homotopy.

• Hence

$$\{G\text{-bundles on } X\} = [X, B]$$

where $[X, B]$ means homotopy classes of maps (X is assumed to be CW-complex).

3.9 We will show that a universal G-bundle exists.

• Notation $EG \to BG$, should be understood as a homotopy type, which has various realizations.

• A G bundle $P \to B$ is universal if and only if E is contractible.

• Proof: Assume that P is contractible. Suppose $P' \to B'$ be an arbitrary G-bundle. We construct a mapping by induction on skeleta. We assume that P' is a CW-complex, glued from cells with trivial stabilizers, i.e. each cell is of the form $D^n \times G$.

$$\begin{array}{ccccc} S^{n-1} \times G & \longrightarrow & D^n \times G & \dashrightarrow & EG \\ \downarrow & & \downarrow & & \downarrow \\ S^{n-1} & \longrightarrow & D^n & \dashrightarrow & BG \end{array}$$

it is enough to construct a mapping $S^{n-1} \times \{1\} \to P$ do $D^n \times \{1\} \to EG$ and use G-action to spread the definition on the whole tube $D^n \times G$. Similarly we construct a homotopy between two maps.

Hence if P is contractible then it is universal. If we have another bundle $P' \to B'$ which is universal, then there are G maps $P' \to P$ and $P \to P'$ and their compositions are homotopic to identities (this is a general nonsens about universal objects).

3.10 Corollary: by the homotopy exact sequence for $G \subset EG \to BG$ we have homotopy group isomorphism $\pi_k(BG) \simeq \pi_{k-1}(G)$. In particular, if G is connected, then BG is 1-connected.

3.11 Since any nontrivial compact Lie group contains torus, hence elements of finite orders, the space EG cannot be of finite dimension (by Euler characteristic argument).

3.12 Examples:
$ES^1 = S^\infty \to \mathbb{P}^\infty = BS^1$ (of the type $K(\mathbb{Z}, 2)$)
$E(S^1)^r = (S^\infty)^r \to (\mathbb{P}^\infty)^r = B(S^1)^r$
$BU(n) = \lim_{N\to\infty} Gras_n(\mathbb{C}^N)$

3.13 For $G = \mathbb{T}$ or $U(n)$ one can approximate BG by compact algebraic manifolds, which admit a decomposition into algebraic cells (BB-decomposition's).

3.14 For all linear algebraic groups $G \subset \mathrm{GL}_m(C)$ we can take EG =Steel manifold

$$St_m(\mathbb{C}^N) := \text{Monomorphisms}(\mathbb{C}^m, \mathbb{C}^N) \subset \text{Hom}(\mathbb{C}^m, \mathbb{C}^N)$$

See [Anderson-Fulton, Ch.2, Lemma 2.1]

• Exercise: Show that

$$\lim_{N\to\infty} \text{codim}(\text{Hom}(\mathbb{C}^m, \mathbb{C}^N) \setminus St_m(\mathbb{C}^N)) = \infty\,.$$

• For any algebraic group Totaro constructs approximation of BG by algebraic varieties in a more systematic way.

3.15 If $H \subset G$, then as a model for EH we can take EG. Hence we get a fibration $G/H \to BH \to BG$.

3.16 If $H \lhd G$ is a normal subgroup, $K = G/H$ then there is a fibration $BH \to BG \to BK$.
(Take $EH := EG$ and $E'G = EG \times EK$, taking the fibration $E'G/G \to EK/K$ we find that the fiber is $EG \times^G G/H = BH$.)

3.17 Characteristic classes for G-bundles [see e.g. Guillemn-Sternberg §8] Consider two contravariant functors:

$$Gbdl := \{G - bundles\}/\sim : hTop \to sets$$

$$H := H^*(- \, \mathbb{Z}) : hTop \to sets$$

$$Map_{Functors}(Gbdl, H) = H^*(BG; \mathbb{Z})$$

• This is just Yoneda Lemma: if $F, H : \mathcal{C} \to \mathcal{S}\rceil\sqcup\int$ and F is representable by $A \in Ob(\mathcal{C})$, i.e.

$$F(X) = Mor_{\mathcal{C}}(X, A)\,,$$

then

$$Mor_{Functors}(F, H) = F(A)\,.$$

Given a transformation of functors

$$\alpha : Mor_{\mathcal{C}}(-, A) \to H(-)$$

We construct an element in $H(A)$ setting $X = A$

$$\alpha \mapsto \alpha(Id_A) \in H(A)\,.$$

Conversely: given $f : X \to A$ and $\alpha \in H(A)$ define

$$\alpha(f) = f^*(\alpha).$$

3.18 Characteristic classes for n-dimensional vector bundles.

• Each vector bundle is determined by its associated principal bundle. Thus $Vect_n(X) = [X, BGL_n(\mathbb{C})]$ and $BGL_n(\mathbb{C}) = BU_n$. Hence

$$\text{characteristic classes of n-vector bundles} = H^*(BU(n))$$

• $H^*(BU(n), \mathbb{Z}) \simeq \mathbb{Z}[c_1, c_2, \ldots, c_n]$

• The map $H^*(BU(n+1)) \to H^*(BU(n))$ is surjective given by $c_{n+1} := 0$.

3.19 For the torus we have

• $G = \mathbb{C}^*$, $EG = \mathbb{C}^\infty \setminus \{0\}$; $B\mathbb{C}^* = \mathbb{P}^\infty = \bigcup_n \mathbb{P}^n$

• $H^*(B\mathbb{C}^*) \simeq \mathbb{Z}[t]$, it is convenient to take $t = c_1(\mathcal{O}(1))$, where $\mathcal{O}(1)$ is the dual of the tautological bundle.

• For S^1 we can take $ES^1 = S^\infty = \bigcup_n S^{2n-1}$

3.20 Corollary:

$$\{\text{topological vector bundles over } X\} \simeq H^2(X; \mathbb{Z})$$

$$\{\text{characteristic classes of line bundles}\} = H^*(\mathbb{P}^\infty) = \mathbb{Z}[t]$$

3.21 For $\mathbb{T} = (S^1)^n$:

$$H^*(B\mathbb{T}) = \mathbb{Z}[t_1, t_2, \ldots, t_n]$$

3.22 The inclusion $\mathbb{T} \to U(n)$ induces $B\mathbb{T} \to BU(n)$ and $H^*(BU(n)) \to H^*(\mathbb{T})$ which is injective

$$H^*(BU(n)) = \mathbb{Z}[c_1, c_2, \ldots, c_n] = \mathbb{Z}[t_1, t_2, \ldots, t_n]^{S_n} \hookrightarrow \mathbb{Z}[t_1, t_2, \ldots, t_n] = H^*(B\mathbb{T})$$

Compare [Anderson-Fulton, Ch2, Proposition 4.1]

3.23 The above statement and many others in this course follows from **Leray-Hirsch theorem**:

• Let $F \to E \to B$ be a fibration. Assume that $H^*(F)$ is free (in our case over $\mathbb{Z}$). Suppose there is a linear map $\phi : H^*(F) \to H^*(E)$, a splitting of the restriction map $H^*(E) \to H^*(F)$. Then $H^*(E)$ is a free module over $H^*(B)$.

3.24 We have the bundle $E = B\mathbb{T} \to BU_n = B$ the fiber is $F = U_n/\mathbb{T}$. The base and the fiber (F =Flag manifold) admit a cell decompositions into even dimensional cells — see explanation below. Hence we have a cell decomposition of $E\mathbb{T}$ which is compatible with the decomposition of the base. (Note that here as a model of $E\mathbb{T}$ is not taken S^∞.)

• Hence $H^*(E) \to H^*(F)$ is split-surjective.

By the Leray-Hirsh theorem $H^*(B\mathbb{T})$ is a free $H^*(BU_n)$–module of the rank $\dim H^*(F)$,

• $H^*(F) \simeq H^*(E)/(H^{>0}(B))$ as algebras (also we can write $H^*(F) \simeq \mathbb{Z} \otimes_{H^*(B)} H^*(E)$)

3.25 We look at the cell decomposition of the approximation $Gras_n(\mathbb{C}^n)$ of $BU(n)$ (see [Anderson-Fulton, Ch. 4, §5]

• The cells are indexed by the sequences

$$0 < i_1 < i_2 < \ldots i_k \leq n$$

$$\begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \quad i_1 = 1,\ i_2 = 3$$

Equivalently

$$(n - k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0) = \text{number of * in the reduced form of the matrix.}$$

3.26 Computation of $H^*(BU(n))$. The map $H^*(BU(n)) \to H^*(B\mathbb{T})$ is injective. The image is invariant with the symmetric group action S_n, since each permutation $\sigma : \mathbb{T} \to \mathbb{T} \to U_n$ is homotopic to the inclusion.

• First we give an argument over $\mathbb{Q}$. We show that in each gradation $\dim H^{2k}(BU_n) = \dim \mathbb{Q}[t_1, t_2, \ldots, t_n]^{S_n}$.

– $\dim H^{2k}(BU(n))$ = number of sequences $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ (no restriction on λ_1), such that $\sum_i \lambda_i = k$

– $\dim H^{2k}(B\mathbb{T})^{S_n} = \mathbb{Z}[t_1, t_2, \ldots, t_n]^{S_n}_k$= the number of monomials with non-increasing exponents.

• We conclude that $H^{2k}(BU(n); \mathbb{Q}) = H^{2k}(B\mathbb{T}; \mathbb{Q})^{S_n}$

• Moreover $H^*(Fl(n); \mathbb{Z}) = \mathbb{Z}[t_1, t_2, \ldots, t_n]/(H^{>0}(BU_n; \mathbb{Z}))$ is torsion-free. Hence $H^*(BU_n; \mathbb{Z}) = \mathbb{Z}[t_1, t_2, \ldots, t_n]^{S_n}$

3.27 Corollary: We have a description of the cohomology ring

$$H^*(Fl(n)) \simeq \mathbb{Z}[t_1, t_2, \ldots, t_n]/(\mathbb{Z}[t_1, t_2, \ldots, t_n]^{S_n}_{>0})\,.$$

3.28 Exercise: Compute the cohomology ring $H^*(Gras(k,n))$ using the fibration $Gras_k(\mathbb{C}^n) \to B(U_k \times U_{n-k}) \to BU_n$.

3.29 General theorem: if G is connected, $\mathbb{T}$ maximal torus, $W = N\mathbb{T}/\mathbb{T}$ the Weyl group, then $H^*(BG;\mathbb{Q}) = H^*(B\mathbb{T};\mathbb{Q})^W$ is a polynomial ring in the variables of even degrees, e.g.

• $H^*(BSp(n);\mathbb{Q}) = \mathbb{Q}[c_2, c_4, \dots, c_{2n}]$, (valid also over $\mathbb{Z}$),

• $H^*(BO_{2n};\mathbb{Q}) = \mathbb{Q}[p_1, p_2, \dots, p_n, e]/(e^2 = p_n)$, $\deg p_i = 4i$, $\deg e = 2n$ (valid also over $\mathbb{Z}[\frac{1}{2}]$)

• BE_8 is he worst, one has to invert $2, 3, 5$. The generators of $H^{2*}(BE_8)$ are in the degrees $2\times$: 2, 8, 12, 14, 18, 20, 24, 30.

[Burt Totaro: The torsion index of E_8 and other groups, Duke Math. J. 129 (2005), no. 2, 219–248]

4 Recollection on Chern classes

What you need to know about Chern classes

4.1 Let $Vect_1$ denote the functor $hTop \to Sets$

$$Vect_1(X) = \text{Isomorphism classes of line bundles over } X$$

• This functor factors through the category of abelian groups (tensor product of line bundles behaves like addition).

• $Vect(X)$ denotes isomorphism classes of vector bundles. This is a semi-ring. Here $\oplus$ is the addition, $\otimes$ is the multiplication.

4.2 The first Chern class

$$c_1 \in Mor_{Functors}(Vect_1, H^2(-,\mathbb{Z})) = H^2(K(\mathbb{Z},2)) = H^2(BS^1) = H^2(\mathbb{P}^\infty) = H^2(\mathbb{P}^1)$$

We chose the generator of $H^2(\mathbb{P}^1)$ so that $c_1(\mathcal{O}(1)) = [pt]$. Here the bundle $\mathcal{O}(1) = \gamma^*$ is the dual of the tautological bundle.

• In other words: the Chern class c_1 is determined by the choice made for $\mathcal{O}(1)$.

4.3 Chern classes of vector bundles: $c(E) = 1 + c_1(E) + \dots + c_{rk(E)}(E)$.

– functoriality (c is a transformation of functors $Vect(-) \to H^*(-,\mathbb{Z})$

– for line bundles $c(L) = 1 + c_1(L)$

– Whitney formula $c(E \oplus F) = c(E)\, c(F)$

• Note c is not a group homomorphism. One can repair that, but has to use $\mathbb{Q}$ coefficients. The resulting transformation is called Chern character. For line bundles

$$ch(L) = \exp(c_1(L))\,.$$

Chern character is additive and multiplicative

$$ch(E \oplus F) = ch(E) + ch(F)\,,$$

$$ch(E \otimes F) = ch(E)\, ch(F)\,.$$

4.4 If L is a holomorphic line bundle over a complex manifold, with a meromorphic section s, then $c_1(L)$ is equal to Poincaré dual of $Zero(s) - Poles(s)$.

4.5 *Projective bundle theorem.* For a vector bundle $E \to B$ let $\mathbb{P}(E) \to B$ be the projectivization[1], $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ the tautological line bundle, then $H^*(\mathbb{P}(E))$ is a free module over $H^*(B)$

$$h^r + a_1 h^{r-1} + \cdots + ra_r = 0.$$

Then $a_i = c_i(E)$.

• There are other conventions of signs, but let's check: If E is a line bundle, then $L = E^*$. We have relation $h + a_1 = c_1(L) + c_1(E) = 0$.

4.6 Corollary: Chern classes of E and the ring structure of $H^*(B)$ determine the ring structure

$$H^*(\mathbb{P}(E)) = H^*(B)[h]/(h^r + c_1 h^{r-1} + \cdots + c_{r-1}h + c_r).$$

4.7 Splitting principle: for any line bundle $E \to B$ there exists $f : B' \to B$ such that, f^*E is a sum of line bundles and f^* is injective on cohomology. E.g.

$$B' = Flags(E) = B \times_{BU(n)} B\mathbb{T}\,,$$

where $\mathbb{T}$ is the maximal torus in $U(n)$.

4.8 The generator of $H^2(B\mathbb{C}^*)$ is identified with $c_1(\mathcal{O}(1))$. Thus the generators of

$$H^*_T(B\mathbb{T}) = \mathbb{Z}[t_1, t_2, \ldots, t_n]$$

can be presented as

$$t_i = c_1(L_i)\,,$$

where $L_i = E\mathbb{T} \times^{\mathbb{T}} \mathbb{C}_{t_i}$ is the line bundle associated to the representation of T in $\mathrm{GL}_1(\mathbb{C})$ given by the projection oh the i-th factor.

4.9 Let $\chi : \mathbb{T} \to \mathbb{C}^*$ be a character, then $c_1(ET \times^{\mathbb{T}} \mathbb{C}_\chi) = \chi$. Here we identify

$$\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) = \mathfrak{t}^* = H^2(B\mathbb{T})\,.$$

Borel's definition of equivariant cohomology [finally, see [Anderson-Fulton, Ch.2 §2]]

4.10 Borel construction $X_G = EG \times^G X$ sometimes is called the mixing space.

4.11 Basic properties:

• It is a module over $H^*_G(pt) = H^*(BG)$

• Contravariant functoriality with respect to X i G.

• If the action is free then $X_G \to X/G$ is a fibration with the contractible fiber EG, hence $H^*_G(X) = H^*(X/G)$. [Anderson-Fulton, Ch 3, §4]

• For $K \subset G$, $X = G/H$ we have $X_G = EG \times^G G/K \simeq EG/K = BK$.

• More generally $H^*_G(G \times^K X) \simeq H^*_K(X)$ for any K-space X..

• If the action is trivial then $X_G = BG \times X$. If $H^*(BG)$ has no torsion (e.g. $G = T$, $\mathrm{GL}_n(\mathbb{C})$, $Sp_n(\mathbb{C})$) then $H^*_G(X) = H^*(BG) \otimes H^*(X)$. For coefficients in $\mathbb{Q}$ we do need the assumption about the torsion. [Anderson-Fulton, Ch 3, §4]

[1]this is the naive projectivization, i.e. the fiber over $x \in B$ consist of the lines in E_x.

4.12 Basic properties of equivariant cohomology of smooth compact algebraic varieties: (G connected, coefficients of cohomology in $\mathbb{Q}$)

• (*) $H^*_G(X)$ is a free module over $H^*(BG)$ hence $H^*_G(BG) \simeq H^*(BG) \otimes H^*(X)$, the information of the action of G is hidden in the multiplication,

• $H^*_{\mathbb{T}}(X) \to H^*_{\mathbb{T}}(X)^T$ is injective.

4.13 Example: [Anderson-Fulton, Ch.2, §6] $\mathbb{P}^n$ with the standard action of $\mathbb{T} = (\mathbb{C}^*)^{n+1}$. We identify $X_{\mathbb{T}}$ with $\mathbb{P}(\bigoplus_{i=0}^n \mathbb{C}_{t_i})$. By the projective bundle theorem

$$H^*_{\mathbb{T}}(\mathbb{P}^n) = \mathbb{Z}[t_0, t_2, \dots, t_n, h]/(\prod_{i=0}^n (t_i + h))\,.$$

• It is a free module over $H^*_T(pt) = H^*(B\mathbb{T}) = \mathbb{Z}[t_0, t_2, \dots, t_n]$

• The map to $H^*(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{n+1})$ is a surjection.

• We have an isomorphism of modules over $H^*(B\mathbb{T})$

$$H^*_{\mathbb{T}}(\mathbb{P}^n) \simeq H^*(B\mathbb{T}) \otimes H^*(\mathbb{P}^n).$$

We will see that for compact smooth algebraic varieties (or Kähler) the above holds always over $\mathbb{Q}$.

• The map

$$H^*_{\mathbb{T}}(\mathbb{P}^n) \to H^*_{\mathbb{T}}((\mathbb{P}^n)^{\mathbb{T}}) = \bigoplus_{i=0}^n H^*_{\mathbb{T}}(pt) = \bigoplus_{i=0}^n \mathbb{Z}[t_0, t_1, \dots, t_n]$$

by

$$[f(\underline{t}, h)] \;\mapsto\; \{f_i\}_{i=0,1,\dots,n}\,, \qquad f_i(\underline{t}) = f(\underline{t}, -t_i)\,.$$

Exercise: this map is injective.

4.14 Example: $\mathbb{T} = \mathbb{C}^*$ acting on $\mathbb{P}^1 \simeq S^2$ via $[t^\ell z_0, t^k z_1]$

$$X_{\mathbb{T}} = \mathbb{P}(\mathcal{O}(\ell) \oplus \mathcal{O}(k))$$

,

$$H^*_{\mathbb{T}}(\mathbb{P}^1) = \mathbb{Z}[h, t]/((h + kt)(h + \ell t)$$

• The elements 1 and h generate over $\mathbb{Z}[t] = H^*(BT)$. This is a free module
[We have $h^2 = -(k+\ell)th - k\ell t^2$, so any polynomial in t and h can be written modulo the ideal $(h^2 + ht)$ as $f_0(t) + f_1(t)h$.]

• The restriction to the fixed points

$$[f(t, h)] \mapsto (f(t, -\ell t), f(t, -kt))\,.$$

is injective.
[If $f(t, -kt) = 0$, then f is divisible by $h + kt$ ....]

4.15 Let $\mathbb{T} = \mathbb{C}^*$ act on $X = \mathbb{C}^*$ via the multiplication by z^k

• We identify $\mathbb{C}^*$ with the subset of $\mathbb{P}^1$

$$\{[1, z] \in \mathbb{P}^1 \mid z \neq 0\}$$

the action of $\mathbb{C}^*$ is as in 4.14 for $\ell = 0$. To compute $H^*_{\mathbb{T}}(\mathbb{C}^*)$ use the Mayer-Vietoris exact sequence [Anderson-Fulton, Ch. 3, §5]: for even degrees we have

$$0 \to H^{2i-1}_{\mathbb{T}}(\mathbb{C}^*) \to H^{2i}_{\mathbb{T}}(\mathbb{P}^1) \stackrel{\alpha}{\to} H^{2i}_{\mathbb{T}}(\mathbb{C}) \oplus H^{2i}_{\mathbb{T}}(\mathbb{C}) \to H^{2i}_{\mathbb{T}}(\mathbb{C}^*) \to 0$$

$$0 \to ? \to \mathbb{Z}[t,h]/(h(h+kt)) \stackrel{\alpha}{\to} \mathbb{Z}[t] \oplus \mathbb{Z}[t] \to ? \to 0$$

$$\alpha(t) = (t,t)\,, \qquad \alpha(h) = (kt, 0)\,.$$

The restriction map to the open $\mathbb{C}$'s can be identified with the restriction to the fixed points. The one but last map α is injective, thus $H^{2i-1}_{\mathbb{T}}(\mathbb{C}^*) = 0$ and

$$H^{2i}_{\mathbb{T}}(\mathbb{C}^*) = coker(\alpha) = \langle t_1^i, t_2^i \rangle / \langle \alpha(t^a h^b)) \rangle = \langle t_1^i, t_2^i \rangle / \langle t_1^i + t_2^i, kt_1^i \rangle = \mathbb{Z}/k\mathbb{Z}\,.$$

• Corollary:

$$H^i(B\mathbb{Z}_k\, \mathbb{Z}) = H^i_{\mathbb{C}^*}(\mathbb{C}^*_k; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}_k & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

(Here $\mathbb{Z}_k$ denotes $\mathbb{Z}/k\mathbb{Z}$.)

4.16 In general, if G is a finite group $H^{>0}(BG;\mathbb{Z})$ is torsion.

• $p : EG \to BG$ is a finite covering, thus $p_*p^* \in \mathrm{End}(H^i(BG))$ is the multiplication by $|G|$. Since for $i > 0$ it factors through trivial group for we have $|G|H^i(BG) = 0$.

• We will mainly perform computation over $\mathbb{Q}$, so will ignore finite groups.

5 Equivariant formality, localization I

5.1 The condition

(*) $H^*_{\mathbb{T}}(X)$ is a free module over $H^*_{\mathbb{T}}(pt)$

Is called *equivariant formality* It can be reformulated

– $H^*_{\mathbb{T}}(X) \otimes_{H^*_{\mathbb{T}}(pt)} \mathbb{Q} \simeq H^*(X)$

– $H^*(X) \otimes H^*_{\mathbb{T}}(pt) \simeq H^*_{\mathbb{T}}(X)$ (it is enough to know that there is an isomorphism of graded vector spaces)

– $H^*_{\mathbb{T}}(X) \to H^*(X)$ is surjective, compare [Anderson-Fulton, Ch. 6, §3].

5.2 The basic argument is analysis of the fibration $X \subset E\mathbb{T} \times^{\mathbb{T}} X \to B\mathbb{T}$ and Serre spectral sequence

$$E_2^{p,q} = H^p_{\mathbb{T}}(pt) \otimes H^q(X) \Rightarrow H^{p+q}_{\mathbb{T}}(X)\,.$$

5.3 If X is a sum of even dimensional cells then (*) holds. It is enough to assume $H^{odd}(X;\mathbb{Q}) = 0$.

5.4 Theorem: If X is smooth algebraic manifold with an algebraic torus $\mathbb{T} = (\mathbb{C}^*)^r$ action, then X is equivariantly formal.

• See [Anderson-Fulton, Ch. 5, Cor. 3.3]

• (Much more difficult result of McDuff is equivariant formality of X symplectic manifolds with Hamiltonian torus action.)

5.5 To show (5.4) we need some basic tools.

• Fundamental class of a subvariety $Y \subset X$: it is the Poincaré dual of the homology class. We denote it $[Y] \in H^{2\text{codim}Y}(X)$. (We do not have to assume that X is compact.) • Equivariant fundamental class of an equivariant subvariety. Let $E_n \to B_n = (\mathbb{P}^n)^r$ be the approximation of the universal $\mathbb{T}$-bundle. He define $[Y] \in H^*_{\mathbb{T}}(X)$ as the fundamental class of $E_n \times^{\mathbb{T}} Y \subset E_n \times^{\mathbb{T}} X$

$$[E_n \times^{\mathbb{T}} Y] \in H^{2\text{codim}Y}(E_n \times^{\mathbb{T}} X) \simeq H^{2\text{codim}Y}_{\mathbb{T}}(X) \qquad \text{for sufficiently large } n\,.$$

• Exercise: Show that the definition does not depend on $n >> 0$.

• Exercise: Define the equivariant fundamental class not passing through approximation, but using the equivariant normal bundle on Y_{smooth}.

5.6 Correspondences: (for cohomology with rational coefficients). Suppose X and Y are compact C^∞ manifolds. We have

$$\text{Hom}(H^*(Y), H^*(X)) \simeq (H^*(Y))^* \otimes H^*(X) \overset{\text{Poincaré}}{\simeq} H^*(Y) \otimes H^*(X) \overset{\text{Künneth}}{\simeq} H^*(X \times Y)\,.$$

Having a cohomology class $a \in H^k(X \times Y)$ we define $\phi_a : H^*(Y) \to H^*(X)$

$$\begin{array}{ccccccc} H^i(Y) & & H^i(X\times Y) & & H^{i+k}(X\times Y) & & H^{i+k-\dim Y}(X) \\ \alpha & \mapsto & \pi_Y^*\alpha & \mapsto & a\cdot(\pi_Y^*\alpha) & \mapsto & \pi_{X*}(a\cdot(\pi_Y^*\alpha))\,. \end{array}$$

Here $\cdot$ is the product in cohomology. Puritans would denote it by $\cup$. The push-forward (a.k.a Gysin homomorphism)π_{X*} can be defined as the map in homology composed with Poincaré dualities. See [Anderson-Fulton, Ch. 3, §6]

• If a is the class of a graph of $f : X \to Y$, $\dim Y = k$ i.e. $a = [graph(f)] \in H^k(X \times Y)$. Then $\phi_a = f^*$. (Exercise.)

• Suppose X and Y smooth an compact algebraic varieties and $Z \subset X \times Y$ any subvariety. Take $a = [Z]$, $\phi_Z := \phi_a$. Then $\phi_Z : H^i(Y) \to H^{i+2c}(X)$ with $c = \text{codim}Z - \dim Y = \dim X - \dim Z$.

• One can drop the assumption that X is compact. It is enough to assume that the projection $Z \to X$ is proper:

$$\alpha \mapsto \pi_Y^*\alpha \mapsto (\pi_Y^*\alpha)_{|Z} \mapsto \pi_{X*}(\pi_Y^*\alpha)_{|Z})\,.$$

5.7 Proof of 5.4. Let $B_n = (\mathbb{P}^n)^r$, $X_n = (\mathbb{C}^{n+1} - 0)^r \times^{\mathbb{T}} X$ be the approximation of the Borel construction. We show that $H^*(X_n) \to H^*(X)$ surjective. It is enough, since $H^k(X_n) \simeq H^*_{\mathbb{T}}(X)$ for large n.

The bundle $(\mathbb{C}^{n+1} - 0)^r \to (\mathbb{P}^n)^r$ is trivial over the set standard affine open set $U \simeq (\mathbb{C}^n)^r$:

$$U \times X \subset X_n.$$

The projection $p : U \times X \to X$ extends to the correspondence

$$\phi_Z : X_n \to X\,, \qquad Z = closure(graph(p)).$$

The map p^* has a left inverse inverse i'^* induced by $i' : X = \{pt\} \times X \to U \times X$, i.e. $pi' = id_X$

$$\begin{array}{ccccc} & & H^*(X) & & \\ & {\scriptstyle i^*}\nearrow & & \nwarrow{\scriptstyle i'^*} & \\ H^*(X_n) & & \longrightarrow & & H^*(U\times X) \\ & {\scriptstyle \phi_Z}\nwarrow & & \nearrow{\scriptstyle (\phi_Z)_{|U\times X}=p^*} & \\ & & H^*(X) & & \end{array}$$

$i^*\phi_Z = id_{H^*(X)}$ because $i'^*p^* = id_{H^*(X)}$.

• Exercise: show that all works for cohomology with $\mathbb{Z}$ coefficients.

5.8 Example of a space which is not equivariantly formal:
Let $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$ with $\mathbb{T}_i = \mathbb{C}^*$, $X = \mathbb{T}/\mathbb{T}_1 \simeq \mathbb{T}_2$:

$$H^*_{\mathbb{T}}(\mathbb{T}/\mathbb{T}_1) = H^*(E\mathbb{T} \times^{\mathbb{T}} \mathbb{T}/\mathbb{T}_1) = H^*(B\mathbb{T}_1)\,.$$

The map to $H^*(X)$ for *=1 is not surjective.

5.9 Example: $\mathbb{T} = S^1$ acting on $X = S^3$ with the quotient S^2 (the Hopf fibration). Then $H^*_{\mathbb{T}}(S^3) \simeq H^*(S^2)$ cannot be surjective to $H^*(S^3)$.

5.10 If X is a free $\mathbb{T}$ space then X is not equivariantly formal (since $H^*_{\mathbb{T}}(X)$ is of finite dimension, cannot be a free module over a polynomial ring).

5.11 **Localization 1.0:** Let X be a finite $\mathbb{T}$–CW complex. Then the kernel and the cokernel of the restriction to the fixed point set $H^*_{\mathbb{T}}(X) \to H^*_{\mathbb{T}}(X^T)$ are torsion $H^*_T(pt)$-modules.

• Other formulation: Let $\Lambda = H^*_T(pt) = \mathbb{Q}[t_1, t_2, \dots, t_3]$, and $K =$ be the fraction field. Then the restriction

$$K \otimes_\Lambda H^*_{\mathbb{T}}(X) \to K \otimes_\Lambda H^*_{\mathbb{T}}(X^T).$$

is an isomorphism. • It will be clear from the proof what elements of Λ should be inverted.

• Proof in the case of the finite $X^{\mathbb{T}}$, see [Anderson-Fulton, Ch. 5, Th. 1.8]. For nonsingular varieties [Anderson-Fulton, Ch 5. Th. 1.13]

5.12 Let M be a Λ-module (it is enough to assume that Λ is a domain). Localization

$$K \otimes_\Lambda M = \{\frac{m}{a} \,|\, a \neq 0\}/\sim$$

$$\frac{m_1}{a_1} \sim \frac{m_2}{a_2} \;\Leftrightarrow\; \exists b \in \Lambda^*\; ba_2 m_1 = ba_1 m_2\,.$$

5.13 Lemma: The localization functor

$$\Lambda - modules \quad\longrightarrow\quad K - modules$$

is exact. (Exercise)

5.14 Proof of 5.11. Induction with respect to the number of cells: Assume that if $X = Y \cup \mathbb{T} \times_G D$. Then the sequence

$$\to K \otimes_\Lambda H^*_{\mathbb{T}}(X,Y) \to\to K \otimes_\Lambda H^*_{\mathbb{T}}(X) \to K \otimes_\Lambda H^*_{\mathbb{T}}(Y) \to$$

is exact. Assume that $G \neq \mathbb{T}$. We will show that $H^*_{\mathbb{T}}(X,Y)$ is a torsion Λ-module.

$$H^*_{\mathbb{T}}(X,Y) \simeq H^*_{\mathbb{T}}(\mathbb{T} \times_G D, T \times_G S) \simeq H^*_G(D,S)\,,$$

(see (4.11)) The action of Λ on $H^*_G(D,S)$ factorizes through $H^*_{\mathbb{T}}(\mathbb{T}/G) = H^*_G(pt) = \Lambda/(\text{characters anihilating } G)$, hence $H^*_G(pt)$ is a torsion Λ-module.

5.15 Exercise: see what goes wrong for $\mathbb{T}$ replaced by a nonabelian groups. For tori the orbit $H^*_{\mathbb{T}}(\mathbb{T}/G)$ turned out to be a torsion $H^*_{\mathbb{T}}(pt)$–module. (Is $H^*_{GL_n}(GL_n/B_n)$ a torsion $H^*_{GL_n}(pt)$–module?)

5.16 Example: $\mathbb{P}^1$with the standard $\mathbb{T}=(\mathbb{C}^*)^2$ action

$$K\otimes_\Lambda H^*_{\mathbb{T}}(\mathbb{P}^1)=K[h]/((t_0+h)(t_1+h))\stackrel{\simeq}{\longrightarrow}K\oplus K$$

$$f[h]\mapsto\big(f(-t_0)\,,f(-t_1)\big)\,.$$

(Chinese reminder theorem.)

5.17 If X is equivariantly formal, then all mappings below are injective

$$\begin{array}{ccc} H^*_{\mathbb{T}}(X) & \longrightarrow & H^*_{\mathbb{T}}(X^{\mathbb{T}}) \\ \downarrow & & \downarrow \\ K\otimes_\Lambda H^*_{\mathbb{T}}(X) & \stackrel{\simeq}{\longrightarrow} & K\otimes_\Lambda H^*_{\mathbb{T}}(X^{\mathbb{T}}) \end{array}$$

If $|X|<\infty$ then

$$K\otimes_\Lambda H^*_{\mathbb{T}}(X^{\mathbb{T}})\simeq K^{|X^{\mathbb{T}}|}$$

Therefore instead of computation in a possibly difficult ring $H^*_{\mathbb{T}}(X)$ it is enough to make calculations with rational functions.

5.18 Example: (exercise) $X=\mathbb{P}^n$, $\mathbb{T}$ the standard one, the image

$$H^*_{\mathbb{T}}(\mathbb{P}^n)\hookrightarrow\bigoplus_{k=0}^n\Lambda=\Lambda^{n+1}$$

consists of such sequences $(f_0,f_1,\dots,f_n)\in\mathbb{Q}[t_0,t_1,\dots,t_n]^{n+1}$, such that t_i-t_j divides f_i-f_j.

Plans for the future:

5.19 Assume that X is equivariantly formal, $|X^T|<\infty$.
Question: how to describe $H^*_T(X)\hookrightarrow\bigoplus_{x\in X^T}\Lambda$?
(an answer for GKM-spaces is easy and handy to use).

5.20 Assume, that X is equivariantly formal and $|X^{\mathbb{T}}|<\infty$.
Question: how to reconstruct an element $\alpha\in H^*_{\mathbb{T}}(X)$ knowing the restrictions $\alpha_{|\{x\}}\in\Lambda$?
Answer: Atiyah-Bott and Beline-Vergne theorem: assuming that X compact manifold

$$\alpha=\sum_{x\in X^T}(i_x)_*\left(\frac{i_x^*\alpha}{e(T_xX)}\right)\in K\otimes_\Lambda H^*_{\mathbb{T}}(X),$$

where $i_x:\{x\}\to X$, and $e(T_xX)\in\Lambda$ is the equivariant Euler class of $T_xX\to\{x\}$, see 2.7.

5.21 Corollary (with the assumptions as above):

$$\int_X\alpha=\sum_{x\in X^T}\frac{i_x^*\alpha}{e(T_xX)}\,.$$

5.22 Corollary: $X=\mathbb{P}^n$, $\alpha=(c_1(\mathcal{O}(1))^n$

$$\sum_{i=0}^n\frac{(-t_i)^n}{\prod_{j\neq i}(t_j-t_i)}=?$$

6 Localization and integration on manifolds

[Anderson-Fulton, Ch. 5]

6.1 Corollary: If X is equivariantly formal, then $H^{even}(X;\mathbb{Q}) \simeq H^{even}(X^T;\mathbb{Q})$ and $H^{odd}(X;\mathbb{Q}) \simeq H^{odd}(X^T;\mathbb{Q})$

- By elementary arguments we already new that $\chi(X) = \chi(X^{\mathbb{T}})$.

6.2 Remark: From Białynicki-Birula decomposition one can derive more: the correspondences

$$\Gamma_i = closure(X_F^+ \to F) \subset F \times X$$

induce

$$H^*(X;\mathbb{Z}) \simeq \bigoplus_{F \subset X^{\mathbb{T}}} H^{*-2n_F^+}(F,\mathbb{Z})\,,$$

where n_F^+ is the dimension of the fiber of the *limit map* $X_F^+ \to F$. [proof by Carrell].

6.3 Let $f: X \to Y$ be a map of compact oriented manifolds. Then the push-forward (or the Gysin map [Anderson-Fulton, Ch.3, §6]) $f_*: H^*(X) \to H_T^*(Y)$ may be defined by Poincaré duality

$$PD_X: H^k(X) \to H_{\dim X - k}(X)$$

$$a \mapsto a \cap [X],$$

We define f_* to be the composition

$$\begin{array}{ccccccc} H^k(X) & \overset{\simeq}{\to} & H_{\dim X-k}(X) & \to & H_{\dim X-k}(Y) & \overset{\simeq}{\leftarrow} & H^{\dim Y - \dim X + k}(Y) \\ a & \mapsto & a \cap [X] & \mapsto & f_*(a \cap [X]) & \mapsto & f_*(a) \end{array}$$

6.4 Another construction for an embedding: Let U be a tubular neighbourhood of X in Y, i.e. U is diffeomorphic to the space of the normal bundle $\pi: \nu \to X$, $c = \mathrm{codim} X$. Let $\tau \in H^c(U, U \setminus X)$ be the Thom class. This means that τ restricted to any fiber of $U \simeq \nu \to X$ is the generator of $H^c(\nu_x, \nu_x \setminus \{0\}) \simeq H^c(\mathbb{R}^c, \mathbb{R}^c \setminus \{0\})$ (i.e. we have a continuous choice of orientations in the fibers). We define f_*:

$$H^k(X) \xrightarrow{\text{Thom}} H^{c+k}(U, U \setminus X) \xleftarrow[\simeq]{\text{excision}} H^{c+k}(Y, Y \setminus X) \longrightarrow H^{c+k}(Y).$$

The Thom isomorphism is given by $H^k(X) \overset{\simeq}{\to} H^{c+k}(U, U \setminus X)$, $a \mapsto \tau \cdot \pi^*(a)$, where $\pi: U \to X$ is the projection in the bundle $\nu \simeq U \to X$.

6.5 Exercise: show that both constructions of f_* are equivalent. Hint $\tau \cap [U] = [X] \in H_{\dim X}(U) \simeq H_{\dim X}(X)$, where $[U] \in H_{\dim Y}(\overline{U}, \partial U)$ is the orientation class.

6.6 Key formula. Let $e(\nu) \in H^c(X)$ be the Euler class, $i: X \hookrightarrow Y$ the inclusion. We have

$$i^* i_*(a) = e(\nu) \cdot a\,.$$

- Since

$$e(\nu) = i^*(\tau)\,, \qquad \tau \in H^c(\nu, \nu \setminus X) \simeq H^c(Y, Y \setminus X)$$

by the definition, we get $i^* i_*(a) = i^*(\tau \cdot \pi^*(a)) = i^*(\tau) \cdot i^* \pi^*(a) = e(\nu)\, a$.

6.7 If $X \subset Y$ is a $\mathbb{T}$-invariant. Let us define i_* as in (6.4). The equivariant class of an invariant submanifold is defined as $i_*(1_X) \in H^*_{\mathbb{T}}(Y)$.

6.8 Suppose, that X is a $\mathbb{T}$-manifold, $i : X^T \to X$ is an embedding,

$$i^* : K \otimes_\Lambda H^*_T(X) \stackrel{\simeq}{\to} K \otimes_\Lambda H^*_T(X^T)\,.$$

The composition $i_* i^*$ by the Euler class of the normal bundle X^T. (over each component $F \subset X^{\mathbb{T}}$ the normal bundle can have a different dimension.)

6.9 Fundamental Lemma: The Euler class $e(\nu(X^{\mathbb{T}}$ in $X) \in H^*_{\mathbb{T}}(X)$ is invertible in $K \otimes_\Lambda H^*_{\mathbb{T}}(X)$.

- It has to be checked for every component of $F \subset X^{\mathbb{T}}$ that the Euler class in invertible.
- If $F = \{x\}$ is a point,

$$e(\nu_F) = \prod_i w_i \in Z[t_1, t_2, \dots, t_r],$$

where $w_1, \dots, w_c$ are weights of the torus representation $\nu_F = T_x X$. The weights are non-zero, since x is an isolated fixed point.

- E.g. if $x = [0 : \dots : 0 : 1 : 0 : \dots : 0] \in \mathbb{P}^n$ (1 on k-th position), then $e(\nu_{\{x\}}) = \prod_{i \neq k}(t_i - t_k)$.

6.10 Proof of the fundamental lemma in the general case: We decompose $\nu = \bigoplus_{w \in \mathcal{W}} \nu_w$. We can assume that ν_w is a complex bundle. (We do not assume that X is a complex manifold but the torus action allows to define complex structure.) Each summand ν_w has a complement μ_w such that

$$\nu_w \oplus \mu_w = 1\!\!1^{d_w} \qquad \text{a trivial bundle of dimension } d_w$$

The above isomorphism can be made equivariant, when we act on μ_w with the character w Then $e(\nu_w \oplus \mu_w) = w^{d_w}$. Let $\mu = \bigoplus_w \mu_w$. We have

$$e(\nu \oplus \mu) = \prod_{w \in \mathcal{W}} w^{d_w}$$

hence

$$e(\nu) \cdot \left(e(\mu) / \prod_{w \in \mathcal{W}} w^{d_w} \right) = 1.$$

6.11 Localization formula (Atiyah-Bott, Berline-Vergne). Assume, that X is a compact $\mathbb{T}$-manifold, which is equivariantly formal. For $a \in H^*_{\mathbb{T}} * X)$ we have

$$a = \sum_F (i_F)_* \left(\frac{i_F^*(a)}{e(\nu(F))} \right) \tag{1}$$

summation over the connected components $F \subset X^{\mathbb{T}}$. Here $i_F : F \to X$ is the inclusion.

- Proof. Let ϕ be the composition

$$K \otimes_\Lambda H^*_T(X) \stackrel{i^*}{\to} \bigoplus_F K \otimes_\Lambda H^*_T(F) \stackrel{1/e(\nu)}{\longrightarrow} \bigoplus_F K \otimes_\Lambda H^*_T(F)\,.$$

Note, that $i_* \circ \phi = Id$. Since $K \otimes_\Lambda H^*_T(X)$ is of a finite dimension over K, thus $\phi \circ i_* = Id$. Hence we have an equality (1) in $K \otimes_\Lambda H^*_T(X)$.

- Note that we have an expression in $K \otimes_\Lambda H^*_T(X)$, but the sum belongs to $H^*_T(X)$, i.e. it is integral.

• The above argument reproves the statement that the restriction to $X^{\mathbb{T}}$ is an isomorphism after tensoring with K.

• It is enough to invert the weights appearing in the normal bundles ν_F.

• We do not have to assume that X is compact. It is enough to know that $X^{\mathbb{T}}$ is compact and X is formal.

6.12 [Anderson-Fulton, Ch. 5, §2] AB-BV integration formula: Let $p_X : X \to pt$ be the constant map. With the assumption as above

$$\int_X a := (p_X)_*(a) = \sum_F (p_F)_* \left(\frac{i_F^*(a)}{e(\nu(F))} \right) \in \Lambda.$$

• The sum is in Λ although the summands belong to K.

• If $|X^T| < \infty$

$$\int_X a = \sum_{p \in X^T} \frac{a_{|p}}{e(T_p X)}$$

6.13 Example [Anderson-Fulton, Ch 5, Ex. 2.5] $\mathbb{P}^n$. Let $h = c_1(\mathcal{O}(1))$:

• Subexample, $n = 1$

$$\int_{\mathbb{P}^1} h = \frac{-t_0}{t_1 - t_0} + \frac{-t_1}{t_0 - t_1} = \cdots = 1\,.$$

• In general

$$\int_{\mathbb{P}^n} h^{k+n} = \sum_{i=0}^{n} \frac{(-t_i)^{k+n}}{\prod_{j \neq i}(t_j - t_i)}$$

$$= (-1)^k \sum_{i=0}^{n} Res_{z=t_i} \frac{z^{k+n}}{\prod_{j=1}^{n}(z - t_j)} = \dots$$

The result is:

$$(-1)^k S_k(t_0, t_1, \dots, t_n) = (-1)^k \sum_{\ell_0 + \ell_1 + \cdots + \ell_n = k} t_0^{\ell_0} t_1^{\ell_1} \dots t_n^{\ell_n}$$

i.e. the *complete symmetric function.*

• Exercise: Check at least that $\int_{\mathbb{P}^n} h^n = 1$.

Application to compute Euler characteristic of holomorphic bundles.

6.14 Riemann-Roch theorem: Let E be a holomorphic bundle over a compact complex manifold, then

$$\chi(X; E) = \int_X td(TX) ch(E)\,.$$

• Remainder: the Todd class td is a multiplicative characteristic class i.e. $td(E \oplus F) = td(E) td(F)$ and for a line bundle $td(L) = \frac{t}{1 - e^{-t}}$, where $t = c_1(L)$.

• If a torus $\mathbb{T}$ acts on X with a finite number of fixed points, and E is a vector bundle admitting $\mathbb{T}$ action, the $td(TX)$ and $ch(E)$ naturally lift to equivariant cohomology (via Borel construction). Then

$$\chi(X; E) = \sum_{x \in X^{\mathbb{T}}} \frac{i_x^*(td(TX) ch(E))}{e(T_x X)}\,.$$

• For simplicity assume that $E = L$ is a line bundle. Each summand is equal to

$$\frac{\prod_{i=1}^{n} \frac{w_{x,i}}{1 - e^{-w_{x,i}}}}{\prod_{i=1}^{n} w_{x,i}} e^{\alpha_x} = \frac{e^{\alpha}}{\prod_{i=1}^{n}(1 - e^{-w_{x,i}})}\,,$$

where $w_{x,i}$ are the weights of the $\mathbb{T}$ action on the tangent space T_xX and α_x is the weight of $\mathbb{T}$ acting on L_x.

- Exercise: compute from above $\chi(\mathbb{P}^n; \mathcal{O}(k))$.

7 Flag variety and flag bundles

[Anderson-Fulton, Ch.4, §4]

7.1 Let $E \to B$ be a complex vector bundle of rank n, $\pi : \mathcal{F}\ell(E) \to B$ the associated bundle of complete flag varieties. A point of $\mathcal{F}\ell(E)$ mapping to $x \in B$ is a sequence

$$V_\bullet = \{0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = E_x \mid \dim(V_i) = i\}\,.$$

The quotients $L_i = V_i/V_{i-1}$ with $V_\bullet$ varying form a line bundle. Let $x_i = c_1(L_i)$.

7.2 Theorem. Cohomology $H^*(\mathcal{F}\ell(E))$ is generated by x_i as a $H^*(B)$ algebra:

$$H^*(\mathcal{F}\ell(E)) \simeq H^*(B)[x_1, x_2, \ldots, x_n]/I\,,$$

where I is the ideal generated by

$$\sigma_i(x_1, x_2, \ldots, x_n) - \pi^* c_i(E) \qquad \text{for } i = 1, 2, \ldots, n\,,$$

so that in $H^*(\mathcal{F}\ell(E))$

$$\pi^*(c(E)) = \prod_{i=1}^{n}(1 + x_i)\,.$$

7.3 The proof by induction.

- For $n = 1$: $\mathcal{F}\ell(E) = B$, $H^*(B)[x_1]/(x_1 - c_1(E)) = H^*(B)$.
- Let $B' = \mathbb{P}(E)$ with the projection to B denoted by p. The bundle $p*(E)$ fits to the exact sequence

$$0 \to \mathcal{O}(-1) \to p^*(E) \to E'\,.$$

By the projective bundle theorem

$$H^*(B') \simeq H^*(B)[h]/\left(\sum_{i=0}^{n} h^i p^*(c_{n-i}(E))\right)\,.$$

Here $h = c_1(\mathcal{O}(1))$. By Whitney formula

$$c(E') = p^*(c(E))(1-h)^{-1}\,,$$

i.e.

$$c_k(E') = \sum_{i=0}^{k} h^i p^*(c_{k-i}(E))\,.$$

(The expression for $0 = c_n(E')$ is exactly the relation in the Projective Bundle Theorem,.) We identify the flag bundle $\mathcal{F}\ell(E')$ with $\mathcal{F}\ell(E)$. The generators in cohomology of $\mathcal{F}\ell(E)$ correspond to generators for $\mathcal{F}\ell(E')$:

$$x_1 = -h\,, \quad x_2 = x'_1\,, \quad x_3 = x'_2 \quad \ldots \quad x_n = x'_{n-1}\,.$$

We have by the inductive assumption

$$H^*(\mathcal{F}\ell(E')) \simeq H^*(B)[h, x_1', x_2', \dots, x_{n-1}']/J$$

$$J = \left\langle \pi'^*(c_i(E')) - \sigma_i(x_1', x_2', \dots, x_n') \text{ for } i = 1, 2, \dots, n-1\,, \quad \sum_{i=0}^{n} h^i \pi^* c_{n-i}(E) \right\rangle .$$

It is enough to change the name of variables and conclude that $J = I$.

- The inclusion $I \subset J$ follows since (topologically) $E \simeq \bigoplus_{i=1}^n L_i$.
- Example: $n = 4$. The generator of J (we drop pull-backs in the notation)

$$\begin{aligned}
&c_1(E') - \sigma_1(x_1', x_2', x_3') = c_1(E) - x_1 - \sigma_1(x_2, x_3, x_4)\\
&c_2(E') - \sigma_2(x_1', x_2', x_3') = c_2(E) - x_1 c_1(E) + x_1^2 - \sigma_2(x_2, x_3, x_4)\\
&c_3(E') - \sigma_3(x_1', x_2', x_3') = c_3(E) - x_1 c_2(E) + x_1^2 c_1(E) - x_1^3 - \sigma_3(x_2, x_3, x_4)\\
&c_4(E) - x_1 c_3(E) + x_1^2 c_2(E) - x_1^3 c_1(E) + x_1^4
\end{aligned}$$

We perform computations in $H^*(B)[x_1, x_2, \dots, x_n]/I$. By induction show that the generators of J are trivial. We abbreviate $(x_1, x_2, \dots)$ by $\underline{x}$

$$\begin{aligned}
&c_1(E') - \sigma_1(\underline{x}') = c_1(E) - x_1 - \sigma_1(\underline{x}') = c_1(E) - \sigma_1(\underline{x})\\
&c_2(E') - \sigma_2(\underline{x}') = c_2(E) - x_1\sigma_1(\underline{x}) + x_1^2 - \sigma_2(\underline{x}') = c_2(E) - \sigma_2(\underline{x})\\
&c_3(E') - \sigma_3(\underline{x}') = c_3(E) - x_1\sigma_2(\underline{x}) + x_1^2\sigma_1(\underline{x}) - x_1^3 - \sigma_3(\underline{x}') = c_3(E) - \sigma_3(\underline{x})\\
&c_4(E) - x_1\sigma_3(\underline{x}) + x_1^2\sigma_2(\underline{x}) - x_1^3\sigma_1(\underline{x}) + x_1^4 = c_4(E) - \sigma_4(\underline{x})
\end{aligned}$$

We apply the formula

$$\sum_{i=0}^{k} (-1)^i x_1^i \sigma_{k-i}(\underline{x}) = \sigma_k(\underline{x}')$$

and for the last row

$$\sum_{i=0}^{n} (-1)^i x_1^i \sigma_{n-i}(\underline{x}) = 0$$

- Conceptually: the relations in J say that $c(E)(1 + x_1)^{-1}$ lives in the gradations $< n$ and $c(E)(1 + x_1)^{-1} = \prod_{k=2}^n (1 + x_k)$. That follows from the identities of I.

7.4 Corollary: Let $\mathbb{T}$ be the maximal torus in $\mathrm{GL}_n(\mathbb{C})$ acting on

$$\mathcal{F}\ell(\mathbb{C}^n) = \mathrm{GL}_n(\mathbb{C})/(\text{upper-triangular}) \simeq U(n)/(U(n) \cap \mathbb{T})\,.$$

$$H^*_{\mathbb{T}}(\mathcal{F}\ell(\mathbb{C}^n) \simeq \Lambda[x_1, x_2, \dots, x_n]/\langle \sigma_i(\underline{t}) - \sigma_i(\underline{x})\rangle \mid i = 1, 2, \dots, n\rangle\,.$$

$$H^*_{\mathbb{T}}(\mathcal{F}\ell(\mathbb{C}^n) \simeq \Lambda \otimes_{\Lambda^{\Sigma_n}} \Lambda\,.$$

- Note

$$H^*_{\mathrm{GL}_n(\mathbb{C})}(\mathcal{F}\ell(\mathbb{C}^n) \simeq \Lambda\,.$$

7.5 [Anderson-Fulton, Ch. 4, §5] For Grassmannian $Gr_k(\mathbb{C}^n)$ the computation follows. The projection $\mathcal{F}\ell(\mathbb{C}^n) \to Gr_k(\mathbb{C}^n)$ induces the inclusion

$$H^*_{\mathbb{T}}(Gr_k(\mathbb{C}^n)) \hookrightarrow H^*_{\mathbb{T}}(\mathcal{F}\ell(\mathbb{C}^n)) \simeq \Lambda \otimes_{\Lambda^{\Sigma_n}} \Lambda\,,$$

(as for any locally-Zariski trivial fibration). The image lies in

$$\Lambda \otimes_{\Lambda^{\Sigma_n}} \Lambda^{\Sigma_k \times \Sigma_{n-k}}\,.$$

By a dimension consideration there is an isomorphism

$$H^*_{\mathbb{T}}(Gr_k(\mathbb{C}^n)) \simeq \Lambda \otimes_{\Lambda^{\Sigma_n}} \Lambda^{\Sigma_k \times \Sigma_{n-k}}\,.$$

- It follows that for any vector bundle $E \to B$ of rank n

$$H^*_{\mathbb{T}}(E) \simeq H^*(B)[c_1, c_2, \dots, c_k, c'_1, c'_2, \dots, c'_{n-k}]/I\,.$$

The ideal I is generated by the homogeneous components of the identity

$$(1 + c_1 + \dots + c_k)(1 + c'_1 + \dots + c'_{n-k}) = c(E)\,.$$

7.6 We denote the group of invertible upper-triangular matrices by B_n. The fixed points of $\mathbb{T}$ acting on $\mathcal{F}\ell_n = GL_n(\mathbb{C})/B_n$ are given by the permutation matrices. The identity corresponds to the standard flag V_0. The quotient map $GL_n(\mathbb{C}) \to \mathcal{F}\ell_n$ is $\mathbb{T}$ equivariant with respect to the action of $\mathbb{T}$ on $GL_n(\mathbb{C})$ by conjugation. The tangent space of $\mathcal{F}\ell(\mathbb{C}^n) = \mathrm{GL}_n(\mathbb{C})/B_n$ at the point $[id]$ is isomorphic to $\mathfrak{gl}_n/\mathfrak{b}$ with the adjoint action of the torus. The weights are $t_j - t_i$ for $i < j$. At the remaining points corresponding to permutations the weights differ by the action of the permutation.

7.7 Let $X = \mathcal{F}\ell(\mathbb{C}^n)$. We will apply AB-BV formula to integrate the class $\prod_{i=1}^n c_1(L_i)^{\alpha_i}$ for some choice of exponents $\alpha_i \in \mathbb{N}$:

- The integration formula is of the form

$$(\bigstar) = \sum_{\sigma \in \Sigma_n} \frac{\prod_{i=1}^n t_{\sigma(i)}^{\alpha_i}}{\prod_{i<j}(t_{\sigma(j)} - t_{\sigma(i)})} = \frac{\begin{vmatrix} t_1^{\alpha_1} & t_1^{\alpha_2} & \dots & t_1^{\alpha_n} \\ t_2^{\alpha_1} & t_2^{\alpha_2} & \dots & t_2^{\alpha_n} \\ \vdots & & & \\ t_n^{\alpha_1} & t_n^{\alpha_2} & \dots & t_n^{\alpha_n} \end{vmatrix}}{\mathrm{Vandermonde}(t_1, t_2, \dots, t_n)}$$

If α_i is decreasing then we obtain the Schur function S_λ indexed by the sequence λ_i obtained as below

$$\begin{matrix} \alpha_1 & > & \alpha_2 & > & \alpha_3 & > \dots > & \alpha_n \\ \| & & \| & & \| & & \| \\ \lambda_1 + n - 1 & & \lambda_2 + n - 2 & & \lambda_3 + n - 3 & \dots & \lambda_n \end{matrix}$$

$$\alpha_k = \lambda_k + n - k$$

The Schur functions in n variables for $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$ form an additive basis of symmetric functions

$$S_\lambda = \frac{\begin{vmatrix} t_1^{n-1+\lambda_1} & t_1^{n-2+\lambda_2} & \dots & t_1^{\lambda_n} \\ t_2^{n-1+\lambda_1} & t_2^{n-2+\lambda_2} & \dots & t_2^{\lambda_n} \\ \vdots & & & \\ t_n^{n-1+\lambda_1} & t_n^{n-2+\lambda_2} & \dots & t_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} t_1^{n-1} & t_1^{n-2} & \dots & 1 \\ t_2^{n-1} & t_2^{n-2} & \dots & 1 \\ \vdots & & & \\ t_n^{n-1} & t_n^{n-2} & \dots & 1 \end{vmatrix}} = \pm \frac{\text{Generalized Undermined}}{\text{Vandermonde}}$$

It is equal $(-1)^{\frac{n(n-1)}{2}}(\bigstar)$.

7.8 Exercise (but maybe not for this course): Check that

$$S_\lambda = \det\left(h_{\lambda_i+j-i}\right)_{i,j=1,\dots,length(\lambda)}$$

where h_i is the complete symmetric function and $h_i = 0$ for $i < 0$.

• The fixed points of $G(k,n)$ are the coordinate subspaces (exercise), they correspond to k-element subsets of $\underline{n} = \{1, 2, \dots, n\}$. The weights at the point corresponding to $I_0 = \{1, 2, \dots, k\}$ can be computed from the isomorphism

$$T_{I_0}G(k,n) \simeq \mathfrak{gl}_\mathfrak{n}/\mathfrak{p}$$

where. $\mathfrak{p} = Lie(P)$, P is the stabilizer of $\mathrm{lin}\{\epsilon_1, \epsilon_2, \dots, \epsilon_k\}$. This set is equal to

$$\{t_j - t_i \mid i \le k < j\}.$$

• At the point p_I corresponding to the set $I \subset \{1, 2, \dots, n\}$ the set of weights is equal to$\{t_j - t_i\}_{i\in I,\, j\notin I}$.

7.9 Let $a \in H^*_{\mathbb{T}}(G(k,n))$ be given by a polynomial $W(c_1(\gamma), c_2(\gamma), \dots c_k(\gamma), c_1(Q), c_2(Q), \dots, c_{n-k}(Q))$ written as a polynomial in $x_1, x_2, \dots, x_n$, symmetric with respect to $\Sigma_k \times \Sigma_{n-k}$. Then

$$\int_{G(k,n)} a = \sum_{I\subset\underline{n}\ |I|=k} \frac{W(t_I, t_{I^\vee})}{\prod_{i\in I}\prod_{j\in I^\vee}(t_j - t_i)}$$

where $I^\vee = \underline{n} \setminus I$.

7.10 Let $L = \Lambda^k\gamma^*$ be the top exterior power of the dual tautological bundle on $G(k,n)$. (This bundle is the pull-back of $\mathcal{O}(1)$ for the Plücker embedding).

• Exercise: Compute the degree of $G(k,n)$ under Plücker embedding: let $m = \dim(G(k,n) = k(m-k)$

$$\int_{G(k,n)} c_1(L)^m = (-1)^m \sum_{I\subset\underline{n}\ |I|=k} \frac{\left(\sum_{i\in I} t_i\right)^m}{\prod_{i\in I}\prod_{j\in I^\vee}(t_j - t_i)}.$$

• In particular

$$\frac{(t_1+t_2)^4}{(t_3-t_1)(t_4-t_1)(t_3-t_2)(t_4-t_2)} + \text{other 5 summands} = 2.$$

Check it.

7.11 Tangent bundle of the Grassmannian $Gr_k(\mathbb{C}^n) = G(k,n)$: let $\gamma \overset{\iota}{\hookrightarrow} \mathbb{1}^n$ be the tautological bundle and let $Q = \mathbb{1}^n/\gamma$ be the quotient bundle. There is an equivariant isomorphism

$$TG(k,n) \simeq \mathrm{Hom}(\gamma, Q).$$

• Proof. We define a map of vector bundles

$$\mathrm{Hom}(\gamma, \mathbb{1}^n) \to TG(k,n)$$

constructing a curve: for $V \in G(k,n)$ let $f \in \mathrm{Hom}(V, \mathbb{1}^n)$. The cure $x_f : (-\epsilon, \epsilon) \to G(k,n)$ is given by

$$x_f(t) = image(\iota + tf) \in G(k,n)$$

(well defined for small t). The bundle map is given by

$$\Phi(f) = \dot{x}_f(0)\,.$$

This map invariant with respect to automorphisms of $\mathbb{C}^n$. At a point $V \in G(k,n)$ decompose $\mathbb{C}^n = V \oplus W$. In the affine neighbourhood of V

$$\{V' \in G(k,n) \mid V' \text{ is transverse to } W\}$$

every element is a graph of a map $V \to W$. The kernel of Φ is equal to $\mathrm{Hom}(\gamma,\gamma) \subset \mathrm{Hom}(\gamma, \mathbb{1}^n)$ (i.e. at the point V the kernel is equal to $\mathrm{Hom}(V,V) \subset \mathrm{Hom}(V, V \oplus W)$). Thus we have (equivariant) short exact sequence of bundles

$$0 \to \mathrm{Hom}(\gamma,\gamma) \to \mathrm{Hom}(\gamma, \mathbb{1}^n) \stackrel{\Phi}{\to} TG(k,n) \to 0$$

Hence

$$TG(k,n) \simeq \mathrm{Hom}(\gamma, Q)\,.$$

8 Application of the integration formula

8.1 Let $\mathbb{T} \subset B \subset \mathrm{GL}_n(\mathbb{C})$ be the diagonal torus, B – the group of upper-triangular matrices. For a character $e^\lambda : \mathbb{T} \to \mathbb{C}^*$ define a line bundle $\mathcal{L}_\lambda = \mathrm{GL}_n(\mathbb{C}) \times^B \mathbb{C}_{-\lambda}$. Here B acts on $\mathbb{C}_{-\lambda}$ via the surjection $B \twoheadrightarrow \mathbb{T} \stackrel{e^{-\lambda}}{\to} \mathbb{C}^*$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, then the diagonal torus acts via the multiplication by $t^{-\lambda_1} t^{-\lambda_2} \dots t^{-\lambda_n}$.

• If $n = 2$, then for $\lambda = (1,0)$ the bundle $\mathcal{L}_\lambda$ is isomorphic to $\mathcal{O}(1)$.

• Borel-Weil-Bott theorem: Suppose $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, then $V_\lambda = H^0(G/B; \mathcal{L}_\lambda)$ is an irreducible representation of $\mathrm{GL}_m(\mathbb{C})$ and $H^k(G/B; \mathcal{L}_\lambda) = 0$ for $k > 0$, [Fulton-Harris, p.392-394]

8.2 Character of a representation V is denoted by χ_V, it is the function from $G = \mathrm{GL}_n \to \mathbb{C}$:

$$\chi_V(g) = tr(g : V \to V)\,.$$

• Since $\chi_V(g) = \chi_V(hgh^{-1})$ the values of χ_V on the maximal torus determine χ_V.

• Let $R(\mathrm{GL}(n)$ be the representation ring. The map

$$\chi : R(\mathrm{GL}(n)) \to \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]^{\Sigma_n}$$

is an isomorphism after $\otimes \mathbb{C}$.

8.3 The construction of the representation ring is generalized to the equivariant K-theory of an algebraic variety (or to any category with exact sequences)

•

$$K_G(X) = \bigoplus \mathbb{Z}[\text{ Isomorphism classes of equivariant vector bundles }]/(\text{short exact sequences})$$

$$0 \to E_1 \to E_2 \to E_3 \to 0 \qquad \Rightarrow \qquad [E_2] = [E_1] + [E_3]\,.$$

• We take the algebraic version of the K-theory, but there is a variant for topological spaces.

• If complex algebraic group G is reductive (all representations split into a direct sum of irreducible representations), then $K_G(pt) = R(G)$. We will consider G reductive only, e.g. $G = \mathrm{GL}_n(\mathbb{C})$.

8.4 Instead of vector bundles we can take the isomorphism classes of coherent sheaves. If X is smooth, then we obtain isomorphic K-theory.

8.5 Let $f : X \to Y$ be a proper G-equivariant map of smooth algebraic G-varieties. We define $f_! : K_G(X) \to K_G(Y)$

$$f_!(E) = \sum_{k=0}^{\dim X} (-1)^k [R^k f_*(E)]$$

• The sheaf $R^k f_*(E)$ is a coherent sheaf, should be replaced by its resolution by locally free sheaves, i.e. by vector bundles. We take $Y = pt$, then

$$f_!(E) = \sum_{k=0}^{\dim X} (-1)^k H^k(X; E) \in R(G) \simeq K_G(pt)\,.$$

8.6 Equivariant Hirzebruch-Riemann-Roch theorem. Let G be an algebraic group acting on X.

$$\begin{array}{ccccc} & K_G(X) & \stackrel{td(X)ch(-)}{\longrightarrow} & \hat{H}^*_G(X) \\ & {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_*} \\ R(G) \simeq & K_G(pt) & \underset{ch}{\longrightarrow} & \hat{H}_G(pt) \end{array}$$

Here $ch : R(G) \to \hat{H}_G(pt)$ maps a representation V to $ch(EG \times^G V)$. We need to take

$$\hat{H}^*_G(pt) := \prod_{k=0}^{\infty} H^k_G(pt)$$

since the Chern character lives in infinite gradations.

• If $G = \mathbb{T}$ the image of $ch : R(\mathbb{T}) \to \hat{H}^*\mathbb{T}(pt) = \mathbb{Z}[[t_1, t_2, \dots, t_n]]$ lies in the ring of Laurent polynomial $\mathbb{Z}[e^{\pm t_1}, e^{\pm t_2}, \dots, e^{\pm t_n}]$.

8.7 There is a coincidence of standard notations:
— $\chi(X; \mathcal{L})$=Euler characteristic of G/B with coefficients in the sheaf $\mathcal{L}$
— if a group G acts on X, then naturally $\chi(X; \mathcal{L}) \in R(G)$.
— $\chi(V) = \chi_V \in R(G)$ character of a representation.

8.8 We will compute the character of the representation V_λ using localization theorem for $\mathbb{T}$-equivariant cohomology.

$$\chi(\mathcal{F}\ell_n; \mathcal{L}_\lambda) = \sum_{p \in (\mathcal{F}\ell_n)^T} \frac{td(T\mathcal{F}\ell_n)_{|p}}{eu(T\mathcal{F}\ell_n)_{|p}} ch(\mathcal{L}_\lambda)$$

$$= \sum_{\sigma \in \Sigma_n} \frac{1}{\prod_{i<j}(1 - e^{-(t_{\sigma(j)} - t_{\sigma(i)})})} \prod_{i=1}^n e^{-\lambda_i t_{\sigma(i)}}\,.$$

With new variables $x_i = e^{-t_i}$:

$$\chi(\mathcal{F}\ell_n; \mathcal{L}_\lambda) = \sum_{\sigma \in \Sigma_n} \frac{1}{\prod_{i<j}(1 - x_{\sigma(j)}/x_{\sigma(i)})} \prod_{i=1}^n x_{\sigma(i)}^{\lambda_i}\,.$$

We introduce the notation

$$x^\lambda = \prod_{i=1}^n x_i^{\lambda_i}\,, \qquad \sigma(x^\lambda) = \prod_{i=1}^n x_{\sigma(i)}^{\lambda_i}\,,$$

$$x^\rho = \prod_{i=1}^n x_i^{n-i+1}\,, \qquad x^{\lambda+\rho} = \prod_{i=1}^n x_i^{\lambda_i+n-i+1}\,.$$

Then

$$\chi_{V_\lambda} = \chi(\mathcal{F}\ell_n;\mathcal{L}_\lambda) = \sum_{\sigma\in\Sigma_n} \frac{\sigma(x^{\lambda+\rho})}{\prod_{i<j}(x_{\sigma(i)}-x_{\sigma(j)})} = S_\lambda(x_1,x_2,\dots,x_n)\,.$$

• This is Weyl character formula describing the character of the representation V_λ

Goresky-Kottwitz-MacPherson: GKM spaces

8.9 Lemma [Chang, Skjelbred]. Suppose a torus acts on a topological space. Let $F = X^{\mathbb{T}}$ and let Y be the sum of F and 1-dimensional orbits. Assume that X is equivariantly formal space. Then the sequence

$$0 \to H^*_{\mathbb{T}}(X) \to H^*_{\mathbb{T}}(F) \to H^{*+1}_{\mathbb{T}}(Y,F)$$

is exact.

• The lemma is equivalent to:

$$ker(H^*_{\mathbb{T}}(F) \to H^{*+1}_{\mathbb{T}}(Y,F)) = ker(H^*_{\mathbb{T}}(F) \to H^{*+1}_{\mathbb{T}}(X,F))\,.$$

• We do not prove CS Lemma in full generality (see Matthias Franz, Volker Puppe, Exact sequences for equivariantly formal spaces, arXiv:math/0307112). The proof will be given for spaces, which are of special interest for geometers.

8.10 Definition of GKM-space: The torus $\mathbb{T} = (\mathbb{C}^*)^r$ acting algebraically on X – a compact algebraic variety (there is a topological version as well). We assume że $|X^{\mathbb{T}}| < \infty$ and there are only finitely many 1-dimensional orbits. We assume that X is equivariantly formal, e.g. X is smooth.

8.11 Assume X is smooth $|X^{\mathbb{T}}| < \infty$. For any $x \in X^{\mathbb{T}}$ no two weights of T_xX are proportional if and only if there are only finitely many 1-dimensional orbits orbits.

8.12 Graph GKM (V, E, w),
- $V = X^{\mathbb{T}}$ vertices
- E edges = 1-dimensional orbits. After fixing an isomorphism of the orbit with $\mathbb{C}^*$ we get an oriented graph
- edges are labeled with weights $w : \mathbb{T} \to \mathbb{C}^*$ of the action of $\mathbb{T}$ on $\mathbb{C}^*$ $\simeq$orbit.

All cohomologies are with coefficients in $\mathbb{Q}$.

8.13 Basic Lemma: suppose $X = \mathbb{P}^1$, $\mathbb{T}$ acts via $w \in \mathfrak{t}^* \simeq H^2_{\mathbb{T}}(pt)$. Then

$$H^*_{\mathbb{T}}(X) = \{(u_0, u_\infty) \in \Lambda^2 \mid u_0 \equiv u_\infty \mod w\}$$

• It follows from the long exact sequence of the pair $(\mathbb{P}^1, \{0,\infty\})$, since

$$H^*_{\mathbb{T}}(\mathbb{P}^1, \{0,\infty\}) \simeq \Lambda/(w) \qquad \text{with a shift of gradation by } 1\,.$$

8.14 Description of $H^*_{\mathbb{T}}(X)$ for GKM-spaces:

$$0 \to H^*_{\mathbb{T}}(X) \to \bigoplus_{x\in F} \Lambda \to \bigoplus_{1-orbits} \Lambda/(w_\ell)$$

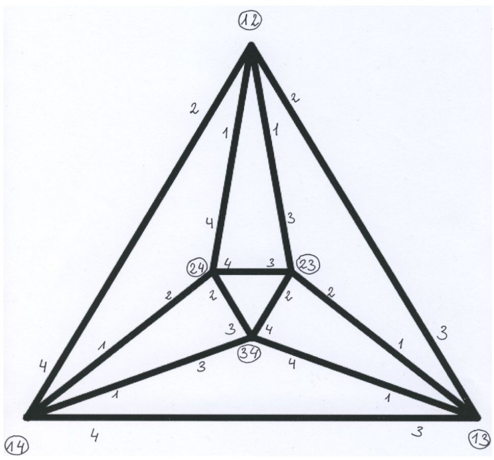
8.15 GKM-algebra associated with a graph $(V, E, w : E \to \mathfrak{t}^*_{\mathbb{Z}})$

$$A(V,E,w) := ker\left(\bigoplus_{v\in V} \Lambda \to \bigoplus_{e\in E} \Lambda/(w_\ell)\right)$$

$$\{a_v\}_{v\in V} \mapsto \{a_{t(e)} - a_{s(e)}\}_{e\in E}$$

(this description does not depend on the orientation of edges)

- The GKM-graph of Grassmannian $Gr_2(\mathbb{C}^4)$



The weight associated to the edge with numbers $i \dots j$ is equal to $t_i - t_j$ or $t_j - t_i$ depending on the choice of the orientation.

8.16 Original reference: Goresky-Kottwitz-MacPherson Equivariant cohomology, Koszul duality, and the localization theorem, Invent. math. 131, (1998). See [Anderson-Fulton, §7].

9 GKM spaces, differential model of equivariant cohomology

9.1 GKM graphs of Grassmannians $Gr_k(\mathbb{C}^n)$:

- vertices V: fixed points are the coordinate subspaces; bijection with subsets $I \subset \{1..n\}$
- edges E if I differs from J by one element; say $i \in I$ is replaced by $j \in J$, then let

$$W = lin\{\varepsilon_i + \varepsilon_j, \varepsilon_k \; k \in I \cap J\}\,.$$

The stabilizer of W has the equation $t_i = t_j$. Hence the orbit of W is 1-dimensional, with the weight equal to $t_i - t_j$.

- Exercise: there are no other edges.

9.2 Moment map: GKM-graph of the Grassmannian can be realized in $\mathbb{R}^n$. Let $m = \binom{n}{k}$, we identify $\mathbb{R}^m$ with $\wedge^k\mathbb{R}^m$:

- We have a map:

$$Gr_k(\mathbb{C}^n) \overset{\text{Plücker}}{\longrightarrow} \mathbb{P}(\wedge^k\mathbb{C}^n) = \mathbb{P}^m \overset{\mu}{\longrightarrow} \mathbb{R}^n\,,$$

where

$$\mu : [\dots, z_I, \dots] \mapsto \frac{1}{||z||^2}(\dots, |z_I|, \dots) \mapsto \frac{1}{|z|^2}(\dots, \sum_{I \ni i} |z_I|^2, \dots)\,.$$

This map is the composition of the standard moment map from $\mathbb{P}^m$ to m-dimensional simplex

$$[\dots : z_1 : \dots] \mapsto \tfrac{1}{||z||^2}(\dots, |z_I|^2, \dots)$$

with a linear map $\mathbb{R}^m \to \mathbb{R}^n$.

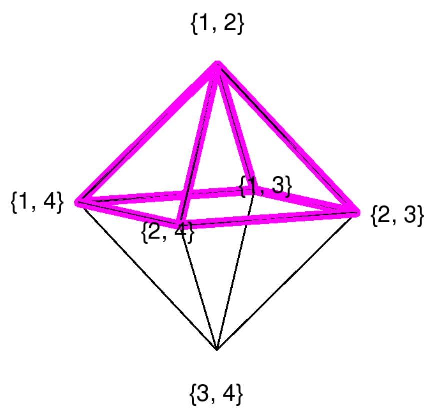
- The 1-dimensional orbits are mapped to intervals.
- The image is contained in $\{x_1 + x_2 + \dots + x_m\} = k$.
- For $\mathbb{P}^n$ the GKM graph is the 1-skeleton of the standard n-simplex.
- For $n = 4$, $m = 2$ we get octahedron in $\{x_1 + x_2 + x_3 + x_4 = 2\}$

$$\begin{array}{rcl} \varepsilon_1 \wedge \varepsilon_2 & \mapsto & (1,1,0,0) \\ \varepsilon_1 \wedge \varepsilon_3 & \mapsto & (1,0,1,0) \\ \varepsilon_1 \wedge \varepsilon_4 & \mapsto & (1,0,0,1) \\ \varepsilon_2 \wedge \varepsilon_3 & \mapsto & (0,1,1,0) \\ \varepsilon_2 \wedge \varepsilon_4 & \mapsto & (0,1,0,1) \\ \varepsilon_3 \wedge \varepsilon_4 & \mapsto & (0,0,1,1) \end{array}$$

- It will follow from differential methods, that the GKM graph of a projective manifold is canonically realized as a graph in $\mathfrak{t}^*$.

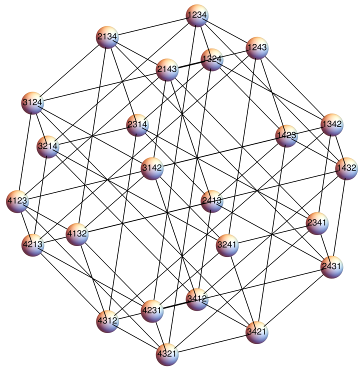
9.3 If X is smooth of dimension n, then there are n edges at each vertex. For singular spaces can be more edges from one vertex:

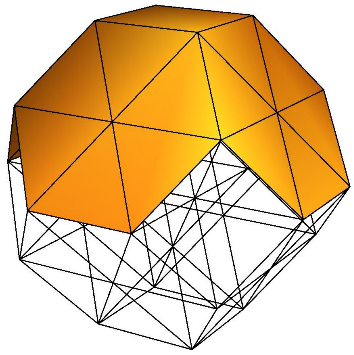
- GKM graph for the Schubert variety $X_1 = \{W \in Gr_2(\mathbb{C}^4) \mid W \cap \mathrm{lin}\{\varepsilon_1, \varepsilon_2\} \neq 0\}$. The point $\{1,2\}$ is singular.



9.4 GKM-graph for the flag variety $\mathcal{F}\ell(n)$

- The vertices V are labeled by permutations
- Since $\mathcal{F}\ell(n) \subset \prod_{k=1}^{n-1} Gr_k(\mathbb{C}^n)$ we see that one dimensional orbits join permutations if and only permutations differ by a transposition $\tau_{i,j}$
- One can realize the GKM graph in $\{\sum_{i=1}^n x_i = \frac{n(n+1)}{2}\} \subset \mathbb{R}^n$. The permutation $\sigma \mapsto (\sigma(1), \sigma(2), \dots, \sigma(n))$. Note that there are internal edges.
- For $n = 4$





Proof of Chang-Skjelbred lemma for smooth GKM spaces.

9.5 Notation:

• $H^*_{\mathbb{T}}(pt) = \Lambda = \mathbb{Q}[t_1, t_2, \dots, t_r]$

• $w : Edges \to \Lambda = \mathbb{Q}[t_1, t_2, \dots, t_r]$, $\ell \mapsto w_\ell$

• $\phi \in \Lambda$ the least common multiple of all weights appearing as in the stabilizers (up to a coefficient in $\mathbb{Q}$). For each weight appearing in the product let $\psi_w := \phi/w$.

• Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s$ be a basis over Λ of the free module $H^*_{\mathbb{T}}(X)$. By the first localization theorem $H^*_{\mathbb{T}}(X^{\mathbb{T}}) \simeq \bigoplus_{x \in X^{\mathbb{T}}} \Lambda$. The isomorphism is induced by the inclusion $\iota : X^{\mathbb{T}} \to X$. The set $\iota^*\varepsilon_1, \iota^*\varepsilon_2, \dots, \iota^*\varepsilon_s$ is a basis of $K \otimes_\Lambda H^*_{\mathbb{T}}(X^{\mathbb{T}})$ over the quotient field $K = (\Lambda)$. Any element $\underline{u} \in H^*_{\mathbb{T}}(X^{\mathbb{T}})$ can be written as

$$\underline{u} = \{u_x\}_{x \in X^{\mathbb{T}}} = \sum \frac{r_i}{s_i} \iota^* \varepsilon_i \,,$$

i.e. a sum of the basis vectors with the coefficients presented as irreducible fractions $\frac{r_i}{s_i}$ (it is unique up to a $\mathbb{Q}$-factor). The denominators s_i are products of w_ℓ's.

Goal: Show that the coefficients $\frac{r_i}{s_i}$ are integral, i.e. $s_i = 1$, provided that the divisibility condition is satisfied.

9.6 Suppose $\underline{u} \in H^*_{\mathbb{T}}(X^{\mathbb{T}}) \simeq \bigoplus_{x \in X^{\mathbb{T}}} \Lambda$ satisfies the divisibility condition

$$w_\ell \,|\, u_{s(\ell)} - u_{t(\ell)} \,,$$

where $s(\ell)$ is the source, and $t(\ell)$ is the target of the edge in the GKM graph.

• Define

$$X_w = X^{\mathbb{T}} \cup (\text{sum of the orbits with } \mathbb{T}\text{-action via } kw \,,\ k \in \mathbb{Q}).$$

With our assumptions $X_w = X^{\mathbb{T}} \cup$(disjoint union of $\mathbb{P}^1$'s)

We claim, that the product of $\psi_w \underline{u}$ belongs to the image of $H^*_{\mathbb{T}}(X_w)$ in $H^*_{\mathbb{T}}(X)$.

• Proof of the claim:

— If no edges adjacent to x is proportional to w, then x is isolated in X_w. Then $\psi_w u_x$ is equal to $(\iota_x)_*(\frac{\psi_w}{e(x)}\, u_x)$,where $e(x)$ is the Euler class at x and $\frac{\psi_w}{e(x)} \in \Lambda$.

— If x and y are connected by the edge ℓ i.e. an orbit with $\mathbb{T}$-action having the weight $w_\ell = qw$, $q \in \mathbb{Q}$, then $e(\nu_x) = \frac{e(x)}{qw} \in \Lambda$ i $e(\nu_y) = \frac{e(y)}{qw} \in \Lambda$ are the Euler classes of the normal bundle of the closure[2] of the orbit$\simeq \mathbb{P}^1$:

$$\nu = f_\ell^*(TX) - T\mathbb{P}^1 \,, \quad f_\ell : \mathbb{P}^1 \hookrightarrow X \quad e(\nu) = f_\ell^*(e(TX))/e(T\mathbb{P}^1) \,.$$

[2]In fact one has to take the normalization of the orbit.

Hence

$$e(\nu_x) = e(\nu_y) \mod w \tag{2}$$

Let $\alpha_x = \frac{\psi_w}{e(\nu_x)} \in \Lambda$, $\alpha_y = \frac{\psi_w}{e(\nu_y)} \in \Lambda$. We have $\alpha_x e(\nu_x) = \alpha_y e(\nu_y)$, and w is not proportional to any factor of that. From (2) it follows

$$\alpha_x = \alpha_y \mod w\,.$$

Since by the assumption

$$u_x = u_y \mod w$$

we have

$$\alpha_x u_x = \alpha_y u_y \mod w\,.$$

We deduce that $\{\alpha_x u_x, \alpha_y u_y\}$ defines an element of the cohomology of the closure of the orbit joining x with y. The push-forward to X restricted to x is equal to $\psi_w u_x$ and restricted to y respectively $\psi_w u_y$.
$\diamondsuit$

9.7 *The end of the proof of CS Lemma*: The coefficients of $\psi_w \underline{u} = \sum \frac{\psi_w r_i}{s_i} \iota^* \varepsilon_i$ belong to Λ. The weight w does not divide ψ_w, hence w does not divide s_i. Since w was arbitrary, $s_i = 1$. Finally we conclude that $\underline{u} = \iota^* \left(\sum r_i \epsilon_i\right)$.
$\square$

Differential model of equivariant cohomology — an overview of the next few lectures

9.8 A model of $\Omega^*(E\mathbb{T})$: It should be a differential graded algebra $A^\bullet$

- a module over $H^*(B\mathbb{T}) \simeq Sym^\bullet(\mathfrak{t}^*) = Polynomials(\mathfrak{t})$
- acyclic, i.e. $H^*(A^\bullet) \simeq H^*(pt) \simeq \mathbb{R}$
- an action of $\lambda \in \mathfrak{t}$ lowering degree by one - an analogue of the contraction of a form with the vector field generated by λ.
- Economic solution: the Weil algebra $W^\bullet(\mathfrak{t}) := Sym^\bullet \mathfrak{t}^* \otimes \wedge^\bullet \mathfrak{t}^*$. For $\xi \in \mathfrak{t}^* = \wedge^1 \mathfrak{t}^* = Sym^1 \mathfrak{t}^*$

$$1 \otimes \xi \in W^1(\mathfrak{t})\,, \qquad \xi \otimes 1 \in W^2(\mathfrak{t})\,.$$

To define the differential let us fix a basis of $\mathfrak{t}$: α_1, α_2,... α_r and the dual basis of $\mathfrak{t}^*$: α_1^*, α_2^*,... α_r^*. For $f \in Sym^\bullet \mathfrak{t}^*$, $\xi \in \Lambda^\bullet \mathfrak{t}^*$

$$d(f \otimes \xi) := \sum_{i=1}^{r} f \cdot \alpha_i^* \otimes \iota_{\alpha_i} \xi\,,$$

where ι_{α_i} is the contraction of the form ξ with the vector α_i

- Exercise: show that $d^2 = 0$ and that the differential does not depend on the choice of a basis.
- Example $n = 1$. Let $\xi = \alpha_1^*$:

$$W(\mathfrak{t}) \simeq \mathbb{R}[t] \otimes (\mathbb{R} \oplus \mathbb{R}\xi)$$

$$d(t^k \otimes \xi) = t^{k+1} \otimes 1\,, \qquad d(t^k \otimes 1) = 0$$

9.9 There is a map from $W^\bullet(\mathfrak{t})$ to the forms on approximations of $E\mathbb{T}$:

$$\Omega^\bullet(E\mathbb{T}) := \lim_{\overleftarrow{m}} \Omega^\bullet((\mathbb{C}^m \setminus \{0\})^r)$$

sending the generators of $Sym^\bullet(\mathfrak{t}^*)$ to pull-backs of forms living on $B\mathbb{T}$ and the generators of $\wedge^\bullet(\mathfrak{t}^*)$ to connection forms. (It will be explained later.)

9.10 Similarly to the model of $\Omega^\bullet(EG)$ a model of $\Omega^*(E\mathbb{T}\times^{\mathbb{T}} X)$ is obtained. The exterior algebra $\wedge^\bullet\mathfrak{t}^*$ which serve as $H^*(\mathbb{T})=\Omega^\bullet(\mathbb{T})^{\mathbb{T}}$ is replaced by $\Omega^\bullet(X)^{\mathbb{T}}$. The complex of twisted differential forms is defined as

$$Sym^\bullet\mathfrak{t}^*\otimes\Omega^\bullet(X)^{\mathbb{T}}$$

with the differential $\tilde{d}$, which is a map of $Sym^\bullet\mathfrak{t}^*$-modules. For a form $\alpha\in\Omega^k(X)^{\mathbb{T}}$ let

$$\tilde{d}(1\otimes\alpha)\in\mathbb{R}\otimes\Omega^{k+1}(X)^{\mathbb{T}}\oplus\mathfrak{t}^*\otimes\Omega^{k-1}(X)^{\mathbb{T}}$$

$$\tilde{d}(1\otimes\alpha)=1\otimes d\alpha+\sum_{i=1}^r\alpha_i^*\otimes\iota_{v_\lambda}\alpha\,,$$

where v_λ is the fundamental field generated by $\lambda\in\mathfrak{t}$.

9.11 If $\mathbb{T}=S^1$ then we obtain the model constructed by Witten. The equivariant differential forms are defined as $Sym^\bullet\mathfrak{t}^*\otimes\Omega^\bullet(X)^{\mathbb{T}}=\Omega^\bullet(X)^{\mathbb{T}}[h]$, i.e.polynomials in h with coefficients in $\Omega^\bullet(X)^{\mathbb{T}}$. The standard differential is perturbed by the contraction

$$\tilde{d}(\alpha)=d\alpha-h\iota_v\alpha\,.$$

We think of h as something very small.

• From the Cartan formula expressing the Lie derivative $\mathcal{L}_v=\iota_v d+d\iota_v$ we compute $\tilde{d}=0$.

10 De Rham model of equivariant cohomology

Main reference:

Atiyah, M. F.; Bott, R. *The moment map and equivariant cohomology,* Topology 23 (1984), no. 1, 1-28.

Text-book: Guillemin, Victor W.; Sternberg, Shlomo. *Supersymmetry and equivariant de Rham theory.* Springer, 1999

10.1 Basics about differential forms $\Omega^\bullet(M)$ on a C^∞ manifolds

• $(\Omega^\bullet(M),d)$ is a CDGA i.e. a graded-commutative algebra with a differential satisfying the Leibniz rule

• vector fields act on forms: for $X\in\Gamma(TM)$ there is a contraction operator:

$$\iota_X:\Omega^k(M)\to\Omega^{k-1}(M)\,.$$

such that for a function $f\in\Omega^0(M)=C^\infty(M)$

$$\iota_X df=Xf\,.$$

The contraction is an odd derivative

$$\iota_X(a\wedge b)=\iota_X a\wedge b+(-1)^{\deg a}a\wedge\iota_X b\,,$$

$$\iota_X\circ\iota_X=0\,.$$

• Lie derivative $\mathcal{L}_X$:

$$\mathcal{L}_X f=Xf\,,\qquad\text{for } f\in\Omega^0(M)\,,$$

$$\mathcal{L}_X(a\wedge b)=\mathcal{L}_X a\wedge b+a\wedge\mathcal{L}_X b\,,$$

$$d\circ\mathcal{L}_X=\mathcal{L}_X\circ d\,.$$

$$[\mathcal{L}_X,\mathcal{L}_Y]=\mathcal{L}_{[X,Y]}\,.$$

10.2 Cartan formula

$$\boxed{\mathcal{L}_X=d\iota_X+\iota_X d\,.}$$

• Proof: it is enough to check that it agrees for functions (YES) and both sides of equations commute with the differential and satisfy the (even) Leibniz rule:

$$d(d\iota_X+\iota_X d)=d^2\iota_X+d\iota_X d=d\iota_X d=d\iota_X d+\iota_X d^2=(d\iota_X+\iota_X d)d\,.$$

• Leibniz rule: this is a general phenomenon, that the super-commutator of two odd differentiations is an even differentiation. Set $U=\iota_X$, $V=d$. We skip $\wedge$ and write $|a|$ for $\deg a$

$$[U,V]=UV+VU\,,$$

$$\begin{aligned}UV(ab)&=U((Va)b+(-1)^{|a|}a(Vb))\\&=(UVa)b+(-1)^{|a|-1}(Va)(Ub)+(-1)^{|a|}(Ua)(Vb)+(-1)^{2|a|}a(UVb)\end{aligned}$$

$$\begin{aligned}VU(ab)&=V((Ua)b+(-1)^{|a|}a(Ub))\\&=(VUa)b+(-1)^{|a|-1}(Ua)(Vb)+(-1)^{|a|}(Va)(Ub)+(-1)^{2|a|}a(VUb)\end{aligned}$$

Hence

$$(UV+VU)(ab)=((UV+VU)a)b+a((UV+VU)b)\,.$$

10.3 We study manifolds with an action of a compact, connected Lie group G. Each element $\lambda\in\mathfrak{g}=Lie(G)$ generates a vector field, denoted v_λ. .

• Taking the fundamental field

$$\mathfrak{g}\xrightarrow{v}\{\text{vector fields on } M\}\,.$$

is a map of Lie algebras, i.e.

$$[v_\lambda,v_\mu]=v_{[\lambda,\mu]}\,.$$

• The contraction with v_λ will be denoted by ι_λ.

10.4 The structure which will be relevant in what follows is:

— M a graded vector space or an algebra

— M is equipped with a differential d of degree 1 and operations $\mathcal{L}_\lambda$ of degree 0 and ι_λ of degree -1. All together satisfy the commutative relations as described above.

• In other words M is a representation of the graded Lie algebra $\mathfrak{g}\oplus\mathfrak{g}\oplus\mathbb{R}d$

$$[\iota_\lambda,\iota_\mu]=0\,,\qquad[\mathcal{L}_\lambda,\iota_\mu]=\iota_{[\lambda,\mu]}\,,\qquad[d,\iota_\lambda]=\mathcal{L}_\lambda\,,$$

$$[\mathcal{L}_\lambda,\mathcal{L}_\mu]=\mathcal{L}_{[\lambda,\mu]}\,,\qquad[\mathcal{L}_\lambda,d]=0\,,\qquad[d,d]=0\,.$$

• Later we will assume that $\mathfrak{g}=\mathfrak{t}$ is commutative, i.e. $[\lambda,\mu]=0$.

10.5 The group G acts on $\Omega^\bullet(M)$. If G is connected

$$\Omega^\bullet(M)^G = \{\alpha \in \Omega^\bullet(M) \mid \forall_\lambda \in \mathfrak{g} \;\; \mathcal{L}_\lambda \alpha = 0\} =: \Omega^\bullet(M)^{\mathfrak{g}}\,.$$

10.6 Assume G is connected. For all $g \in G$ and $[\alpha] \in H^*(M)$ the transported form has the same cohomology class $[g^*\alpha] = [\alpha]$.

10.7 If G is compact, every form can be averaged. Hence

$$H^*(\Omega^*(M)^G) = H^*(\Omega^*(X))\,.$$

Principal bundles

10.8 Let $p : P \to B = M/G$ be a principal bundle. The group is assumed to be compact and connected. Let us define *basic forms* [Guillemin-Sternberg §2.3.5]:

$$\Omega^*(P)_{bas} = \{\alpha \in \Omega^*(P) \mid \forall v_0 \in \mathfrak{g}\; \mathcal{L}_v \alpha = 0\,,\; \iota_v \alpha = 0\} = \{\alpha \in \Omega^*(P) \mid \forall v_0 \in \mathfrak{g}\; \iota_v \alpha = 0\,,\; \iota_v d\alpha = 0\}\,.$$

This is a subcomplex.

10.9 Theorem:

$$\Omega^*(P)_{bas} = p^*\Omega^*(B) \simeq \Omega^*(B)\,.$$

10.10 For M with an action of $\mathbb{T} = S^1$. For short let $\iota = \iota_\lambda$ for a fixed $\lambda \in \mathfrak{t}$. Let us define a differential in $\mathbb{R}[h] \otimes \Omega^*(M)^{\mathbb{T}}$

$$d_h(\omega) = d - h\iota\,.$$

This is called the Cartan construction, also appears in a Witten's paper [Supersymmetry and Morse theory, J. Differential Geometry 17 (1982), no. 4, 661-692]. The symbol h stands for an independent variable, which lives in the gradation 2. If we specialize h to a number, then we obtain a $\mathbb{Z}_2$-graded complex. (Sometimes it is more convenient to have $+h\iota$, but we obtain an isomorphic complex).

10.11 The cohomology $H^*_{\mathbb{T},dR}(M) = H^*(\Omega^*(M)^{\mathbb{T}}[h], d_h)$ is a module over the polynomial ring $\mathbb{R}[h]$. If $M = pt$ then $H^*_{\mathbb{T},dR}(M) = \mathbb{R}[h]$.

10.12 We will show, that $H^*_{\mathbb{T},dR}(M) \simeq H^*_{\mathbb{T}}(M;\mathbb{R})$, first constructing a map on the level of differential forms.

- There is a mapping $\mathbb{R}[h] \to \Omega^2(\mathbb{P}^n)$, $h \mapsto \omega_n$, where ω_n is the Fubini-Study form. (It is enough to assume that $[\omega_n]$ generates $H^2(\mathbb{P}^n)$ and $(\omega_{n+1})_{|\mathbb{P}^n} = \omega_n$ to get a map to $\varprojlim$.)
- Define $M_{\mathbb{T},n} = S^{2n+1} \times^{\mathbb{T}} M$, an approximation of the Borel construction. The polynomial ring $\mathbb{R}[h]$ acts on $\Omega^*(M_{\mathbb{T},n})$, h acts as the pull back of ω_n.

10.13 We will construct a map of $\mathbb{R}[h]$ modules

$$\mathbb{R}[h] \otimes \Omega^*(M)^{\mathbb{T}} \to \Omega^*(M_{\mathbb{T},n}) = \Omega^*(S^{2n+1} \times M)_{bas}$$

First approximation: For $\alpha \in \Omega^*(M)^{\mathbb{T}}$

$$1 \otimes \alpha \mapsto p^*\alpha\,,$$

where $p : S^{2n+1} \times M \to M$ is the projection.

• We check if the image is a basic form:

— $p^*\alpha$ is $\mathbb{T}$-invariant (YES)

— $\iota(p^*\alpha) = 0$? (NO)

Some correction needs to be done.

10.14 The principal bundle and its connection: Suppose $P \to P/\mathbb{T} = B$ is a principal bundle. The tangent space of the fiber at each point is canonically isomorphic to $\mathfrak{t}$. With fixed $\lambda \in \mathfrak{t}$, the vector v_λ spans that fiber.

• The connection is a $\mathbb{T}$-invariant 1-form θ, such that $\theta(v_\lambda) = 1$. Such form can be constructed having a $\mathbb{T}$-invariant metric.

$$\theta(w) = \frac{(v_\lambda, w)}{(v_\lambda, v_\lambda)} .$$

This is just the orthogonal projection from TP to the tangent space of the fiber, i.e. to $ker(TP \to TB)$

• In general a connection is a 1-form with values in $\mathfrak{g}$, which is G invariant, with G acting on $\mathfrak{g}$ via the adjoint representation..

10.15 Let $\theta \in \Omega^1(S^{2n+1})^{\mathbb{T}}$, be the connection. This is equivalent to $\iota\theta = 1$. It is elementary to check that

$$\theta = -\frac{i}{2\pi}\partial \log ||z||^2$$

is a good choice. When restricted to the points of the form $(z_0, 0, \ldots, 0)$ it is equal to

$$-\frac{i}{2\pi}\frac{\bar{z}_0\, dz_0}{|z_0|^2} = -\frac{i}{2\pi}\frac{dz_0}{z_0} .$$

For the parametrization of the orbit $\gamma_z(t) = e^{2\pi i t}z$ we compute

$$\theta(\dot{\gamma}(0)) = \left\langle -\tfrac{i}{2\pi}\gamma_z^*(\tfrac{dz}{z}), \tfrac{d}{dt}\right\rangle = \left\langle -\tfrac{i}{2\pi}\tfrac{2\pi i e^{2\pi i t} z\, dt}{e^{2\pi i t} z}, \tfrac{d}{dt}\right\rangle = 1$$

The differential $d\theta$ is a basic form and it is the Kähler form ω_n on $\mathbb{P}^n$.

• It follows that in general $d\theta$ is a basic form: $[d\theta] \in H^2(P/\mathbb{T})$ is the first Chern class of the line bundle associated to P (up to a scalar).

10.16 Correction: We identify θ_n with its pull-back to $S^{2n+1} \times M$.

• Let

$$\alpha' = p^*\alpha - \theta_n \wedge p^*\iota\alpha .$$

We have

$$\iota\alpha' = \iota p^*\alpha - \iota(\theta_n \wedge p^*\iota\alpha) = \iota p^*\alpha - 1 \wedge p^*\iota\alpha + \theta_n \wedge \iota p^*\iota\alpha = 0 .$$

• We check that the map $\phi : f(h) \otimes \alpha \mapsto f(\omega_n) \wedge (p^*\alpha - \theta \wedge p^*(\iota\alpha))$ is a chain map. It is enough to check for $f(h) = 1$

$$\begin{aligned} d\phi(1 \otimes \alpha) &= d\big(p^*\alpha - \theta_n \wedge p^*(\iota\alpha)\big) \\ &= dp^*\alpha - d\theta_n \wedge p^*(\iota\alpha) + \theta_n \wedge dp^*(\iota\alpha) , \\ \phi(d_h(1 \otimes \alpha)) &= \phi(1 \otimes d\alpha - h \otimes \iota\alpha) = \phi(1 \otimes d\alpha) - \phi(h \otimes \iota\alpha) \\ &= p^*d\alpha - \theta_n \wedge p^*(\iota d\alpha) - \omega_n \wedge p^*(\iota\alpha) \end{aligned}$$

Since α is $\mathbb{T}$ invariant

$$dp^*(\iota\alpha) = p^*(d\iota\alpha) = p^*(-\iota d\alpha)$$

we obtain that $d\phi(1 \otimes \alpha) = \phi(d_h(1 \otimes \alpha))$.

10.17 Theorem: the map $\phi : \mathbb{R}[h] \otimes \Omega^\bullet(M)^{\mathbb{T}} \to \varprojlim \Omega^\bullet(S^{2n+1} \times M)_{bas}$ is a quasiisomorphism, i.e. an isomorphism of cohomologies.

- Proof:

— The complex $\mathbb{R}[h] \otimes \Omega^\bullet(M)$ is filtered (a decreasing filtration) by the powers the ideal (h).

— The complex $\varprojlim \Omega^\bullet(S^{2n+1} \times M)_{bas}$ is filtered by

$$ker\big(\varprojlim \Omega^\bullet(S^{2n+1} \times M)_{bas} \to \Omega^\bullet(S^{2n+1} \times M)_{bas}\big)\,.$$

The map ϕ is a quasiisomorphism on the associated graded complexes. Hence it is a quasiisomorphism. (This is an exercise in homological algebra.)

11 Models for higher dimensional Lie groups. Moment map $M \to \mathfrak{t}^*$

11.1 Reference to general theory of G^* modules: Guillemin-Sternberg §2. **We make the assumption $G = \mathbb{T}$ simplifying radically the formulas.**

11.2 Let $p : P \to B$ be a S^1-principal bundle (i.e. S^1 acts freely on P and $B = P/S^1$). We identify S^1 with the image

$$\mathbb{R} \to \mathbb{C}\,, \qquad t \mapsto e^{2\pi i t}\,,$$

hence we have determined the choice of $\lambda \in \mathfrak{t} \simeq \mathbb{R}$.

- Let $\theta \in \Omega^1(P; \mathfrak{t})^{\mathbb{T}} \simeq \Omega^1(P)^{\mathbb{T}}$ be a connection, i.e. $\iota\theta = 1$.
- The form $d\theta$ is closed. We check that $d\theta$ is a basic form

$$\iota d\theta = \mathcal{L}\theta - d\iota\theta = 0 - d1 = 0\,.$$

Hence $d\theta$ defines an element of $H^2(B)$.

- Exercise: $[d\theta] = c_1(L)$, where L is the associated line bundle $L = P \times^{S^1} \mathbb{C}$. In particular the cohomology class does not depend on the choice of the connection. Hint for $B = \mathbb{P}^n$ we have $d\theta = -\omega_{FS}$.

11.3 The case of a higher dimensional torus $\mathbb{T} = (S^1)^n$ acting on a smooth manifold M:

- Set $A = \Omega^\bullet(M)$. Let

$$\tilde{A} = \text{Polynomial functions}(\mathfrak{t}, A)^{\mathbb{T}} \simeq Sym\,\mathfrak{t}^* \otimes A^{\mathbb{T}}$$

Here

$$Sym\,\mathfrak{t}^* = \bigoplus_{k=0}^{\infty} Sym^k \mathfrak{t}^* = \text{Polynomial functions on } \mathfrak{t}\,.$$

- The constructions below are purely algebraic. Thus we consider a G^* module A i.e a graded vector space equipped with operations d, ι_λ, $\mathcal{L}_\lambda$ for $\lambda \in \mathfrak{t}$ satisfying the relations 10.3.
- We set

$$A_{hor} = \{\alpha \in A \mid \forall \lambda \in \mathfrak{t}\ \iota_\lambda \alpha = 0\} \qquad \text{horizontal submodule}$$

and

$$A_{bas} = A_{hor}^{\mathbb{T}} = \{\alpha \in A \mid \forall \lambda \in \mathfrak{t}\ \iota_\lambda \alpha = 0,\ \iota_\lambda d\alpha = 0\}\,.$$

• The differential in $\tilde{A}$ is $Sym\,\mathfrak{t}^*$-linear and for $\alpha \in A^k$

$$\tilde{d}(1 \otimes \alpha)(\lambda) = d\alpha - \iota_\lambda \alpha$$

viewed as a function on $\mathfrak{t}$, which is linear with respect to λ, i.e. it belongs to

$$\mathbb{R} \otimes A^{k+1} \oplus \mathfrak{t}^* \otimes A^{k-1}\,.$$

In a basis $\lambda_1, \dots \lambda_n$ of $\mathfrak{t}$

$$\tilde{d}(1 \otimes \alpha) = 1 \otimes d\alpha - \sum_{i=1}^{n} \lambda_i^* \otimes \iota_{\lambda_i} \alpha\,.$$

• We will use physicists notation. The vectors will have superscripts, and functionals subscripts. Also the running index will be a instead if i, which can easily confused with ι. We write

$$\tilde{d}(1 \otimes \alpha) = 1 \otimes d\alpha - \sum_{a=1}^{n} \lambda_a \otimes \iota_{\lambda^a} \alpha$$

or according to the Einstein notation

$$\tilde{d}(1 \otimes \alpha) = 1 \otimes d\alpha - \lambda_a \otimes \iota_{\lambda^a} \alpha\,.$$

11.4 [Guillemin-Sternberg §3.2] If $A = \Omega^\bullet(\mathbb{T})$, then $A^{\mathbb{T}} = \wedge \mathfrak{t}^*$. The resulting $\tilde{A}$ is the Weil algebra of $\mathfrak{t}$

$$W(\mathfrak{t}) = Sym(\mathfrak{t}^*) \otimes \wedge \mathfrak{t}^*\,.$$

• Theorem: $H^0(W(\mathfrak{t})) = \mathbb{R}$ and $H^k(W(\mathfrak{t})) = 0$ for $k > 0$.
Proof: Since $W(\mathfrak{t}_1 \oplus \mathfrak{t}_2) = W(\mathfrak{t}_1) \otimes W(\mathfrak{t}_2)$ as dg-algebra, it is enough to compute cohomology for $\mathfrak{t}$ of dimension 1. This was an easy check.
• Since $\Omega^\bullet(\mathbb{T})^{\mathbb{T}} = \wedge \mathfrak{t}^*$, if $\dim \mathbb{T} = 1$ an explicit map from $W(\mathfrak{t}) = \mathbb{R}[h] \otimes (\mathbb{R} + \mathfrak{t}^*)$ to

$$(\Omega^\bullet(S^{2m+1} \setminus 0) \times \wedge \mathfrak{t}^*)_{bas}$$

was already given in the previous section:

$$f \otimes \xi \mapsto f(\omega_{FS})(\xi - \theta \wedge \iota\xi)\,.$$

For higher dimensional tori we take the product of these maps and obtain a quasiisomorphism

$$W(\mathfrak{t}) \to \Omega^\bullet(E\mathbb{T} \times^{\mathbb{T}} \mathbb{T}) \overset{qis}{\simeq} \Omega^\bullet(E\mathbb{T})\,.$$

The right hand side is understood as the inverse limit of forms on finite dimensional representations. Note that $W(\mathfrak{t})$ is a very economic model of forms on $E\mathbb{T}$.

Mathai-Quillen twist See [Mathai-Quillen: Superconnections, Thom classes, and equivariant differential forms. Topology25(1986), no.1, 85-110], [Guillemin-Sternberg §7.2]
We construct an explicit map of complexes

$$\tilde{A} \to (W(\mathfrak{t}) \otimes A)_{bas} \overset{qis}{\simeq} (\Omega^\bullet(EG) \otimes A)_{bas}\,,$$

which for $A \simeq \Omega^\bullet(M)$ will provide a convenient model for the equivariant cohomology.

11.5 [Guillemin-Sternberg §2.3.4] Let A be a $\mathbb{T}^*$ module. We say that A is locally free if there exists a connection, i.e. $\theta \in \mathfrak{t} \otimes (A^1)^{\mathbb{T}}$, in a basis of $\mathfrak{t}$ it can be written as

$$\sum_{a=1}^{n} \lambda^a \otimes \theta_a \,.$$

such that for

$$\theta_a(\lambda^b) = \delta_a^b \,.$$

• Differential forms $\Omega^\bullet(M)$ is a locally free $\mathbb{T}^*$ module if the action of T is locally free, i.e. the stabilizers of points are finite.

11.6 Mathai-Quillen twist: consider $\mathbb{T}^*$-algebras W and A, with W locally free (e.g. $W = W(\mathfrak{t})$. Let

$$\gamma = \sum \theta_a \otimes \iota_{\lambda^a} ,$$

$$\phi = exp(\gamma) \in Aut(W \otimes A) = 1 + \gamma + \tfrac{1}{2}\gamma \circ \gamma + \dots$$

It is well defined since $\gamma^{n+1} = 0$ for $n = \dim(\mathbb{T})$.

11.7 The map γ, hence also ϕ, is T-invariant.

• **Theorem.** [Guillemin-Sternberg , chapter 4, Theorem 4.1.1] For any $\lambda \in \mathfrak{t}$

$$\phi \circ (\iota_\lambda \otimes 1 + 1 \otimes \iota_\lambda) \circ \phi^{-1} = \iota_\lambda \otimes 1$$

$$\phi \circ (d \otimes 1 + 1 \otimes d) \circ \phi^{-1} = (d \otimes 1 + 1 \otimes d) - \sum \nu_a \otimes \iota_{\lambda^a} + \sum \theta_a \otimes \mathcal{L}_{\lambda^a}$$

where $\nu_a = d\theta_a$

• This is a direct computation. See [W. Greub, S. Halperin, S, Vanstone: Curvature, Connections and Cohomology, vol. III Academic Press New York. (1976)] Prop. V, p.286,, or better compute it manually. This is an **Exercise.**

11.8 After the twist

$$\phi((W \otimes A)_{hor}) = W_{hor} \otimes A$$

For $W = W(\mathfrak{t})$

$$\phi((W \otimes A)_{bas}) = S(\mathfrak{t}) \otimes A$$

with the differential

$$\tilde{d} = 1 \otimes d - \sum \lambda^a \otimes \iota_{\lambda_a}$$

That is exactly the **Cartan model** of equivariant cohomology. [Guillemin-Sternberg §4.2]

11.9 The construction can be carried out for noncommutative connected groups. The action of G on $\mathfrak{g}$ has to be taken into account. Then the cohomology of

$$(Sym\, \mathfrak{g}^* \otimes \Omega^\bullet(M))^G$$

with an appropriate differential serves, as a model for equivariant cohomology. Reference: Guillemin-Sternberg §3-4

Moment map

11.10 Assume $T = S^1$. Let $\alpha \in \Omega^2(M)^{\mathbb{T}}$. Suppose $d\alpha = 0$. An equivariant enhancement of α is a function $f \in \Omega^0(M)$, such that

$$d_h(1 \otimes \alpha - h \otimes f) = 0\,,$$

i.e.

$$1 \otimes d\alpha - h \otimes \iota\alpha + h \otimes df = 0\,.$$

This reduces to

$$\iota\alpha = df\,.$$

11.11 Basic example: Moment map $f : \mathbb{P}^1 \to \mathbb{R}$.

• Suppose $\mathbb{T} = S^1$ acts on $\mathbb{P}^1$ with the weights (λ_0, λ_1). In the 0-th affine standard chart the action is linear and the weights are $\lambda_1 - \lambda_0$. The fundamental field at the point z is equal to

$$v = \frac{d}{dt}(e^{(\lambda_1-\lambda_0)2\pi i t} z)_{|t=0} = 2\pi i(\lambda_1 - \lambda_0)z = 2\pi(\lambda_1 - \lambda_0)(-y + ix)$$

i.e.

$$v = 2\pi(\lambda_1 - \lambda_0)\left(-y\tfrac{d}{dx} + x\tfrac{d}{dy}\right)\,.$$

Let $\alpha = \omega_{FS}$. In the affine coordinate

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar\partial\log(1 + |z|^2) = \frac{i}{2\pi}\frac{dz \wedge d\bar z}{(1 + |z|^2)^2} = \frac{1}{\pi}\frac{dx \wedge dy}{(1 + x^2 + y^2)^2}.$$

We compute the contraction

$$\iota_v\omega_{FS} = 2\pi(\lambda_1 - \lambda_0)\left(-y\iota_x\omega_{FS} + x\iota_y\omega_{FS}\right) = -2\pi(\lambda_1 - \lambda_0)\frac{ydy + xdx}{(1 + x^2 + y^2)^2}\,.$$

Let

$$f = \frac{\lambda_0 + \lambda_1|z|^2}{1 + |z|^2} = \frac{\lambda_0 + \lambda_1(x^2 + y^2)}{1 + x^2 + y^2}\,,$$

$$df = (\lambda_1 - \lambda_0)\frac{2xdx + 2ydy}{(1 + x^2 + y^2)^2}\,.$$

The form

$$1 \otimes \omega_{FS} - h \otimes \pi f$$

is a closed equivariant form.

• Globally f is defined by the formula

$$f([z_0, z_1]) = \frac{\lambda_0|z_0|^2 + \lambda_1|z_1|^2}{||z||^2}\,.$$

11.12 In general, if the action on $\mathbb{P}^n$ has weights $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ we set

$$f([z]) = \frac{\sum_{i=0}^n \lambda_i|z_i|^2}{||z||^2}$$

Then $1 \otimes \omega_n - h \otimes \pi f$ is an equivariant d_h-closed form.

• An element $f \in \mathfrak{t}^* \otimes \Omega^0(M) = \mathrm{Hom}(\mathfrak{t}, C^\infty(M))$ by adjunction is the same as a map $\mu : M \to \mathfrak{t}^*$

$$\langle \mu(x), \lambda\rangle = f(\lambda)(x)\,.$$

• For $\mathbb{T} = (S^1)^{n+1}$ acting on $\mathbb{P}^n$ we obtain the map

$$\mu([z]) = \frac{1}{||z||^2}(|z_0|^2, |z_1|^2, \dots, |z_n|^2)\,.$$

Symplectic geometry [Guillemin-Sternberg §9], but before beginning see [V. I. Arnold, Mathematical Methods Of Classical Mechanics. Graduate Texts in Mathematics 60. Springer 1989] chapter 8.

11.13 The most interesting case is when M is a symplectic manifold e.g. Kähler manifold and the symplectic ω has a lift to an equivariant form, then $\mu : M \to \mathfrak{t}^*$ is defined.

• Of course μ is constant on the components of $X^{\mathbb{T}}$.

11.14 Symplectic manifold (M, ω) such that ω is a nondegenerate 2-form, $d\omega = 0$

• basic examples:

— M complex Kähler manifold,

— $M = T^*N$, where N is a real smooth manifold, $\omega = d($Liouville form

• ω induces an isomorphism $TM \simeq T^*M$: $v \mapsto \iota_v\omega$

– a function f defines a vector field X_f. It is the field, such that $\iota_{X_f}\omega = df$

– the symplectic structure defines a structure of a Lie algebra of functions (Poisson bracket)

$$\{f, g\} = \omega(X_f, X_g) = (\iota_{X_f}\omega)(X_g) = df(X_g) = X_g f\,.$$

• Definition: Action of S^1 is Hamiltonian iff the fundamental field v is equal to X_f for some f

$$\iota_v\omega = df \quad \text{i.e.} \quad v = X_f\,.$$

If that is so then $\omega + h\,f$ is a closed equivariant form.

12 Hamiltonian action and the moment map

[Dusa McDuff, Dietmar Salamon ; Introduction to Symplectic Topology (Oxford Mathematical Monographs) §5]

[Anna Cannas da Silva Lectures on Symplectic Geometry.]

12.1 Physical motivation:

• Hamiltonian system q position, $p = mv$ momentum, $H(p, q)$ a C^∞ function

$$\begin{cases} \dot q = \frac{\partial H}{\partial p} \\ \dot p = -\frac{\partial H}{\partial q} \end{cases}$$

• Motion of a particle in the constant gravitation field, H=energy, $q = h$ height:

$$H(q, p) = \tfrac{mv^2}{2} + mgq = \tfrac{p^2}{2m} + mgq\,, \qquad \begin{cases} \dot q = \frac{p}{m} = v \\ \dot p = -mg \end{cases}$$

• Conservation energy law: H is constant along trajectories

12.2 Poisson bracket in local Darboux coordinates

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i\,, \qquad \{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\,.$$

• The Hamiltonian equations take the form $\dot q = \{q, H\}$, $\dot p = \{p, H\}$.

12.3 Let ω be a symplectic form on M and $f : M \to \mathbb{R}$. Then ω is invariant with respect to the Hamiltonian flow generated by f

$$\mathcal{L}_{X_f}\omega = d\iota_{X_f}\omega + \iota_{X_f}d\omega = d\iota_{X_f}\omega = ddf = 0\,.$$

We also note that $\iota_{X_f}\omega$ is closed.

12.4 The commutator of the Hamiltonian fields is related with the Poisson bracket

$$[X_f, X_g] = -X_{\{f,g\}}\,.$$

• We have to show that

$$\iota_{[X_f,X_g]}\omega = d\{g,f\} \quad \text{which is by definition } d(\omega(X_g, X_f))\,.$$

• We compute the Lie derivative

$$\mathcal{L}_{X_f}(\iota_{X_g}\omega) = \iota_{\mathcal{L}_{X_f}X_g}\omega = \iota_{[X_f,X_g]}\omega$$

since $\mathcal{L}_{X_f}\omega = 0$. By the Cartan formula

$$\mathcal{L}_{X_f}(\iota_{X_g}\omega) = d\iota_{X_f}\iota_{X_g}\omega + \iota_{X_f}d\iota_{X_g}\omega = d(\omega(X_g, X_f))\,.$$

12.5 Let $C^\infty(M;TM)$ be the space of smooth vector fields. It is a Lie algebra with respect to the Poisson bracket. The map

$$-X : C^\infty(M) \to C^\infty(M;TM)\,, \qquad f \mapsto -X_f$$

is a map of Lie algebras. (Applying alternative conventions we can get rid of ,,–".)

• For an arbitrary Lie group: The G-action defines a map of Lie algebras

$$v : \mathfrak{g} \to C^\infty(M;TM)\,.$$

We say that the action is Hamiltonian if there exists a linear map of Lie algebras $\tilde\mu : \mathfrak{g} \to C^\infty(M)$ making the following diagram commutative up to a sign

$$\begin{array}{ccc} & & C^\infty(M) \\ & {\scriptstyle\tilde\mu}\nearrow & \downarrow{\scriptstyle X} \\ \mathfrak{g} & \xrightarrow{v} & C^\infty(M;TM) \end{array}$$

Existence of the map $\tilde\mu$ is equivalent to having a map $\mu : M \to \mathfrak{t}^*$, called the moment map.

12.6 From now on we assume that $G = \mathbb{T} = (S^1)^n$. The moment map is given in coordinates $\mu = (\mu_1, \ldots, \mu_n) \in \mathfrak{t}^* = \mathbb{R}^n$. The Hamiltonian flows associated to μ_i commute, moreover we assume $\{\mu_i, \mu_j\} = 0$, so that $\tilde\mu : \mathfrak{t} \to C^\infty(M)$ is a map of Lie algebras.

12.7 The map μ restricted to the fixed points is locally constant. The moment map $\mu \in C^\infty(M, \mathfrak{t}^*)$ evaluated at $\lambda \in \mathfrak{t}$ is a function whose differential vanishes at zeros of the fundamental vector field:

$$d\mu(\lambda)(x) = 0 \quad \text{iff} \quad v_\lambda(x) = 0\,.$$

12.8 The map μ is constant on the orbits:

$$d\mu_i(v_{\lambda_j}) = (\iota_{v_{\lambda_i}}\omega)(v_{\lambda_j}) = \omega(v_{\lambda_i}, v_{\lambda_j}) = \{\mu_i, \mu_j\} = 0\,.$$

12.9 Theorem [Atiyah, Guillemin-Sternberg]. If M is compact, then $\Delta_{M,\mathbb{T}} := \mu(M)$ is a convex polytope

$$\Delta_{M,\mathbb{T}} = Conv(\mu(M^T))\,.$$

See [McDuff-Salamon §5.5, Theorem 5.47]

- Note that the image of the moment map μ restricted to a 1-dimensional $\mathbb{T}_\mathbb{C} = \mathbb{T} \otimes \mathbb{C}$ orbit is an interval.

12.10 Assume $M \subset \mathbb{P}^m$ is a smooth projective variety, $\omega = (\omega_{FS})_{|M}$.

12.11 The most important example $M = \mathbb{P}^n$, $\mathbb{T} = (S^1)^{n+1}$, $\mu = const\frac{1}{||z||^2}(\dots, |z_i|^2, \dots) \in \mathbb{R}^{n+1}$. The constant depends on the convention.

12.12 If M is a smooth projective variety with an algebraic action of $\mathbb{T}_\mathbb{C} \simeq (\mathbb{C}^*)^n$ then it can be equivariantly embedded into $\mathbb{P}(V)$ for some representation V of a finite cover of $\mathbb{T}$. Hence it admits a moment map (possibly after a modification of ω).

- If M is a smooth projective toric variety (i.e. M has a dense and open orbit of $\mathbb{T}_\mathbb{C}$), then $M/\mathbb{T} = \Delta_{M,\mathbb{T}}$.

12.13 Suppose M is equivariantly embedded into $\mathbb{P}(V)$, $L = \mathcal{O}(1)_{|M}$ an equivariant vector bundle. The form $\omega = \omega_{FS|M}$ represents $c_1(L) \in H^2_\mathbb{T}(M)$. Let $x \in M^\mathbb{T}$ be a fixed point. Then $c_1(L)_{|x} \in H^2_\mathbb{T}(pt) \simeq \mathrm{Hom}(\mathbb{T}, S^1)$ is the character of the action of $\mathbb{T}$ on L_x. We claim that

$$\mu(x) = c_1(L) \in \mathrm{Hom}(\mathbb{T}, S^1) \otimes \mathbb{R} = \mathfrak{t}^*\,.$$

- That is true for $M = \mathbb{P}^n$ with the action of $(S^1)^{n+1}$, since

$$\mu([0 : \dots : 0 : 1 : 0 : \dots : 0]) = (0, \dots, 0, 1, 0, \dots, 0) \qquad \text{with the preferred normalization.}$$

In general chose coordinates of $V = \mathbb{C}^{m+1}$, such that $\mathbb{T}$ action is diagonal. Consider the embedding $\mathbb{T} \hookrightarrow \mathbb{T}_{big} = (S^1)^{m+1}$ and the natural maps

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{t}^* \\ \downarrow & & \uparrow \\ \mathbb{P}^m & \xrightarrow{\mu_{big}} & \mathfrak{t}^*_{big} \end{array}$$

The claim follows from the commutativity of the diagram.

12.14 (!!!) Note that the moment polytope does not depend on the C^∞ consideration with the symplectic form. It only depends on the action of $\mathbb{T}$ on L. It can be defined purely in the realm of algebraic geometry as

$$\Delta_{M,\mathbb{T}} = Conv\{\chi(L_x) \mid x \in M^{\mathbb{T}}\}\,.$$

12.15 Example. Let $M = \mathcal{F}\ell(n)$ be the flag manifold. We have an equivariant embedding

$$\mathcal{F}\ell(n) \hookrightarrow \prod_{k=1}^{n-k} Gr_k(\mathbb{C}^n) \hookrightarrow \prod_{k=1}^{n-k} \mathbb{P}(\wedge^k\mathbb{C}^n)\,.$$

Let $p_i : \mathcal{F}\ell(n) \to \mathbb{P}(\wedge^k\mathbb{C}^n)$ be the projection and let ω_k be the Fubini-Study form on $\mathbb{P}(\wedge^k\mathbb{C}^n)$. For a sequence of positive numbers $a_i \in \mathbb{R}^n$ let

$$\omega_{\underline{a}} = \sum_{k=1}^{n-1} a_k p_k^*(\omega_k)\,.$$

This is a symplectic form and the $\mathbb{T}$ action admits a moment map

$$\mu_{\underline{a}} = \sum_{k=1}^{n-1} a_k\, \mu_k \circ p_k\,,$$

where μ_k is the moment map for $\mathbb{P}(\wedge^k\mathbb{C}^n)$.

12.16 Suppose

$$(V_1 \subset \cdots \subset V_{n-1}) \in \mathcal{F}\ell(n)^{\mathbb{T}}\,.$$

Such a point corresponds to a permutation $\sigma \in \Sigma_n$

$$V_1 = lin\{\epsilon_{\sigma(1)}\}\,, \quad V_2 = lin\{\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}\}\,, \quad \dots, \quad V_{n-1} = lin\{\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}, \dots, \epsilon_{\sigma(n-1)}\}\,.$$

Denote it by V_σ

12.17 The value of the map $Gr_k(\mathbb{C}^n) \to \mathbb{P}(\wedge^k\mathbb{C}^n) \stackrel{\mu_k}{\to} \mathbb{R}^n$ restricted at the point

$$lin\{\epsilon_{\sigma(i)} \mid i \le k\}$$

is equal to

$$-\sum_{i=1}^{k} \epsilon_{\sigma(i)}\,.$$

• For $n = 4$ the moment polytopes for $Gr_1(\mathbb{C}^4)$ and $Gr_3(\mathbb{C}^4)$ are tetrahedra, and $Gr_2(\mathbb{C}^4)$ is the octahedron.

12.18 Take $\underline{a} = (1, 1, \dots, 1)$ then

$$\mu_{\underline{a}}(V_\sigma) = -\sum_{k=1}^{n-1}\sum_{i=1}^{k} \epsilon_{\sigma(i)} = -\sum_{k=1}^{n-1}(n-k)\epsilon_{\sigma(k)}\,,$$

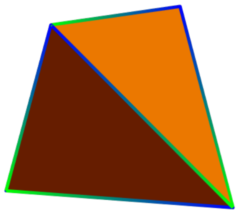
which is equal up to the shift by $n\sum_{k=1}^n \epsilon_k$ to $\sum_{k=1}^n k\epsilon_{\sigma(k)}$.

• This way we obtain the permutohedron in $\mathbb{R}^n$ which can also be defined as the convex hull of Σ_n orbit of $(1, 2, \dots, n)$.

12.19 Taking various values of a_i we obtain deformations of the permutohedron

$$Conv(\Sigma_n(a_1, a_1 + a_2, \dots, a_1 + a_2 + \cdots + a_n)) \quad \text{up to a shift.}$$

The extreme values with some a_i's equal to 0, the images are moment polytopes for partial flag varieties.



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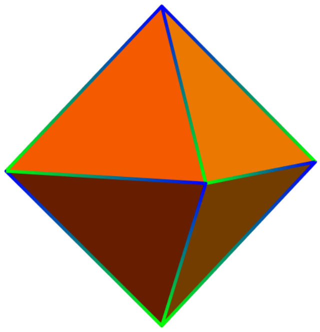

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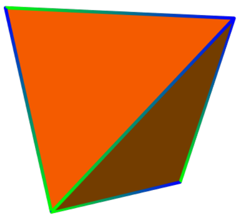

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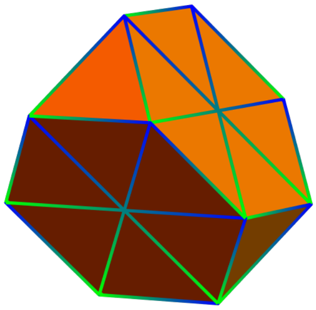

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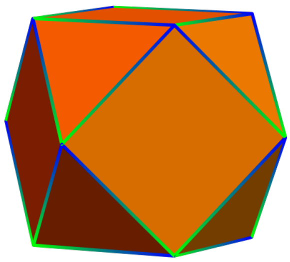

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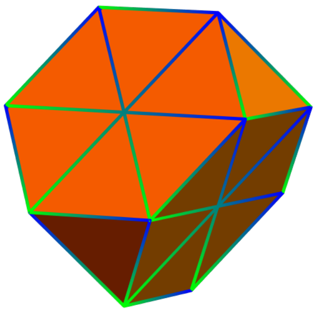

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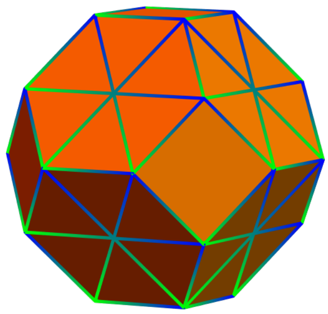

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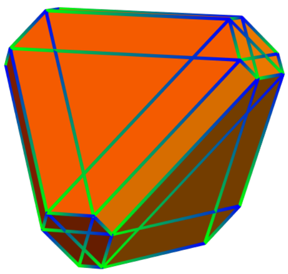
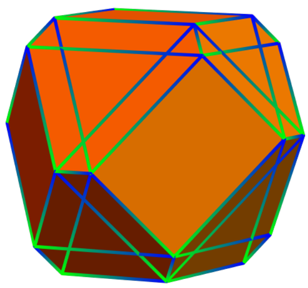
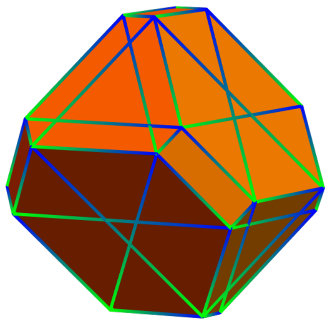


a={0.5, 1, 4}



a={3, 1, 4}



a={3, 4, 1}

13 Moment map and quotients

13.1 Suppose a compact group G acts on a symplectic manifold (M, ω) with a moment map $\mu : M \to \mathfrak{g}^*$. Recall that ω is G invariant $\mathcal{L}_\lambda \omega = 0$ and μ is G invariant with respect to the coadjoint action on $\mathfrak{g}^*$.

13.2 Symplectic reduction [Guillemin-Sternberg §9.6], [McDuff,Salamon §5.4]

• Assume that $a \in \mathfrak{g}^*$ is an invariant element with respect to the coadjoint action. Then $\mu^{-1}(a)$ is G-invariant manifold.

• Furthermore assume that G action on $\mu^{-1}(a)$ is free. Then the quotient $X = \mu^{-1}(a)/G$ is denoted by $M//_{\mu,a}G$. Often a is assumed to be 0 and we write $M//_\mu G$. This is called the symplectic quotient. We will assume that $a = 0$.

13.3 Let $x \in \mu^{-1}(0)$. The tangent space T_xGx is coisotropic and $(T_xGx)^{\perp_\omega} = T_x\mu^{-1}(0)$.

• For $\lambda \in \mathfrak{g}$, $v \in T_x\mu^{-1}(0)$ compute $\omega(X_\lambda, v) = d\mu_\lambda(v)$, where $\mu_\lambda(x) = \mu(x)(\lambda)$. But since $\mu^{-1}(0)$ is mapped by μ to 0, the tangent vectors are mapped to 0 as well. Hence $(T_xGx)^{\perp_\omega} \subset T_x\mu^{-1}(0)$. Since $\dim((T_xGx)^{\perp_\omega}) = \dim G$ and $T_x\mu^{-1}(0) = \dim M - \dim G$ and ω is nondegenerate, the opposite inclusion holds.

13.4 The manifold X has a canonical symplectic structure induced from M: For $v, w \in T_yX$ find the lifts $\tilde{v}, \tilde{w} \in T_xM$ (with x mapping to y) and apply ω. It is well defined because ω is G-invariant and the orbits lie in the kernel of ω. Moreover the induced form is nondegenerate (it is an exercise in the linear algebra).

13.5 Example 1. $M = \mathbb{C}^n$ with the standard form, $G = S^1$ acting by scalar multiplication, $\mu(z) = |z|^2$, $a \in 1$. Then

$$\mathbb{C}^n//_{\mu,a}S^1 = \mathbb{P}^{n-1}$$

with the Fubini-Study form.

13.6 Example 2 (slightly more general): $M = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$, $k < n$ with the action of $U(k)$. Let $A^* = \overline{A}^T$. Note that $\mathfrak{u}(k) = \{X \in \mathfrak{gl}_k \mid X^* = -X\}$. The moment map is defined by

$$\mu(A) = iA^*A \in \mathfrak{u}(k) \simeq \mathfrak{u}(k)^* \,.$$

$a = iI$. Then $\mu^{-1}(a)$ is equal to unitary k-tuples of vectors in $\mathbb{C}^n$, and $X//_{\mu,a}U(k)$ is equal to the Grassmannian $Gr_k(\mathbb{C}^n)$.

• Exercise: Compute that this is a moment map.

13.7 Kirwan [Cohomology of Quotients in Symplectic and Algebraic Geometry] compared symplectic quotients with GIT quotients in algebraic geometry. They basically coincide: the symplectic quotient by a compact group G is equal to the GIT quotient by the complexification $G_\mathbb{C}$ (as C^∞ manifolds). The symplectic quotients depends on the choice of the moment map (and $a \in \mathfrak{g}$) and GIT quotient depends on the linearization and stability condition. These notions can be translated one to another.

13.8 Example 3 (still more general): We want to obtain $\mathcal{F}\ell_n = \mathrm{GL}_n/B_n$ as a symplectic quotient. The Borel group is not a complexification of a compact group. Thus we take a presentation of the flag manifold in terms of a quiver:

$$1 \to 2 \to \cdots \to n-1 \to \boxed{n}$$

• Let $M = \prod_{k=1}^{n-1} \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{k+1})$, $G = \prod_{k=1}^{n-1} U(k)$. The moment map is given by

$$(A_1, A_2, \ldots, A_{n-1}) \mapsto (A_1^* A_1, A_2^* A_2, \ldots, A_{n-1}^* A_{n-1})$$

and a is the sequence of i times the identity matrices.

• $\mu^{-1}(a)$ is a sequence of isometric embeddings $\mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1}$, the quotient is the flag variety. Taking the quotient we forget about the particular coordinates on $V_k \subset \mathbb{C}^n$.

13.9 [Kirwan] If M is a compact symplectic manifold with a G action admitting a moment map μ, $X = M//_{\mu,a}$, then the map

$$\kappa : H_G^*(M) \to H_G^*(\mu^{-1}(a)) \simeq H^*(X)$$

is surjective.

[D. Mumford, J. Fogarty, and F. Kirwan. Geometric Invariant Theory, volume 34 of Results in Mathematics and Related Areas (2). Springer-Verlag, third edition, 1994. §8], compare [Megumi Harada, Gregory D. Landweber, Surjectivity for Hamiltonian G-spaces in K-theory, Trans. Amer. Math. Soc. 359 (2007), 6001-6025]

• The assumptions of the theorem can be relaxed. Just assume that μ is proper.

• A double-equivariant version: Assume that a group $\mathbb{T}$ acts on M, and $\mathbb{T}$ action commutes with G-action, then

$$\kappa : H_{\mathbb{T}\times G}^*(M) \to H_{\mathbb{T}\times G}^*(\mu^{-1}(a)) \simeq H_{\mathbb{T}}^*(X)$$

is surjective.

13.10 Back to Example 1:

$$\kappa : H_{\mathbb{C}^*}^*(\mathbb{C}^n) \simeq \mathbb{Q}[h] \twoheadrightarrow H^*(\mathbb{P}^{n-1}) \simeq \mathbb{Q}[h]/(h^n)$$

$$\kappa : H_{\mathbb{T}\times\mathbb{C}^*}^*(\mathbb{C}^n) \simeq \mathbb{Q}[t_1, t_2, \ldots t_n, h] \twoheadrightarrow H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) \simeq \mathbb{Q}[t_1, t_2, \ldots, t_n.h]/(\prod(h+t_i))$$

13.11 Back to Example 2:

$$\kappa : H_{U(k)}^*(\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)) \simeq \mathbb{Q}[c_1, c_2, \ldots, c_k] \twoheadrightarrow H^*(Gr_k(\mathbb{C}^n))$$

$$\kappa : H_{\mathbb{T}\times U(k)}^*(\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)) \simeq \mathbb{Q}[t_1, t_2, \ldots t_n, c_1, c_2, \ldots, c_k] \twoheadrightarrow H_{\mathbb{T}}^*(Gr_k(\mathbb{C}^n))$$

13.12 Projective toric varieties (without fans, but via polytopes), compare [Anderson-Fulton, Ch 8].

• Let X be a smooth compact algebraic manifold with a torus action. Assume that $\dim X = \dim \mathbb{T}_\mathbb{C}$ and $\mathbb{T}_\mathbb{C}$ has an open orbit and dense. We can assume that $\mathbb{T}_\mathbb{C}$ action is free on the open orbit. Then X is determined by a certain combinatorial data involving characters.

• Assume that the action of $\mathbb{T}$ admits a moment map to $\mathfrak{t}^* \simeq \mathbb{R}^n$. If the moment map is the restriction of the standard moment map $X \hookrightarrow \mathbb{P}^N \to \mathfrak{t}_N^* \to \mathfrak{t}^*$, then the moment polytope Δ_X has integral vertices.

• Since we assume that X is smooth, thus locally, around any fixed point X looks like $\mathbb{C}^n$ with the standard action of $(\mathbb{C}^*)^n$, so the moment polytope locally is linearly isomorphic to a neighbourhood of $0 \in \mathbb{C}^n/(S^1)^n \simeq \mathbb{R}^n_{\geq 0}$.

• Each facet F_i (a codimension 1 face) of $\Delta_X \subset \mathfrak{t}^*$ we set $v_i \in (\mathfrak{t}^*)^* = \mathfrak{t}$, the normal vector (integral, minimal length). Let $\mathbb{T}_i$ be the 1-dimensional subtorus corresponding to v_i

13.13 For $p \in F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_\ell}$ let $\mathbb{T}_p = \mathbb{T}_{i_1}\mathbb{T}_{i_2}\dots\mathbb{T}_{i_\ell} \simeq (S^1)^\ell$. Topologically $X = \Delta_X \times (S^1)^n/\sim$. The pairs (p,t) and (p,t') are identified if and only if $t't^{-1} \in \mathbb{T}_p$.

13.14 The inverse images $\mu^{-1}(x_i)$ are divisors (=codimension 1 subvarieties) in X.

13.15 Theorem [Daniłov, Jurkiewicz, Davis-Januszkiewicz] The cohomology ring is generated by the classes of $[D_i] \in H^2(X)$. Assume that Δ_X has d facets:

$$H^*(X) = \mathbb{Z}[x_1, \dots, x_d]/(I + J)\,,$$

$$I = (x_{i_1}x_{i_2}\dots x_{i_\ell} \mid F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_\ell} \text{ is not a codimension } \ell \text{ face of } \Delta_X)\,.$$

$$J = (\sum \langle u, v_i\rangle x_i \mid u \in \mathfrak{t}^*_{\mathbb{Z}})\,.$$

Here the left hand side is written in the additive notation, but it concerns the monomials.

• The quotient $\mathbb{Z}[x_1, \dots, x_d]/I$ is called the Stanley Reisner ring. [Anderson-Fulton, §8.3] • Similarly the equivariant cohomology. Let $\Lambda = Sym(\mathfrak{t}^*_{\mathbb{Z}}) = H^*_{\mathbb{T}}(pt)$

$$H^*_{\mathbb{T}}(X) = \Lambda[x_1, \dots, x_d]/(I' + J')\,,$$

$$I' = \Lambda \otimes I\,.$$

$$J' = (u - \sum \langle u, v_i\rangle x_i \mid u \in \mathfrak{t}^*_{\mathbb{Z}})\,.$$

• Note that

$$\mathbb{Z}[x_1, \dots, x_d]/I \simeq \Lambda[x_1, \dots, x_d]/(I' + J')$$

and

$$\mathbb{Z}[x_1, \dots, x_d]/(I + J) \simeq \Lambda[x_1, \dots, x_d]/(I' + J') \otimes_\Lambda \mathbb{Z}\,.$$

13.16 Connection with the Kirwan map: any toric variety can be obtained by the Cox construction

$$X = U/\mathbb{T}'\,,$$

Where $U \subset \mathbb{C}^d$,

$$U = \mathbb{C}^d \setminus \bigcup_I V_I$$

where sum runs over the sequences $i_1, i_2, \dots, i_\ell$ such that $\bigcap_{j=1}^\ell F_{i_j}$ is not a face and

$$V_I = \{x_{i_1} = x_{i_2} = \cdots = x_{i_\ell} = 0\}\,,$$

$\mathbb{T}'$=some subtorus of $(\mathbb{C}^*)^d$. Decomposing $(\mathbb{C}^*)^d = \mathbb{T}' \times \mathbb{T}$ we obtain an action of $\mathbb{T}$ on $U/\mathbb{T}'$.

13.17 Example $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/(\text{diagonal torus})$. Let $\mathbb{T} = \{t \in (\mathbb{C}^*)^{n+1} \mid t_0 = 1\}$.

$$H^*_{\mathbb{T}}(\mathbb{P}^n) = \mathbb{Z}[x_0, x_1, \dots, x_n]/(x_0 x_1 \dots x_n,)$$

The Λ-module structure is given by the relations in J': the vectors v_i consists of the standard basis vectors ϵ_i, $v_0 = -\sum \epsilon_i$. For the generator $t_i \in \Lambda$, $i > 0$

$$\langle t_i, v_j \rangle = \begin{cases} -\delta_{i,j} & \text{for } j > 0 \\ 1 & \text{for } j = 0 \end{cases}$$

hence

$$t_i \mapsto x_i - x_0 \qquad \text{for } i > 0\,.$$

13.18 The ranks of $H^*_{\mathbb{T}}(X)$ can be easily computed inductively from the exact sequence of a pair: for a smooth closed invariant submanifold $N \subset M$ we have

$$\to H^{*-2\text{codim}N}_{\mathbb{T}}(N) \to H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}(M \setminus N) \to H^{*-2\text{codim}N+1}_{\mathbb{T}}(N) \to \,.$$

Note that if X is a sum of $\mathbb{T}$ orbits, then each $H^{odd}_{\mathbb{T}}(orbit) = 0$ and the sequence splits.

- $$H^*_{\mathbb{T}}(X) \simeq \bigoplus_{\mathcal{O} \text{ orbit}} H^{*-2\text{codim}\mathcal{O}}_{\mathbb{T}}(B\mathbb{T}_{\mathcal{O}})\,, \qquad \mathbb{T}_{\mathcal{O}} \simeq (\mathbb{C}^*)^{\text{codim}\mathcal{O}}$$

- Let us compute the equivariant Poincaré polynomial: set $q = t^2$

$$P_{\mathbb{T}}(X) = \sum_{\mathcal{O}} q^{\text{codim}O}(1-q)^{-\text{codim}O}$$

- The nonequivariant Poincaré polynomial can be computed due to equivariant formality:

$$P_{\mathbb{T}}(X) = P(X)P(B\mathbb{T})\,,$$

hence

$$P(X) = P_{\mathbb{T}}(X)P(B\mathbb{T})^{-1} = \left(\sum_{\mathcal{O}} q^{\text{codim}O}(1-q)^{-\text{codim}O}\right)(1-q)^n = \sum_{\mathcal{O}} q^{\text{codim}O}(1-q)^{\dim O}$$

13.19 Example: $X = \mathbb{P}^2$
3 fixed points $\to \quad 3q^2$
3 lines $\to \quad 3q(1-q)$
1 open orbit $\to \quad (1-q)^2$

$$3q^2 + 3q(1-q) + (1-q)^2 = 3q^2 + 3q - 3q^2 + 1 - 2q + q^2 = q^2 + q + 1$$

14 Equivariant Schubert Calculus on Grassmannians

This section contains mainly the example of the calculus on Grassmannian $Gr_2(\mathbb{C}^4)$. See [Anderson-Fulton, Chapter 9] for the explanation.

14.1 The Grassmannian $Gr_d(\mathbb{C}^n) = \mathrm{GL}_n/B_n$ is the union of Schubert cells Ω°_λ, $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_d \geq 0$ with $i_1 \leq n-d$. For convenience we set $\lambda_{d+1} = 0$. Set $e = n-d$. We fix the standard flag $E_\bullet$ preserved by the Borel group and define

$$\Omega^\circ_\lambda(E_\bullet) = \{V \subset \mathbb{C}^n \mid \dim(E_q \cap V) = k \;\text{ for } q \in [e+k-\lambda_k, e+k-\lambda_{k+1}]\}\,,$$

i.e. the sets Ω°_λ are defined by the strict Schubert conditions. • For $n=4$, $d=2$,

$$\Omega^\circ_{00}(E_\bullet) = \left\{V \subset \mathbb{C}^4 \;:\; \begin{array}{ll} \dim(E_q \cap V) = 1 & \text{for } q \in [3-0, 3-0] \\ \dim(E_q \cap V) = 2 & \text{for } q \in [4-0, 4-0] \end{array}\right\}.$$

(The dimensions of the intersections are generic.)

$$\Omega^\circ_{22}(E_\bullet) = \left\{V \subset \mathbb{C}^4 \;:\; \begin{array}{ll} \dim(E_q \cap V) = 1 & \text{for } q \in [3-2, 3-2] \\ \dim(E_q \cap V) = 2 & \text{for } q \in [4-2, 4-2] \end{array}\right\}.$$

(The dimensions are the maximal possible, i.e. $\Omega^\circ_{22} = \{E_2\}$.)

$$\Omega^\circ_{10}(E_\bullet) = \left\{V \subset \mathbb{C}^4 \;:\; \begin{array}{ll} \dim(E_q \cap V) = 1 & \text{for } q \in [3-1, 3-0] \\ \dim(E_q \cap V) = 2 & \text{for } q \in [4-0, 4-0] \end{array}\right\}.$$

(The only nontrivial condition is $\dim(E_2 \cap V) = 1$ but $E_1 \not\subset V$, $V \not\subset E_3$)

$$\Omega^\circ_{11}(E_\bullet) = \left\{V \subset \mathbb{C}^4 \;:\; \begin{array}{ll} \dim(E_q \cap V) = 1 & \text{for } q \in [3-1, 3-1] \\ \dim(E_q \cap V) = 2 & \text{for } q \in [4-1, 4-0] \end{array}\right\}.$$

(This means, that $V \subset E_3$.)

$$\Omega^\circ_{20}(E_\bullet) = \left\{V \subset \mathbb{C}^4 \;:\; \begin{array}{ll} \dim(E_q \cap V) = 1 & \text{for } q \in [3-2, 3-0] \\ \dim(E_q \cap V) = 2 & \text{for } q \in [4-0, 4-0] \end{array}\right\}.$$

(This means $E_1 \subset V$, $V \neq E_2$.)

$$\Omega^\circ_{21}(E_\bullet) = \left\{V \subset \mathbb{C}^4 \;:\; \begin{array}{ll} \dim(E_q \cap V) = 1 & \text{for } q \in [3-2, 3-1] \\ \dim(E_q \cap V) = 2 & \text{for } q \in [4-1, 4-0] \end{array}\right\}.$$

($E_1 \subset V$ and $V \subset E_3$.)

14.2 For the standard flag the Schubert cells are the B_n orbits of the torus-fixed points. Let $x_{i,j} = lin\{\epsilon_i, \epsilon_j\}$

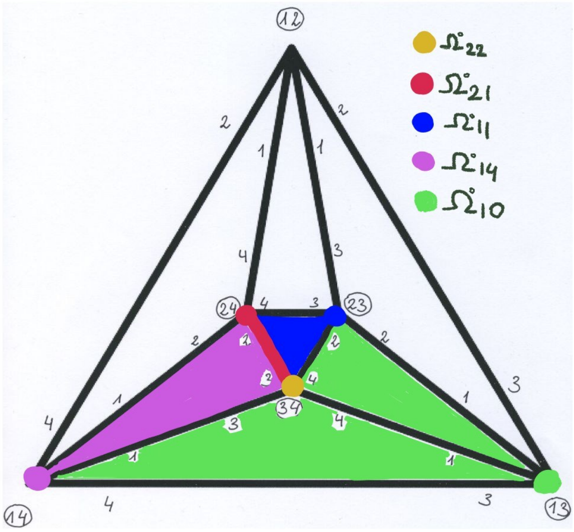
$$\begin{array}{rcl} \Omega^\circ_{00}(E_{st}) = B_4 x_{34}\,, & & \text{open cell} \\ \Omega^\circ_{22}(E_{st}) = B_4 x_{12}\,, & & \text{a point} \\ \Omega^\circ_{10}(E_{st}) = B_4 x_{24}\,, & & \text{divisor} \\ \Omega^\circ_{11}(E_{st}) = B_4 x_{23}\,, & & \text{dim=2, closure} \simeq \mathbb{P}^2 \\ \Omega^\circ_{20}(E_{st}) = B_4 x_{14}\,, & & \text{dim=2, closure} \simeq \mathbb{P}^2 \\ \Omega^\circ_{21}(E_{st}) = B_4 x_{13}\,, \text{dim=1, closure} \simeq \mathbb{P}^1 & & \end{array}$$

14.3 If we reverse the reference flag, then the Schubert cells are the orbits of the opposite Borel group B^-_n, consisting of the lower triangular matrices.

$$\begin{array}{rl} \Omega^\circ_{00}(E_{op}) = B^-_4 x_{12}\,, & \text{open cell} \\ \Omega^\circ_{22}(E_{op}) = B^-_4 x_{34}\,, & \text{a point} \\ \Omega^\circ_{10}(E_{op}) = B^-_4 x_{13}\,, & \text{divisor} \\ \Omega^\circ_{11}(E_{op}) = B^-_4 x_{23}\,, & \text{dim=2, closure} \simeq \mathbb{P}^2 \\ \Omega^\circ_{20}(E_{op}) = B^-_4 x_{14}\,, & \text{dim=2, closure} \simeq \mathbb{P}^2 \\ \Omega^\circ_{21}(E_{op}) = B^-_4 x_{24}\,, & \text{dim=1, closure} \simeq \mathbb{P}^1 \end{array}$$

(we replace $x_{i,j}$ by $x_{5-j,5-i}$).

• Let us work with the opposite flag. We set $\sigma_\lambda = [\overline{\Omega^\circ_\lambda(E_{op})}]$.



14.4 The main statements of nonequivariant Schubert calculus are the following:

• The Giambelli formula says, that the classes of Schubert varieties can be expressed by the Chern classes of the (dual) tautological bundle V^*

$$[\Omega_\lambda] = S_\lambda(V^*)\,.$$

• The rules how to multiply $\sigma_\lambda[\Omega_\lambda]$'s: Pieri rule and more general Littlewood-Richardson rule.

14.5 For example for $d = 1$, $Gr_1(\mathbb{C}^n) = \mathbb{P}^{n-1}$, $V^* = \mathcal{O}(1)$ and $[\Omega_i] = [\mathbb{P}^{n-1-i}] = c_1(\mathcal{O}(1))^i$.

14.6 Nonequivariant multiplication for $Gr_2(\mathbb{C}^4)$

	σ_{00}	σ_{10}	σ_{11}	σ_{20}	σ_{21}	σ_{22}
σ_{00}	σ_{00}	σ_{10}	σ_{11}	σ_{20}	σ_{21}	σ_{22}
σ_{10}	σ_{10}	$\sigma_{11}+\sigma_{20}$	σ_{21}	σ_{21}	σ_{22}	0
σ_{11}	σ_{11}	σ_{21}	σ_{22}	0	0	0
σ_{20}	σ_{20}	σ_{21}	0	σ_{22}	0	0
σ_{21}	σ_{21}	σ_{22}	0	0	0	0
σ_{22}	σ_{22}	0	0	0	0	0

14.7 The product $\sigma_\lambda \cdot \sigma_\mu$ can be written as $\sum_\nu c^\nu_{\lambda\mu}\sigma_\nu$. The coefficients are called the Littlewood-Richardson coefficients. They are nonnegative integers:

$$c^\nu_{\lambda\mu} = |g_1\Omega_\lambda(F_{st}) \cap g_2\Omega_\mu(F_{st}) \cap g_3\Omega_{\nu^\vee}(F_{st})|\,,$$

where $\nu^\vee$ is the opposite partition $\nu^\vee = \mathit{Reverse}((n-k)^k - \nu)$, g_i are general elements of GL_n. In the equivariant calculus the coefficients $c^\nu_{\lambda\mu}$ are polynomials in $t_1, t_2, \ldots, t_n$.

14.8 In the nonequivariant case the reference flag is irrelevant for computing cohomology classes. Instead of B_n orbits one can take the orbits of the opposite Borel group B_n^-.

14.9 Equivariant cohomology contains more information. There are at least three important bases of $H^*_{\mathbb{T}}(Gr_d(\mathbb{C}^n)$:

• The basis on $[\sigma_\lambda]$ — the natural choice;

• The bases of Schur classes of V^* — convenient for functorial reasoning;

• The basis of the fixed point classes (this is a basis after the localization in $S = \langle t_i - t_j \mid i \neq j \rangle$) — here the multiplication is easy.

14.10 The analogues of the Giambelli formulas are the Kempf-Laksov formulas. In [Anderson-Fulton, 9.2] given for B_n^- orbit closures.

14.11 Table of the restrictions of Schubert classes at the fixed points

	x_{34}	x_{24}	x_{23}	x_{14}	x_{13}	x_{12}
σ_0	1	1	1	1	1	1
σ_{10}	$t_1+t_2-t_3-t_4$	t_1-t_4	t_2-t_4	t_1-t_3	t_2-t_3	0
σ_{11}	$(t_1-t_3)(t_1-t_4)$	$(t_1-t_2)(t_1-t_4)$	0	$(t_1-t_2)(t_1-t_3)$	0	0
σ_{20}	$(t_1-t_4)(t_2-t_4)$	$(t_1-t_4)(t_3-t_4)$	$(t_2-t_4)(t_3-t_4)$	0	0	0
σ_{21}	$(t_1-t_3)(t_1-t_4)(t_2-t_4)$	$(t_1-t_2)(t_1-t_4)(t_3-t_4)$	0	0	0	0
σ_{22}	$(t_1-t_3)(t_2-t_3)(t_1-t_4)(t_2-t_4)$	0	0	0	0	0

14.12 The formula for σ_{10}: in nonequivariant cohomology $\sigma_1 = c_1(V^*) = c_1(\mathcal{O}(1))$ (the bundle $\mathcal{O}(1)$ comes from the Plücker embedding).

- The equivariant formula is of the form

$$\sigma_{10} = c_1(V^*) + \text{linear form}(t_1, t_2, t_3, t_4)\,.$$

The form is chosen in such way that $(\sigma_{10})_{|x_{1,2}} = 0$, i.e. it is equal $t_1 + t_2$. This reasoning works in general.

14.13 Equivariant multiplication table.

- Multiplication by σ_{10}

$$\begin{aligned}
\sigma_{10}\sigma_{22} &= (t_1+t_2-t_3-t_4)\,\sigma_{22}\\
\sigma_{10}\sigma_{21} &= (t_1-t_4)\,\sigma_{21}+\sigma_{22}\\
\sigma_{10}\sigma_{20} &= (t_2-t_4)\,\sigma_{20}+\sigma_{21}\\
\sigma_{10}\sigma_{11} &= (t_1-t_3)\,\sigma_{11}+\sigma_{21}\\
\sigma_{10}^2 &= (t_2-t_3)\,\sigma_{10}+\sigma_{11}+\sigma_{20}
\end{aligned}$$

According to the equivariant Monk formula

$$\sigma_{10}\sigma_\lambda = \sum_{\lambda^+}\sigma_{\lambda^+} + (\sigma_{10})_{|x_\lambda}\,\sigma_\lambda\,,$$

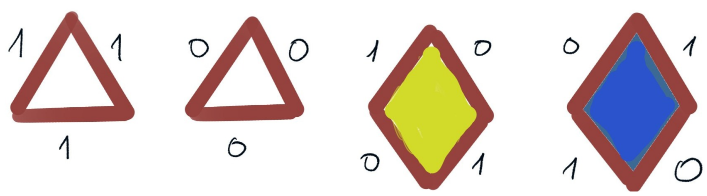
where x_λ is the fixed point in $\Omega^\circ_\lambda(E_{op})$.

- The remaining multiplications

$$\begin{aligned}
\sigma_{22}^2 &= (t_1-t_3)(t_2-t_3)(t_1-t_4)(t_2-t_4)\,\sigma_{22}\\
\sigma_{21}\sigma_{22} &= (t_1-t_3)(t_1-t_4)(t_2-t_4)\,\sigma_{22}\\
\sigma_{20}\sigma_{22} &= (t_1-t_4)(t_2-t_4)\,\sigma_{22}\\
\sigma_{11}\sigma_{22} &= (t_1-t_3)(t_1-t_4)\,\sigma_{22}\\
\sigma_{21}^2 &= (t_1-t_4)^2\sigma_{22}+(t_1-t_2)(t_1-t_4)(t_3-t_4)\,\sigma_{21}\\
\sigma_{20}\sigma_{21} &= (t_1-t_4)\,\sigma_{22}+(t_1-t_4)(t_3-t_4)\,\sigma_{21}\\
\sigma_{11}\sigma_{21} &= (t_1-t_2)(t_1-t_4)\,\sigma_{21}+(t_1-t_4)\,\sigma_{22}\\
\sigma_{20}^2 &= (t_2-t_4)(t_3-t_4)\,\sigma_{20}+(t_3-t_4)\,\sigma_{21}+\sigma_{22}\\
\sigma_{11}\sigma_{20} &= (t_1-t_4)\,\sigma_{21}\\
\sigma_{11}^2 &= (t_1-t_2)(t_1-t_3)\,\sigma_{11}+(t_1-t_2)\,\sigma_{21}+\sigma_{22}
\end{aligned}$$

14.14 Knutson-Tao puzzles: we draw a triangle with all edges of length n and fill them with pieces of the following shapes

- Three nonequivariant puzzles and one equivariant:

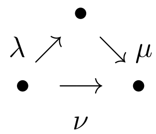


The last one is not rotatable.

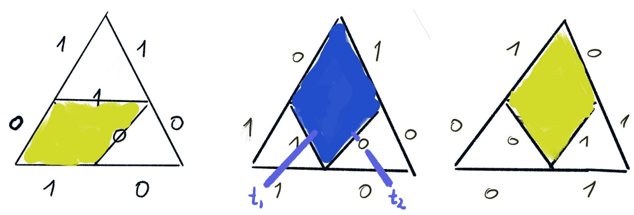
- We change the coding of Schubert varieties. Instead of partitions we use 0-1 sequences of length n. We walk along the edges of Young diagram $NE \to SW$: the sequence has 1 if we go S, 0 if we go W.

$$\begin{array}{ccc} 00 & \to & 0011 \\ 10 & \to & 0101 \\ 11 & \to & 0110 \\ 20 & \to & 0110 \\ 21 & \to & 1010 \\ 22 & \to & 1100 \end{array}$$

We label the edges of the triangle with the codes

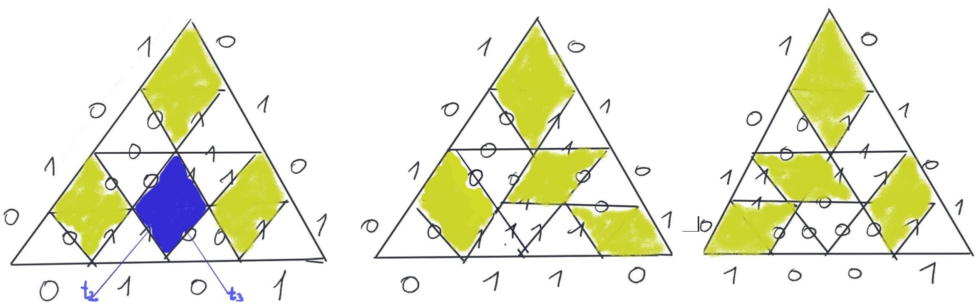


14.15 Multiplication in $\mathbb{P}^1 = Gr_1(\mathbb{C}^2)$



$$\sigma_0\sigma_1 = \sigma_1 \qquad \sigma_1\sigma_1 = (t_1 - t_2)\sigma_1 \qquad \sigma_0\sigma_0 = \sigma_0\,.$$

14.16 Multiplication in $Gr_2(\mathbb{C}^4)$



Three coefficients of the expansion of $\sigma_{10}\sigma_{10}$ in $H^*_{\mathbb{T}}(Gr_2(\mathbb{C}^4)$

$$c^{10}_{10,10} = t_2 - t_3\,, \qquad c^{11}_{10,10} = 1\,, \qquad c^{20}_{10,10} = 1\,.$$

14.17 [Anderson-Fulton, §9, Theorem 8.4] The equivariant Littlewood-Richardson coefficient is equal to

$$c^{\nu}_{\lambda\mu} = \sum_{\text{puzzle fillings}} \prod_{\text{special pieces}} \left(t_{\text{left leg}} - t_{\text{right leg}}\right).$$

- In [Anderson-Fulton, §9] the signs of the variables are reversed, due to a different convention.