Problem list 9— 5 December

1 Write all the entries of the Serre spectral sequence of the fibration $E\mathbb{T}\times X \to E\mathbb{T}\times^{\mathbb{T}} X$ for $X=\mathbb{P}^1$ or S^3 (with the usual torus actions).

2 Consider the spectral sequence from the previous problem for X arbitrary, and $\mathbb{T}=S^1$. Show that $d_2^{p,1}: H^p_{\mathbb{T}}(X)=E_2^{p,1}\to E_2^{p+2,0}=H^{p+2}_{\mathbb{T}}(X)$ can be identified with the multiplication by the generator of $H^2_{\mathbb{T}}(pt)$. What happens when X is equivariantly formal?

3 Write all the entries of the spectral sequence $E_2^{p,q}=H^p(B\mathbb{T},H^q(X))\Rightarrow H^{p+q}_{\mathbb{T}}(X)$ for $\mathbb{T}=S^1$ acting on $S^3\subset\mathbb{C}^2$ as the scalar multiplication.

4 Consider the spectral sequence from the previous problem for X arbitrary, and $\mathbb{T}=S^1$. Assuming that $\mathbb{T}=S^1$ recognize the differential

$$H^q(X)=E_2^{0,q}\to E_2^{2,q-1}=H^2(B\mathbb{T})\otimes H^{q-1}(X)\simeq H^1(\mathbb{T})\otimes H^{q-1}(X)$$

as a map induced by the multiplication $S^1\times X\to X$.

(Taking $X=\mathbb{T}$ the differential $H^1(\mathbb{T})=E_2^{0,1}\to E_2^{2,0}=H^2(B\mathbb{T})$ is an isomorphism.)

5 Let γ be the tautological bundle over the Grassmannian $Gr_2(\mathbb{C}^4)$. Compute the push-forward to the point

$$\int_{Gr_2(\mathbb{C}^4)} c_2(\gamma^*)^n\,.$$

Show that the result is equal to the Schur function for $\lambda=(n-2,n-2,0,0)$ if $n\geqslant 2$. If possible – generalize this calculus.

Hint $c_2(L_1\oplus L_2)^n=S_\lambda(c_1(L_1),c_1(L_2))$ for $\lambda=(n,n)$. Use Laplace block-expansion.

6 The space $\mathbb{C}^{2n}$ is equipped with the canonical non-degenerate antisymmetric 2-form

$$\omega=\sum_{i=1}^{n} dx_i\wedge dx_{n-i+1}$$

(i.e. the symplectic form). Let $LG_n\subset Gr_n(\mathbb{C}^n)$ be the Lagrangian Grassmannian, i.e. the set of isotropic n-subspaces. The torus

$$diag(t_1,t_2,\dots,t_n,t_n^{-1},\dots,t_2^{-1},t_1^{-1})$$

acts on $\mathbb{C}^{2n}$ preserving ω, hence it acts on LG_n. Find the fixed points and the GKM-graph.