Problem list 7— 21 November

1 Show that the localization functor

$$\Lambda - modules \quad \longrightarrow \quad K - modules$$

is exact.

2 See what goes wrong with the proof of localization theorem for $\mathbb{T}$ replaced by a nonabelian groups. In the torus case for any proper subgroup $K \subset \mathbb{T}$ the orbit $H^*_{\mathbb{T}}(\mathbb{T}/K)$ turned out to be a torsion $H^*_{\mathbb{T}}(pt)$–module. What happens for noncommutative groups?

3 Let $X = \mathbb{P}^n$ with the standard action of $\mathbb{T} = (\mathbb{C}^*)^{n+1}$. Show that the image

$$H^*_{\mathbb{T}}(\mathbb{P}^n) \hookrightarrow \bigoplus_{k=0}^{n} \Lambda = \Lambda^{n+1}$$

consists of such sequences $(f_0, f_1, \dots, f_n) \in \mathbb{Q}[t_0, t_1, \dots, t_n]^{n+1}$, such that $t_i - t_j$ divides $f_i - f_j$. (Of course not using the general theorem, but by a direct computation.)

4 The torus $\mathbb{T} = (\mathbb{C}^*)^{n+1}$ acts in the standard way on $\mathbb{P}^n$. Let $h = c_1(\mathcal{O}(1)) \in H^2_{\mathbb{T}}(\mathbb{P}^n)$. Using AB-BV localization fromula compute

$$p_*(h^{n+m}) \in H^*_{\mathbb{T}}(pt) = \mathbb{Z}[t_0, t_1, \dots t_n]\,.$$

(The restriction of h at the standard fixed point $p_k \in \mathbb{P}^n$ is equal $h_{|p_k} = -t_k$.)

5 Let $L = \mathcal{O}(m)$, $m \geqslant 0$. Using Riemann-Roch theorem and the localization theorem compute $\chi(\mathbb{P}^n; L)$. Check if it agrees with the formula well known for algebraic geometers:

$$\chi(\mathbb{P}^n; L) = \dim H^0(\mathbb{P}^n; \mathcal{O}(m)) = \mathbb{C}[t_0, t_1, \dots, t_n]_{\deg=m}\,.$$

(The restriction of L at the standard fixed point $p_k \in \mathbb{P}^n$ is equal to $\mathbb{C}_{-m\,t_k}$, i.e. $\mathbb{C}$ with $\mathbb{T}$ acting with the weight $-m\,t_k$.)