Problem list 5 — 7 November

1 Let $NT \subset U(n)$ be the normalizer of the maximal torus. Compute $H^*(U(n)/NT;\mathbb{Q})$.

Hint: Use the fact[1] that $H^*(X/G;\mathbb{Q}) \xrightarrow{\simeq} H^*(X;\mathbb{Q})^G$ for a finite group G acting on a topological space. Apply it to $X = BT$, $G =$ the permutation group.

BTW: What is it $U(2)/NT$?

2 Compute cohomology of $BSU(n)$. (Or equivalently $BSL_n(\mathbb{C})$.)

3 Cohomology of Grassmannians:

- Show that the Chern classes of the tautological bundle generate $H^*(Gras_k(\mathbb{C}^n))$. Find the relations.
- Let $E \to B$ be a complex vector bundle, $0 < k < rank(E)$. By $Gr_k(E)$ we denote the associated Grassmann bundle (we replace the fibers E_x by the Grassmannians of k-dimensional subspaces in E_x). Let γ be the tautological bundle over $Gr_k(E)$. Show that $H^*(Gr_k(E))$ is generated over $H^*(B)$ by the Chern classes $c_i(\gamma)$, $i = 1, 2, \ldots, k$.
- As a corrolary compute $H^*_T(Gr_k(\mathbb{C}^n))$, where T is the maximal torus in $\mathrm{GL}_n(\mathbb{C})$.
- Compare the answer with Projective Bundle Theorem $H^*(\mathbb{P}(E)) \simeq H^*(B)[h]/(\textit{well known relation})$.

4 Let G be a Lie group of dimension d. Suppose $P \to B$ is a principal bundle. Assume that P is n-connected, i.e. its homotopy groups are trivial in degrees $\leqslant n$. Show that $H^k(B) \simeq H^k(BG)$ for $k \leqslant n$.

[1] Glen E. Bredon - Sheaf Theory-Springer-Verlag New York (Graduate Texts in Mathematics 170), Theorem 19.2.