Problem list 4 — 31 October

1 Find a CW-decomposition of $\mathbb{P}^n$ with the standard action of $(S^1)^{n+1}$ (at least with $n = 2$.)

2 Show that for a finite group G and a G-space X

$$H^*_G(X;\mathbb{Q}) \simeq H^*(X/G;\mathbb{Q}) .$$

3 a) Find a presentation of the cohomology ring $H^*(Gras_k(\mathbb{C}^n))$ using the fibration

$$Gras_k(\mathbb{C}^n) \to B(U_k \times U_{n-k}) \to BU_n .$$

Hint: use a corollary from Leray-Hirsh theorem, sayin that the fibration $F \to E \to B$ satisfying the appropriate assumptions $H^*(F;\mathbb{Q}) = \mathbb{Q} \otimes_{H^*(B;\mathbb{Q})} H^*(E;\mathbb{Q})$.

b) Show that the Chern classes of the tautological bundle generate $H^*(Gras_k(\mathbb{C}^n))$.

4 Let $\mathbb{T} = (\mathbb{C}^*)^2$ act on $\mathbb{P}^3$ by the formula

$$(t_0, t_1) \cdot [z_0 : z_1 : z_2 : z_3] = [t_0 z_0 : t_1 z_1 : t_1^{-1} z_2 : t_0^{-1} z_3] .$$

Let X be the quadric $z_0 z_3 = z_1 z_2$. compute $H^*_{\mathbb{T}}(X)$. It is an algebra over $\mathbb{Z}[t_0, t_1]$. Find generators and their relations. Check that it is free as a module.

Hint: $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

5 Simplicial model of EG using bar-construction [mini-talk KP]
Let $X_i = G^{i+1}$ and $d_{i,k}k : X_i \to X_{i-1}$ is the projection, forgetting about the k-th component $k = 0, 1, \ldots i$. (This is a presimplicial topological space.) The geometric realization is defined as

$$|X_\bullet| = \left(\bigsqcup_{i \geqslant 0} X_i \times \Delta^i \right) \Big/ \sim$$

$$(d_{i,k}(a), b) \sim (a, \partial_{i-1,k}(b)) \quad \text{for} \quad a \in X_i,\ b \in \Delta^{i-1}$$

where $\partial_{i-1,k} : \Delta_{i-1} \hookrightarrow \Delta_i$ is the inclusion of the k-th facet in the standard symplex. Show that $E = |X_\bullet|$ is contractible and the diagonal action of G is free.

Present E/G as a geometric realization of a (pre-)simplicial set $Y_\bullet$, such that $Y_i = G^i$.