Problem list 3 — 24 October

1 Let

$$\Sigma_r(n,m) = \{ f \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m) \ : \ \dim(ker(f)) = r \} .$$

Compute the dimension of $\Sigma_r(n,m)$.

Deduce that for any fixed $i \in \mathbb{N}$ and sufficiently large m the homotopy groups $\pi_i(\Sigma_0(n,m))$ is trivial.

2 Find a CW-decomposition of $\mathbb{P}^n$ with the standard action of $(S^1)^{n+1}$ (at least with $n = 2$.)

3 Find a presentation of the cohomology ring $H^*(G(k,n))$ using the fibration $Gras_k(\mathbb{C}^n) \to B(U_k \times U_{n-k}) \to BU_n$.

Hint: use a corollary from Leray-Hirsh theorem, sayin that the fibration $F \to E \to B$ satisfying the appropriate assumptions $H^*(F;\mathbb{Q}) = \mathbb{Q} \otimes_{H^*(B;\mathbb{Q})} H^*(E;\mathbb{Q})$.

4 Suppose $H \triangleleft G$ is a normal subgroup, $K = G/H$. Construct a fibration $BH \to BG \to BK$.

(Take $EH := EG$ and $E'G = EG \times EK$, taking the fibration $E'G/G \to EK/K$ we find that the fiber is $EG \times^G G/H = BH$.)

5 Milnor construction of EG.
For topological spaces X and Y let $X * Y$ denote its join, i.e.

$$CX \times Y \cup_{X \times Y} X \times CY ,$$

where CX denotes the cone over X. Let G be a topological group, show that n-fold join $X * X * \cdots * X$ has homotopy groups π_i trivial for $i < n-1$ and G acts freely on that space. Taking the infinite join we obtain a model of EG.

6 Simplicial model of EG using bar-construction.
Let $X_i = G^{i+1}$ and $d_{i,k}k : X_i \to X_{i-1}$ is the projection, forgetting about the k-th component $k = 0, 1, \ldots i$. (This is a presimplicial topological space.) The geometric realization is defined as

$$|X_\bullet| = \left(\bigsqcup_{i \geqslant 0} X_i \times \Delta^i \right) \Big/ \sim$$

$$(d_{i,k}(a), b) \sim (a, \partial_{i-1,k}(b)) \quad \text{for} \quad a \in X_i,\ b \in \Delta^{i-1}$$

where $\partial_{i-1,k} : \Delta_{i-1} \hookrightarrow \Delta_i$ is the inclusion of the k-th facet in the standard symplex. Show that $E = |X_\bullet|$ is contractible and the diagonal action of G is free.

Present E/G as a geometric realization of a (pre-)simplicial set $Y_\bullet$, such that $Y_i = G^i$.