List 2 — 17 October

1 [Done ???] For a representation V of $\mathbb{T}$ consider an action of $\tilde{\mathbb{T}} = \mathbb{T} \times S^1$ on $\tilde{V} = V$, where S^1 acts by the scalar multiplication. Denote by $\hbar$ the weight corresponding to the character $\tilde{T} \to S^1$, which is the projection. Show that

$$c(V) = e(\tilde{V})_{|\hbar=1}\,.$$

2 Let[1] $\mathbb{G}_m = Spec(\mathbb{F}[t, t^{-1}])$ (simply meaning $\mathbb{F}^*$ as an algebraic group). Show that for any field $\mathbb{F} = \overline{\mathbb{F}}$ any linear action of $\mathbb{T}_{\mathbb{F}} = (\mathbb{G}_m)^r$ on the vector space $\mathbb{F}^n$ can be diagonalized.

3 Let A be an algebra over a field $\mathbb{F}$ and $X = Spec(A)$. Defining an action of $\mathbb{G}_m$ on X is equivalent to defining a $\mathbb{Z}$-gradation of A. Prove this correspondence and generalize it to an action of the algebraic torus $\mathbb{G}_m^r$.

4 Let G be a group, H a subgroup, $E \to G/H$ be a vector bundle with G-action, such that for any $g \in G$, $x \in G/H$ the map $g : E_x \to E_{gx}$ is linear. Show that $E \simeq G \times^H E_{[e]}$. Here $[e]$ denotes the coset eH (i.e. any equivariant bundle over a homogeneous space G/H is induced from a H-representation).

5 Let G be a topological group, H a subgroup. Conastruct a honeomorphism

$$G \times^H G/H \simeq G/H \times G/H\,.$$

Is it G-equivariant with respect to a suitable G-actions?

[1]The multiplicative group