Problem list 14 — 23 January

1 Describe the moment polytopes of homogeneous spaces for $SO(5)$ and $Sp(3)$, in particular for the Lagrangian Grassmannian $LG(3) \subset Gr_3(\mathbb{C}^6)$ and for the generalized flag manifold $Sp(3)/B \simeq p^{-1}(LG(3))$, where $p : Fl_{1,2,3}(\mathbb{C}^6) \to Gr_3(\mathbb{C}^6)$. Here $Fl_{1,2,3}(\mathbb{C}^6)$ denotes the partial flags $V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^6$. Make some colourful pictures!

2 Let G be a Lie group acting on a symplectic manifold M with a moment map $\mu : M \to \mathfrak{g}^*$. Show that μ is G invariant, i.e. $\mu(gx) = Ad_g^*(\mu(x))$.

3 (Angular momentum) Consider the natural action of $G = SO(3)$ on $\mathbb{R}^6 = T^*\mathbb{R}^3$ (with the standard symplectic structure). Find the moment map.

Here we can identify $\mathfrak{so}_3$ with $\mathbb{R}^3$ with the vector product $\times$, moreover we identify $\mathfrak{so}_3$ with $\mathfrak{so}_3^*$ via the scalar product $trace(X^TY)$ (for $X, Y \in \mathfrak{so}_3$) which becomes 2 times the standard scalar product in $\mathbb{R}^3$.

(Read more about the moment map see: Dusa McDuff, Dietmar Salamon - Introduction to symplectic topology-Oxford University Press, 1999, Section 5.2)

4 Let $U(k)$ act on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ (assuming $k < n$). Show that $\mu(A) = iA^A$ is a moment map.

5 Suppose $\mathbb{T} = (\mathbb{C}^*)^2$ acting of $\mathbb{C}^2$ in the standard way. We extend this action to $\mathbb{P}^2$. Let X be the blow-up at 0. Pick an equivariant embedding of X into a projective space, and find the resulting moment polytope for a choice of $\mathbb{T}$ action on $L = \mathcal{O}(1)_{|X}$.