

## Problem list 13 — 16 January

**1** Let  $G = SU(2)$ ,  $\mathfrak{g} = \mathfrak{su}_2 = \text{lin}\{i, j, k\}$  with the well known commutation relations  $[i, j] = 2k$ , etc. Write down explicitly the Chevalley complex computing  $H^*(SU(2))$ . Compare it with  $H^*(BSU(2))$ , check that indeed  $H^*(BG) \simeq (\text{Sym } \mathfrak{g}^*)^G \simeq (\text{Sym } \mathfrak{t}^*)^W$ .

**2** Define the equivariant intersection form

$$H_{\mathbb{T}}^*(M) \times H_{\mathbb{T}}^*(M) \rightarrow H_{\mathbb{T}}^*(pt)$$

$$(a, b) \mapsto \int_M ab \in \mathbb{H}_T^*(pt).$$

Compute the intersection form in the basis  $[\mathbb{P}^0], [\mathbb{P}^1], [\mathbb{P}^2]$ , where  $\mathbb{P}^i = \mathbb{P}(\text{lin}\{\varepsilon_0, \dots, \varepsilon_i\})$ .

[Use Wolfram Mathematica or your favourite formal algebra software for higher dimension  $\mathbb{P}^n$ 's.]

**3** Describe the moment polytopes of homogeneous spaces for  $SO(5)$  and  $Sp(3)$ , in particular for the Lagrangian Grassmannian  $LG(3) \subset Gr_3(\mathbb{C}^6)$  and for the generalized flag manifold  $Sp(3)/B \simeq p^{-1}(LG(3))$ , where  $p : Fl_{1,2,3}(\mathbb{C}^6) \rightarrow Gr_3(\mathbb{C}^6)$ . Here  $Fl_{1,2,3}(\mathbb{C}^6)$  denotes the partial flags  $V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^6$ . Make some colourful pictures!

**4** Let  $G$  be a Lie group acting on a symplectic manifold  $M$  with a moment map  $\mu : M \rightarrow \mathfrak{g}^*$ . Show that  $\mu$  is  $G$  invariant, i.e.  $\mu(gx) = \text{Ad}_g^*(\mu(x))$ .

**5** Let  $(M, \omega)$  be a symplectic manifold with a Hamiltonian  $S^1$  action. Let  $H : M \rightarrow \mathfrak{t}^* \simeq \mathbb{R}$  be the moment map. Prove the Duistermaat-Heckman formula

$$\int_M e^{\hbar f} \frac{\omega^n}{n!} = \sum_{p \in S^1} \frac{e^{\hbar f(p)}}{e(T_p M)}.$$