Problem list 11 — 9 January

1 Let $\mathfrak{g}$ be a graded vector space with a binary operation $[-,-]$ which is graded-antisymmetric, i.e.

$$[x,y] = -(-1)^{\deg(x)\deg(y)}[y,x]\,.$$

We say that $[x,-]$ satisfies the graded Leibniz formula if

$$[x,[y,z]] = [[x,y],z] + (-1)^{\deg(x)\deg(y)}[y,[x,z]]\,.$$

Show, that in the classical case (i.e. $\mathfrak{g}$ lives only in even gradations) the above identity is equivalent to the Jacobi identity.

2 Let $\mathfrak{g}$ be a Lie algebra (in the usual sense). Show that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ with

$$\iota_\lambda \in \tilde{\mathfrak{g}}_{-1} = \mathfrak{g}\,, \quad \mathcal{L}_\lambda \in \tilde{\mathfrak{g}}_0 = \mathfrak{g} \qquad d \in \tilde{\mathfrak{g}}_1 = \mathbb{R}$$

with the commutation relations

$$[\iota_\lambda, \iota_\mu] = 0\,, \qquad [\mathcal{L}_\lambda, \iota_\mu] = \iota_{[\lambda,\mu]}\,, \qquad [d, \iota_\lambda] = \mathcal{L}_\lambda\,,$$
$$[\mathcal{L}_\lambda, \mathcal{L}_\mu] = \mathcal{L}_{[\lambda,\mu]}\,, \qquad [\mathcal{L}_\lambda, d] = 0\,, \qquad [d,d] = 0\,,$$

is a graded Lie algebra.

3 Mathai-Quillen twist: Assume that $P \to B$ is a $\mathbb{T}$-principal bundle with a connection

$$\theta \in \mathrm{Hom}(TP, \mathfrak{t}) \simeq \mathfrak{t} \otimes \Omega^1(P)\,.$$

That is:

1. θ is $\mathbb{T}$-invariant
2. for $\lambda \in t$, $x \in P$ the value of θ at x in the fundamental vector field v_λ is equal to λ.

In a basis of $\{\lambda^a\} \subset \mathfrak{t}$ we write

$$\theta = \sum_a \lambda^a \otimes \theta_a\,.$$

Let X be a $\mathbb{T}$ manifold.

$$\gamma = \sum \theta_a \otimes \iota_{\lambda^a} \in \mathrm{End}(\Omega^\bullet(P) \otimes \Omega^\bullet(X))\,.$$

Show, that γ does not depend on the choice of a basis.

4 **Assume that** $\dim T = 1$ **or 2**. Consider $\mathbb{T}^*$-algebras W and A, with W locally free (e.g. $W = W(\mathfrak{t})$. Let

$$\phi = exp(\gamma) \in Aut(W \otimes A) = 1 + \gamma + \tfrac{1}{2}\gamma \circ \gamma + \dots$$

Check that for any $\lambda \in \mathfrak{t}$

$$\phi \circ (\iota_\lambda \otimes 1 + 1 \otimes \iota_\lambda) \circ \phi^{-1} = \iota_\lambda \otimes 1$$
$$\phi \circ (d \otimes 1 + 1 \otimes d) \circ \phi^{-1} = (d \otimes 1 + 1 \otimes d) - \sum \nu_a \otimes \iota_{\lambda^a} + \sum \theta_a \otimes \mathcal{L}_{\lambda^a}$$

where $\nu_a = d\theta_a$

5 Let $G = SU(2)$, $\mathfrak{g} = \mathfrak{su}_2 = lin\{i,j,k\}$ with the well known commutation relations $[i,j] = 2k$, etc. Write down explicitly the Chevalley complex computing $H^*(SU(2))$. Compare it with $H^*(BSU(2))$, check that indeed $H^*(BG) \simeq (Sym\,\mathfrak{g}^*)^G \simeq (Sym\,\mathfrak{t}^*)^W$.

6 Define the equivariant intersection form

$$H^*_{\mathbb{T}}(M) \times H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}(pt)$$
$$(a,b) \mapsto \int_M ab \in \mathbb{H}^*_T(pt)\,.$$

Compute the intersection form in the basis $[\mathbb{P}^0], [\mathbb{P}^1], [\mathbb{P}^2]$, where $\mathbb{P}^i = \mathbb{P}(lin\{\varepsilon_0, \dots, \varepsilon_i\})$.
[Use Wolfram Mathematica or your favourite formal algebra software for higher dimension $\mathbb{P}^n$'s.]