Problem list 11— 19 December

1 Let γ be the tautological bundle over the Grassmannian $Gr_k(\mathbb{C}^n)$. Compute

$$\int_{Gr_2(\mathbb{C}^4)} c_1(\gamma^*)^{(n-k)k},$$

using the AB-BV formula.

Hint: Install Wolfram Mathematica on your laptop and execute for a fixed n and k

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Sum[
 (-Sum[t[a], {a, J}])^(k (n - k))/Product[t[b] - t[a], {a, J}, {b, Complement[Range[n], J]}],
{J, Subsets[Range[n], {k}]}]
Factor[%]
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How far can you go?

2 Let γ be the tautological bundle and $\mathbb{Q}$ be the quotient bundle over the Grassmannian $Gr_k(\mathbb{C}^n)$. We consider generalized Schur classes for arbitrary sequences $\lambda_1, \lambda_2, \ldots, \lambda_d$ for a vector bundle of rank d: if the E bundle splits into direct sum of line bundles L_i, and $t_i = c_1(L_i)$ then

$$S_\lambda(E) = \frac{\det\begin{pmatrix} t_1^{\lambda_1+d-1} & t_1^{\lambda_1+d-2} & \ldots & t_1^{\lambda_d} \\ t_2^{\lambda_2+d-1} & t_2^{\lambda_2+d-2} & \ldots & t_2^{\lambda_d} \\ \vdots & & & \vdots \\ t_d^{\lambda_1+d-1} & t_d^{\lambda_2+d-2} & \ldots & t_d^{\lambda_d} \end{pmatrix}}{\det\begin{pmatrix} t_1^{d-1} & t_1^{d-2} & \ldots & t_1^{0} \\ t_2^{d-1} & t_2^{d-2} & \ldots & t_2^{0} \\ \vdots & & & \vdots \\ t_d^{d-1} & t_d^{d-2} & \ldots & t_d^{0} \end{pmatrix}}$$

Show that

$$\int_{Gr_k(\mathbb{C}^n)} S_\lambda(\gamma) S_\mu(Q) = S_\nu(\mathbb{C}^n)$$

for some ν.

3 Let $X_1 \subset Gr_2(\mathbb{C}^n)$ be (the Schubert variety) defined by

$$X_1 = \{W \in Gr_2(\mathbb{C}^4) \mid W \cap lin\{\varepsilon_1, \varepsilon_2\} \neq 0\}.$$

Compute an equation of X_1 in neighbourhoods of the fixed points $lin\{\varepsilon_i, \varepsilon_j\}$. Compute the restriction of the fundamental class $[X_1]$ to the fixed points.

Hint: to compute the restriction of $[X_1]$ at $lin\{\varepsilon_i, \varepsilon_j\}$ one can use the divisibility relations or the fact that $\int_{Gr_2}(\mathbb{C}^4)[X_1] = 0$.

4 Define the equivariant intersection form

$$H^*_{\mathbb{T}}(M) \times H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}(pt)$$

$$(a, b) \mapsto \int_M ab \in \mathbb{H}^*_T(pt).$$

Compute the intersection form in the basis $[\mathbb{P}^0], [\mathbb{P}^1], [\mathbb{P}^2]$, where $\mathbb{P}^i = \mathbb{P}(lin\{\varepsilon_0, \ldots, \varepsilon_i\})$.

5 Let G be a compact connected Lie group. Prove that

$$H^k(G; \mathbb{R}) \simeq (\Lambda^k \mathfrak{g}^*)^G.$$

BTW: $H^{2k}(BG;\mathbb{R}) \simeq (Sym^k \mathfrak{g}^*)^G$.