Problem list 10— 12 December

The space $\mathbb{C}^{2n}$ is equipped with the canonical non-degenerate antisymmetric 2-form

$$\omega = \sum_{i=1}^{n} dx_i \wedge dx_{n-i+1}$$

(i.e. the symplectic form). Let $LG_n \subset Gr_n(\mathbb{C}^{2n})$ be the Lagrangian Grassmannian, i.e. the set of isotropic n-subspaces. The torus

$$diag(t_1, t_2, \dots, t_n, t_n^{-1}, \dots, t_2^{-1}, t_1^{-1})$$

acts on $\mathbb{C}^{2n}$ preserving ω, hence it acts on LG_n. Let $e_1, e_2, \dots, e_n, f_n, \dots, f_2, f_1$ be the Darboux basis. We have found that the fixed points correspond to the choices of subsets $I \subset \{1, 2, \dots, n\}$

$$p_I = lin\{e_i\,, f_j : i \in I,\ j \notin I\}$$

1 Compute the GKM-graph of $LG(n)$.

2 Let $\mathbb{C}^* \simeq \mathbb{T}_0 \subset \mathbb{T} = (\mathbb{C}^*)^{2n}$ be the 1-dimensional subtorus

$$\mathbb{T}_0 = \{(t^n, t^{n-1}, \dots, t, t^{-1}, \dots, t^{1-n}, t^{-n}) \mid t \in \mathbb{C}^*\}\,.$$

Compute the dimension of the Białynicki-Birula cell attached to $p_I \in LG(n)$.

3 Finish the proof, that the only 1-dimensional orbits in $Gr_k(\mathbb{C}^n)$ correspond to exchanging one element in the subset $I \subset \{1, 2, \dots, n\}$, $|I| = k$.

4 Using AB-BV formula compute the equivariant $\int_{LG(2)} c_1(\gamma^*)^k$ for $k = 3, 4$, where γ is the tautological bundle on $LG(2)$.

5 Let γ be the tautological bundle over the Grassmannian $Gr_2(\mathbb{C}^4)$. Compute the equivariant push-forward to the point

$$\int_{Gr_2(\mathbb{C}^4)} c_2(\gamma^*)^n\,.$$

Show that the result is equal to the Schur function for $\lambda = (n-2, n-2, 0, 0)$ if $n \geqslant 2$. If possible – generalize this calculus.

Hint $c_2(L_1 \oplus L_2)^n = S_\lambda(c_1(L_1), c_1(L_2))$ for $\lambda = (n, n)$. Use Laplace block-expansion.

6 Let $X_1 \subset Gr_2(\mathbb{C}^n)$ be (the Schubert variety) defined by

$$X_1 = \{W \in Gr_2(\mathbb{C}^4) \mid W \cap lin\{\varepsilon_1, \varepsilon_2\} \neq 0\}\,.$$

Compute an equation of X_1 in neighbourhoods of the fixed points $lin\{\varepsilon_i, \varepsilon_j\}$. Compute the restriction of the fundamental class $[X_1]$ to the fixed points.

Hint: to compute the restriction of $[X_1]$ at $lin\{\varepsilon_i, \varepsilon_j\}$ one can use the divisibility relations or the fact that $\int_{Gr_2}(\mathbb{C}^4)[X_1] = 0$.