Some problems at the beginning

1 [Done] Let S^1 act on a smooth manifold M, and let $p \in M$ be an isolated fixed point. Let v be the fundamental vector field, e.g.

$$v(x) = \tfrac{d}{dt} tx_{|t=0} \,.$$

Compute the index of v at p.

(You can assume that $M = \mathbb{R}^n$ and the action is linear.)

2 [Done] Let p be a prime number. Let M be a compact smooth manifold with $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ smooth action. Show that $\chi(M) \equiv_p \chi(M^{\mathbb{Z}_p})$ not using triangulations. Generalize the result for p-groups and S^1.

3 [Done] Show that if $\mathbb{Z}_p$ acts without fixed points on a contractible spaces X, then X cannot be a compact manifold nor simplicial complex.
(*) Also, X cannot be a finite dimensional CW-complex (not not necessarily compact).

4 Let X be an algebraic variety over $\mathbb{C}$, Y a subvariety. Show that $\chi(X) = \chi(Y) + \chi(X \setminus Y)$.
(Assume that X and Y are smooth, eventually generalize. See Sullivan, D. *Combinatorial invariants of analytic spaces*, Lecture Notes in Math., Vol. 192 Springer-Verlag, Berlin-New York, 1971, pp. 165–168.)

5 Describe Białynicki-Birula decomposition of the quadrics:
a) $Q = \{z_0z_5 + z_1z_4 + z_2z_3 = 0\} \subset \mathbb{P}^5$ with the action of $\mathbb{C}^*$
(i)

$$t[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] = [z_0 : tz_1 : t^2z_2 : t^3z_3 : t^4z_4 : t^5z_5]$$

(ii)

$$t[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] = [z_0 : tz_1 : tz_2 : tz_3 : tz_4 : t^2z_5]$$

b) $Q = \{z_0z_4 + z_1z_3 + z_2^2 = 0\} \subset \mathbb{P}^4$ with the action of $\mathbb{C}^*$

$$t \cdot [z_0 : z_1 : z_2 : z_3 : z_4] = [z_0 : tz_1 : t^2z_2 : t^3z_3 : t^4z_4]$$

6 Fix integers $w_0, w_1, \dots, w_n$. Let $\mathbb{C}^*$ act on the projective space $\mathbb{P}^n$ by the formula

$$t \cdot [z_0 : z_1 : \dots, z_n] = [t^{w_0}z_0 : t^{w_1}z_1 : \dots, t^{w_n}z_n].$$

What are the fixed points? Describe Białynicki-Birula cells. What are their dimensions.

7 Let $(\mathbb{C}^*)^{n+1}$ act on the projective space $\mathbb{P}^n$ by the formula

$$(t_0, t_1, \dots, t_n) \cdot [z_0 : z_1 : \dots, z_n] = [t_0z_0 : t_1z_1 : \dots, t_nz_n].$$

Consider the compact torus $T_0 = (S^1)^{n+1}$ defined by $|t_i| = 1$. Show that the map

$$\mu : \mathbb{P}^n \to \mathbb{R}^{n+1}$$

$$\mu([z_0 : z_1 : \dots, z_n]) = \tfrac{1}{\sum_{k=0}^n |z_k|^2}(|z_0|^2, |z_1|^2, \dots, |z_n|^2)$$

is well defined and the fibers are exactly the orbits of T_0.